Group of Homography in Real Projective Plane

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Summary. Using the Mizar system [2], we formalized that homographies of the projective real plane (as defined in [5]), form a group.

Then, we prove that, using the notations of Borsuk and Szmielew in [3]

“Consider in space $\mathbb{RP}^2$ points P_1, P_2, P_3, P_4 of which three points are not collinear and points Q_1, Q_2, Q_3, Q_4 each three points of which are also not collinear. There exists one homography h of space $\mathbb{RP}^2$ such that $h(P_i) = Q_i$ for $i = 1, 2, 3, 4$.”

(Existence Statement 52 and Existence Statement 53) [3]. Or, using notations of Richter [11]

“Let $[a], [b], [c], [d]$ in $\mathbb{RP}^2$ be four points of which no three are collinear and let $[a'], [b'], [c'], [d']$ in $\mathbb{RP}^2$ be another four points of which no three are collinear, then there exists a 3×3 matrix M such that $[Ma] = [a'], [Mb] = [b'], [Mc] = [c'],$ and $[Md] = [d']$ ”

Makarios has formalized the same results in Isabelle/Isar (the collineations form a group, lemma statement52-existence and lemma statement 53-existence) and published it in Archive of Formal Proofs¹ [10], [9].

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¹http://isa-afp.org/entries/Tarskis_Geometry.shtml

1. PRELIMINARIES

From now on i, n denote natural numbers, r denotes a real number, r_1 denotes an element of $\mathbb{R}_F$, a, b, c denote non zero elements of $\mathbb{R}_F$, u, v denote elements of $\mathcal{E}_T^3$, p_1 denotes a finite sequence of elements of $\mathbb{R}^1$, p_3, u_4 denote finite sequences of elements of $\mathbb{R}_F$, N denotes a square matrix over $\mathbb{R}_F$ of dimension 3, K denotes a field, and k denotes an element of K .

Now we state the propositions:

- (1) $I_{\mathbb{R}_F}^{3 \times 3} = \langle \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle \rangle$.
- (2) $r_1 \cdot N = r_1 \cdot I_{\mathbb{R}_F}^{3 \times 3} \cdot N$.
- (3) If $r \neq 0$ and u is not zero, then $r \cdot u$ is not zero.

PROOF: $r \cdot u \neq 0_{\mathcal{E}_T^3}$ by [4, (52), (49)]. $\square$

Let us consider elements $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ of $\mathbb{R}_F$ and a square matrix A over $\mathbb{R}_F$ of dimension 3. Now we state the propositions:

- (4) Suppose $A = \langle \langle a_{11}, a_{12}, a_{13} \rangle, \langle a_{21}, a_{22}, a_{23} \rangle, \langle a_{31}, a_{32}, a_{33} \rangle \rangle$. Then
 - (i) $\text{Line}(A, 1) = \langle a_{11}, a_{12}, a_{13} \rangle$, and
 - (ii) $\text{Line}(A, 2) = \langle a_{21}, a_{22}, a_{23} \rangle$, and
 - (iii) $\text{Line}(A, 3) = \langle a_{31}, a_{32}, a_{33} \rangle$.
- (5) Suppose $A = \langle \langle a_{11}, a_{12}, a_{13} \rangle, \langle a_{21}, a_{22}, a_{23} \rangle, \langle a_{31}, a_{32}, a_{33} \rangle \rangle$. Then
 - (i) $A_{\square, 1} = \langle a_{11}, a_{21}, a_{31} \rangle$, and
 - (ii) $A_{\square, 2} = \langle a_{12}, a_{22}, a_{32} \rangle$, and
 - (iii) $A_{\square, 3} = \langle a_{13}, a_{23}, a_{33} \rangle$.

The theorem is a consequence of (4).

- (6) Let us consider elements $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{31}, b_{32}, b_{33}$ of $\mathbb{R}_F$, and square matrices A, B over $\mathbb{R}_F$ of dimension 3. Suppose $A = \langle \langle a_{11}, a_{12}, a_{13} \rangle, \langle a_{21}, a_{22}, a_{23} \rangle, \langle a_{31}, a_{32}, a_{33} \rangle \rangle$ and $B = \langle \langle b_{11}, b_{12}, b_{13} \rangle, \langle b_{21}, b_{22}, b_{23} \rangle, \langle b_{31}, b_{32}, b_{33} \rangle \rangle$. Then $A \cdot B = \langle \langle a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}, a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}, a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \rangle, \langle a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31}, a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32}, a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \rangle, \langle a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31}, a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32}, a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \rangle \rangle$. The theorem is a consequence of (4) and (5).
- (7) Let us consider elements $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, b_1, b_2, b_3$ of $\mathbb{R}_F$, a matrix A over $\mathbb{R}_F$ of dimension 3×3 , and a matrix B over $\mathbb{R}_F$ of dimension 3×1 . Suppose $A = \langle \langle a_{11}, a_{12}, a_{13} \rangle, \langle a_{21}, a_{22}, a_{23} \rangle, \langle a_{31}, a_{32}, a_{33} \rangle \rangle$ and $B = \langle \langle b_1 \rangle, \langle b_2 \rangle, \langle b_3 \rangle \rangle$. Then $A \cdot B = \langle \langle a_{11} \cdot b_1 + a_{12} \cdot b_2 + a_{13} \cdot b_3 \rangle, \langle a_{21} \cdot b_1 + a_{22} \cdot b_2 + a_{23} \cdot b_3 \rangle, \langle a_{31} \cdot b_1 + a_{32} \cdot b_2 + a_{33} \cdot b_3 \rangle \rangle$.

$\langle a_{21} \cdot b_1 + a_{22} \cdot b_2 + a_{23} \cdot b_3 \rangle, \langle a_{31} \cdot b_1 + a_{32} \cdot b_2 + a_{33} \cdot b_3 \rangle$. The theorem is a consequence of (4).

- (8) Let us consider non zero elements a, b, c of $\mathbb{R}_F$, and square matrices M_1, M_2 over $\mathbb{R}_F$ of dimension 3. Suppose $M_1 = \langle \langle a, 0, 0 \rangle, \langle 0, b, 0 \rangle, \langle 0, 0, c \rangle \rangle$ and $M_2 = \langle \langle \frac{1}{a}, 0, 0 \rangle, \langle 0, \frac{1}{b}, 0 \rangle, \langle 0, 0, \frac{1}{c} \rangle \rangle$. Then

(i) $M_1 \cdot M_2 = I_{\mathbb{R}_F}^{3 \times 3}$, and

(ii) $M_2 \cdot M_1 = I_{\mathbb{R}_F}^{3 \times 3}$.

The theorem is a consequence of (1).

- (9) Let us consider non zero elements a, b, c of $\mathbb{R}_F$. Then $\langle \langle a, 0, 0 \rangle, \langle 0, b, 0 \rangle, \langle 0, 0, c \rangle \rangle$ is an invertible square matrix over $\mathbb{R}_F$ of dimension 3. The theorem is a consequence of (8).

- (10) (i) $[1, 0, 0]$ is not zero, and

(ii) $[0, 1, 0]$ is not zero, and

(iii) $[0, 0, 1]$ is not zero, and

(iv) $[1, 1, 1]$ is not zero.

- (11) (i) $[1, 0, 0] \neq 0_{\mathcal{E}_T^3}$, and

(ii) $[0, 1, 0] \neq 0_{\mathcal{E}_T^3}$, and

(iii) $[0, 0, 1] \neq 0_{\mathcal{E}_T^3}$, and

(iv) $[1, 1, 1] \neq 0_{\mathcal{E}_T^3}$.

PROOF: $[1, 0, 0] \neq [0, 0, 0]$ by [7, (2)]. $[0, 1, 0] \neq [0, 0, 0]$ by [7, (2)]. $[0, 0, 1] \neq [0, 0, 0]$ by [7, (2)]. $[1, 1, 1] \neq [0, 0, 0]$ by [7, (2)]. $\square$

- (12) (i) $e_1 \neq 0_{\mathcal{E}_T^3}$, and

(ii) $e_2 \neq 0_{\mathcal{E}_T^3}$, and

(iii) $e_3 \neq 0_{\mathcal{E}_T^3}$.

PROOF: $[1, 0, 0] \neq [0, 0, 0]$ by [7, (2)]. $[0, 1, 0] \neq [0, 0, 0]$ by [7, (2)]. $[0, 0, 1] \neq [0, 0, 0]$ by [7, (2)]. $\square$

Let n be a natural number. Note that $I_{\mathbb{R}_F}^{n \times n}$ is invertible.

Let M be an invertible square matrix over $\mathbb{R}_F$ of dimension n . One can verify that $M^\sim$ is invertible.

Let K be a field and N_1, N_2 be invertible square matrices over K of dimension n . One can check that $N_1 \cdot N_2$ is invertible.

2. GROUP OF HOMOGRAPHY

From now on N , N_1 , N_2 denote invertible square matrices over $\mathbb{R}_F$ of dimension 3 and P , P_1 , P_2 , P_3 denote points of the projective space over $\mathcal{E}_T^3$.

Now we state the propositions:

- (13) (The homography of N_1)((the homography of N_2)(P)) = (the homography of $N_1 \cdot N_2$)(P).

PROOF: Consider u_{12} , v_{12} being elements of $\mathcal{E}_T^3$, u_8 being a finite sequence of elements of $\mathbb{R}_F$, p_{12} being a finite sequence of elements of $\mathbb{R}^1$ such that P = the direction of u_{12} and u_{12} is not zero and $u_{12} = u_8$ and $p_{12} = N_1 \cdot N_2 \cdot u_8$ and $v_{12} = \text{M2F}(p_{12})$ and v_{12} is not zero and (the homography of $N_1 \cdot N_2$)(P) = the direction of v_{12} . Consider u_2 , v_2 being elements of $\mathcal{E}_T^3$, u_6 being a finite sequence of elements of $\mathbb{R}_F$, p_2 being a finite sequence of elements of $\mathbb{R}^1$ such that P = the direction of u_2 and u_2 is not zero and $u_2 = u_6$ and $p_2 = N_2 \cdot u_6$ and $v_2 = \text{M2F}(p_2)$ and v_2 is not zero and (the homography of N_2)(P) = the direction of v_2 . Consider u_1 , v_1 being elements of $\mathcal{E}_T^3$, u_7 being a finite sequence of elements of $\mathbb{R}_F$, p_1 being a finite sequence of elements of $\mathbb{R}^1$ such that (the homography of N_2)(P) = the direction of u_1 and u_1 is not zero and $u_1 = u_7$ and $p_1 = N_1 \cdot u_7$ and $v_1 = \text{M2F}(p_1)$ and v_1 is not zero and (the homography of N_1)((the homography of N_2)(P)) = the direction of v_1 . Consider a being a real number such that $a \neq 0$ and $u_2 = a \cdot u_{12}$. Consider b being a real number such that $b \neq 0$ and $u_1 = b \cdot v_2$. $v_1 = \langle (N_1 \cdot \langle u_7 \rangle^T)_{1,1}, (N_1 \cdot \langle u_7 \rangle^T)_{2,1}, (N_1 \cdot \langle u_7 \rangle^T)_{3,1} \rangle$ by [1, (1), (40)]. $v_2 = \langle (N_2 \cdot \langle u_6 \rangle^T)_{1,1}, (N_2 \cdot \langle u_6 \rangle^T)_{2,1}, (N_2 \cdot \langle u_6 \rangle^T)_{3,1} \rangle$ by [1, (1), (40)]. $v_{12} = \langle (N_1 \cdot N_2 \cdot \langle u_8 \rangle^T)_{1,1}, (N_1 \cdot N_2 \cdot \langle u_8 \rangle^T)_{2,1}, (N_1 \cdot N_2 \cdot \langle u_8 \rangle^T)_{3,1} \rangle$ by [1, (1), (40)]. Reconsider $v_6 = v_2$ as a finite sequence of elements of $\mathbb{R}_F$. Reconsider $i_4 = \frac{1}{b}$ as a real number. $v_6 = i_4 \cdot u_1$ by [4, (49), (52)]. Reconsider $l_{11} = \text{Line}(N_2, 1)(1)$, $l_{12} = \text{Line}(N_2, 1)(2)$, $l_{13} = \text{Line}(N_2, 1)(3)$, $l_{21} = \text{Line}(N_2, 2)(1)$, $l_{22} = \text{Line}(N_2, 2)(2)$, $l_{23} = \text{Line}(N_2, 2)(3)$, $l_{31} = \text{Line}(N_2, 3)(1)$, $l_{32} = \text{Line}(N_2, 3)(2)$, $l_{33} = \text{Line}(N_2, 3)(3)$ as an element of $\mathbb{R}_F$. $N_{2\Box,1} = \langle l_{11}, l_{21}, l_{31} \rangle$ and $N_{2\Box,2} = \langle l_{12}, l_{22}, l_{32} \rangle$ and $N_{2\Box,3} = \langle l_{13}, l_{23}, l_{33} \rangle$ by [1, (1), (45)]. The direction of v_1 = the direction of v_{12} by [5, (7)], [1, (45)], [5, (93)], [7, (8)]. $\square$

- (14) (The homography of $I_{\mathbb{R}_F}^{3 \times 3}$)(P) = P .

- (15) (i) (the homography of N)((the homography of $N^\sim$)(P)) = P , and
(ii) (the homography of $N^\sim$)((the homography of N)(P)) = P .

The theorem is a consequence of (13) and (14).

- (16) If (the homography of N)(P_1) = (the homography of N)(P_2), then $P_1 = P_2$. The theorem is a consequence of (15).

- (17) Let us consider a non zero element a of $\mathbb{R}_F$. Suppose $a \cdot I_{\mathbb{R}_F}^{3 \times 3} = N$. Then (the homography of N)(P) = P .

The functor `EnsHomography3` yielding a set is defined by the term

- (Def. 1) the set of all the homography of N where N is an invertible square matrix over $\mathbb{R}_F$ of dimension 3.

One can check that `EnsHomography3` is non empty.

Let h_1, h_2 be elements of `EnsHomography3`. The functor $h_1 \circ h_2$ yielding an element of `EnsHomography3` is defined by

- (Def. 2) there exist invertible square matrices N_1, N_2 over $\mathbb{R}_F$ of dimension 3 such that h_1 = the homography of N_1 and h_2 = the homography of N_2 and it = the homography of $N_1 \cdot N_2$.

Now we state the propositions:

- (18) Let us consider elements h_1, h_2 of `EnsHomography3`. Suppose h_1 = the homography of N_1 and h_2 = the homography of N_2 . Then the homography of $N_1 \cdot N_2 = h_1 \circ h_2$. The theorem is a consequence of (13).
 (19) Let us consider elements x, y, z of `EnsHomography3`. Then $(x \circ y) \circ z = x \circ (y \circ z)$. The theorem is a consequence of (18).

The functor `BinOpHomography3` yielding a binary operation on `EnsHomography3` is defined by

- (Def. 3) for every elements h_1, h_2 of `EnsHomography3`, $it(h_1, h_2) = h_1 \circ h_2$.

The functor `GroupHomography3` yielding a strict multiplicative magma is defined by the term

- (Def. 4) $\langle \text{EnsHomography3}, \text{BinOpHomography3} \rangle$.

Note that `GroupHomography3` is non empty, associative, and group-like.

Now we state the propositions:

- (20) $\mathbf{1}_{\text{GroupHomography3}} = \text{the homography of } I_{\mathbb{R}_F}^{3 \times 3}$.
 (21) Let us consider elements h, g of `GroupHomography3`, and invertible square matrices N, N_{10} over $\mathbb{R}_F$ of dimension 3. Suppose h = the homography of N and g = the homography of N_{10} and $N_{10} = N^\sim$. Then $g = h^{-1}$. The theorem is a consequence of (20).

3. MAIN RESULTS

The functors: `Dir100`, `Dir010`, `Dir001`, and `Dir111` yielding points of the projective space over $\mathcal{E}_T^3$ are defined by terms

- (Def. 5) the direction of $[1, 0, 0]$,
 (Def. 6) the direction of $[0, 1, 0]$,

(Def. 7) the direction of $[0, 0, 1]$,

(Def. 8) the direction of $[1, 1, 1]$,

respectively. Now we state the proposition:

- (22) (i) $\text{Dir100} \neq \text{Dir010}$, and
 (ii) $\text{Dir100} \neq \text{Dir001}$, and
 (iii) $\text{Dir100} \neq \text{Dir111}$, and
 (iv) $\text{Dir010} \neq \text{Dir001}$, and
 (v) $\text{Dir010} \neq \text{Dir111}$, and
 (vi) $\text{Dir001} \neq \text{Dir111}$.

Let a be a non zero element of $\mathbb{R}_F$. Let us consider N . Note that $a \cdot N$ is invertible as a square matrix over $\mathbb{R}_F$ of dimension 3.

- (23) Let us consider a non zero element a of $\mathbb{R}_F$. Then (the homography of $a \cdot N_1$)(P) = (the homography of N_1)(P). The theorem is a consequence of (2), (13), and (17).
- (24) Suppose P_1, P_2 and P_3 are not collinear. Then there exists an invertible square matrix N over $\mathbb{R}_F$ of dimension 3 such that
- (i) (the homography of N)(P_1) = Dir100, and
 - (ii) (the homography of N)(P_2) = Dir010, and
 - (iii) (the homography of N)(P_3) = Dir001.

PROOF: Consider u_1 being an element of $\mathcal{E}_T^3$ such that u_1 is not zero and P_1 = the direction of u_1 . Consider u_2 being an element of $\mathcal{E}_T^3$ such that u_2 is not zero and P_2 = the direction of u_2 . Consider u_3 being an element of $\mathcal{E}_T^3$ such that u_3 is not zero and P_3 = the direction of u_3 . Reconsider $p_3 = u_1, q_1 = u_2, r_2 = u_3$ as a finite sequence of elements of $\mathbb{R}_F$. Consider N being a square matrix over $\mathbb{R}_F$ of dimension 3 such that N is invertible and $N \cdot p_3 = \text{F2M}(e_1)$ and $N \cdot q_1 = \text{F2M}(e_2)$ and $N \cdot r_2 = \text{F2M}(e_3)$. (The homography of N)(P_1) = Dir100 by [8, (22), (1)], [6, (22)], [5, (75)]. (The homography of N)(P_2) = Dir010 by [8, (22), (1)], [6, (22)], [5, (75)]. (The homography of N)(P_3) = Dir001 by [8, (22), (1)], [6, (22)], [5, (75)]. $\square$

- (25) Let us consider non zero elements a, b, c of $\mathbb{R}_F$. Suppose $N = \langle \langle a, 0, 0 \rangle, \langle 0, b, 0 \rangle, \langle 0, 0, c \rangle \rangle$. Then
- (i) (the homography of N)(Dir100) = Dir100, and
 - (ii) (the homography of N)(Dir010) = Dir010, and
 - (iii) (the homography of N)(Dir001) = Dir001.

PROOF: (The homography of N)(Dir100) = Dir100 by (12), [8, (22), (1)], [7, (8), (2)]. (The homography of N)(Dir010) = Dir010 by (12), [8, (22), (1)], [7, (8), (2)]. (The homography of N)(Dir001) = Dir001 by (12), [8, (22), (1)], [7, (8), (2)]. $\square$

Let us consider a point P of the projective space over $\mathcal{E}_T^3$.

(26) There exist elements a, b, c of $\mathbb{R}_F$ such that

- (i) P = the direction of $[a, b, c]$, and
- (ii) $a \neq 0$ or $b \neq 0$ or $c \neq 0$.

(27) Suppose Dir100, Dir010 and P are not collinear and Dir100, Dir001 and P are not collinear and Dir010, Dir001 and P are not collinear. Then there exist non zero elements a, b, c of $\mathbb{R}_F$ such that P = the direction of $[a, b, c]$. The theorem is a consequence of (26).

(28) Let us consider non zero elements a, b, c, i_1, i_2, i_3 of $\mathbb{R}_F$, a point P of the projective space over $\mathcal{E}_T^3$, and an invertible square matrix N over $\mathbb{R}_F$ of dimension 3. Suppose P = the direction of $[a, b, c]$ and $i_1 = \frac{1}{a}$ and $i_2 = \frac{1}{b}$ and $i_3 = \frac{1}{c}$ and $N = \langle \langle i_1, 0, 0 \rangle, \langle 0, i_2, 0 \rangle, \langle 0, 0, i_3 \rangle \rangle$. Then (the homography of N)(P) = the direction of $[1, 1, 1]$.

PROOF: Consider u, v being elements of $\mathcal{E}_T^3$, u_4 being a finite sequence of elements of $\mathbb{R}_F$, p being a finite sequence of elements of $\mathbb{R}^1$ such that P = the direction of u and u is not zero and $u = u_4$ and $p = N \cdot u_4$ and $v = M2F(p)$ and v is not zero and (the homography of N)(P) = the direction of v . $[a, b, c]$ is not zero by [7, (4)], [1, (78)]. Consider d being a real number such that $d \neq 0$ and $u = d \cdot [a, b, c]$. Reconsider $d_1 = d \cdot a, d_2 = d \cdot b, d_3 = d \cdot c$ as an element of $\mathbb{R}_F$. $v = [i_1 \cdot d_1, i_2 \cdot d_2, i_3 \cdot d_3]$ by [1, (45)]. $\square$

(29) Let us consider a point P of the projective space over $\mathcal{E}_T^3$. Suppose Dir100, Dir010 and P are not collinear and Dir100, Dir001 and P are not collinear and Dir010, Dir001 and P are not collinear. Then there exist non zero elements a, b, c of $\mathbb{R}_F$ such that for every invertible square matrix N over $\mathbb{R}_F$ of dimension 3 such that $N = \langle \langle a, 0, 0 \rangle, \langle 0, b, 0 \rangle, \langle 0, 0, c \rangle \rangle$ holds (the homography of N)(P) = Dir111. The theorem is a consequence of (27) and (28).

(30) Let us consider points P_1, P_2, P_3, P_4 of the projective space over $\mathcal{E}_T^3$. Suppose P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear. Then there exists an invertible square matrix N over $\mathbb{R}_F$ of dimension 3 such that

- (i) (the homography of N)(P_1) = Dir100, and
- (ii) (the homography of N)(P_2) = Dir010, and

- (iii) (the homography of N)(P_3) = Dir001, and
- (iv) (the homography of N)(P_4) = Dir111.

The theorem is a consequence of (24), (29), (9), (25), and (13).

- (31) Let us consider points $P_1, P_2, P_3, P_4, Q_1, Q_2, Q_3, Q_4$ of the projective space over $\mathcal{E}_T^3$. Suppose P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear and Q_1, Q_2 and Q_3 are not collinear and Q_1, Q_2 and Q_4 are not collinear and Q_1, Q_3 and Q_4 are not collinear and Q_2, Q_3 and Q_4 are not collinear. Then there exists an invertible square matrix N over $\mathbb{R}_F$ of dimension 3 such that

- (i) (the homography of N)(P_1) = Q_1 , and
- (ii) (the homography of N)(P_2) = Q_2 , and
- (iii) (the homography of N)(P_3) = Q_3 , and
- (iv) (the homography of N)(P_4) = Q_4 .

The theorem is a consequence of (30), (15), and (13).

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