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Fubini's Theorem on Measure

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Summary. The purpose of this article is to show Fubini's theorem on measure [16], [4], [7], [15], [18]. Some theorems have the possibility of slight generalization, but we have priority to avoid the complexity of the description. First of all, for the product measure constructed in [14], we show some theorems. Then we introduce the section which plays an important role in Fubini's theorem, and prove the relevant proposition. Finally we show Fubini's theorem on measure.

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1. Preliminaries

Now we state the propositions:

(1) Let us consider a disjoint valued finite sequence F, and natural numbers n, m. If $n < m$, then $\bigcup \operatorname{rng}(F \restriction n)$ misses $F(m)$.

(2) Let us consider a finite sequence F, and natural numbers m, n. Suppose $m \leqslant n$. Then $\operatorname{len}(F \restriction m) \leqslant \operatorname{len}(F \restriction n)$.

(3) Let us consider a finite sequence F, and a natural number n. Then $\bigcup \operatorname{rng}(F \restriction n) \cup F(n+1) = \bigcup \operatorname{rng}(F \restriction (n+1))$. The theorem is a consequence of (2).

(4) Let us consider a disjoint valued finite sequence F, and a natural number n. Then $\bigcup (F \restriction n)$ misses $F(n+1)$.

(5) Let us consider a set P, and a finite sequence F. Suppose P is $\cup$-closed and $\emptyset \in P$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) \in P$. Then $\bigcup F \in P$.

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PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \bigcup \operatorname{rng}(F{\restriction}\$_1) \in P$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (13)], [3, (59)], [19, (55)], [3, (82)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

Let A, X be sets. Observe that the functor $\chi_{A,X}$ yields a function from X into $\overline{\mathbb{R}}$. Let X be a non empty set, S be a σ-field of subsets of X, and F be a finite sequence of elements of S. Let us observe that the functor $\bigcup F$ yields an element of S. Let F be a sequence of S. Let us note that the functor $\bigcup F$ yields an element of S. Let F be a finite sequence of elements of $X \dot{\to} \overline{\mathbb{R}}$ and x be an element of X. The functor $F\#x$ yielding a finite sequence of elements of $\overline{\mathbb{R}}$ is defined by

(Def. 1) dom it = dom F and for every element n of $\mathbb{N}$ such that $n \in \operatorname{dom} it$ holds $it(n) = F(n)(x)$.

Now we state the proposition:

(6) Let us consider a non empty set X, a non empty family S of subsets of X, a finite sequence f of elements of S, and a finite sequence F of elements of $X \dot{\to} \overline{\mathbb{R}}$. Suppose $\operatorname{dom} f = \operatorname{dom} F$ and f is disjoint valued and for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = \chi_{f(n),X}$. Let us consider an element x of X. Then $\chi_{\bigcup f,X}(x) = \sum(F\#x)$.

2. Product Measure and Product σ-measure

Now we state the proposition:

(7) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, and a σ-field S_2 of subsets of X_2. Then $\sigma(\text{DisUnion}\,\text{MeasRect}(S_1, S_2)) = \sigma(\text{MeasRect}(S_1, S_2))$.

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, S_2 be a σ-field of subsets of X_2, M_1 be a σ-measure on S_1, and M_2 be a σ-measure on S_2. The functor $\text{ProdMeas}(M_1, M_2)$ yielding an induced measure of $\text{MeasRect}(S_1, S_2)$ and $\text{ProdpreMeas}(M_1, M_2)$ is defined by

(Def. 2) for every set E such that $E \in$ the field generated by $\text{MeasRect}(S_1, S_2)$ for every disjoint valued finite sequence F of elements of $\text{MeasRect}(S_1, S_2)$ such that $E = \bigcup F$ holds $it(E) = \sum(\text{ProdpreMeas}(M_1, M_2) \cdot F)$.

The functor $\text{Prod}\,\sigma\text{-Meas}(M_1, M_2)$ yielding an induced σ-measure of $\text{MeasRect}(S_1, S_2)$ and $\text{ProdMeas}(M_1, M_2)$ is defined by the term

(Def. 3) σ-Meas(the Caratheodory measure determined by $\text{ProdMeas}(M_1, M_2)){\restriction}\sigma(\text{MeasRect}(S_1, S_2))$.

Now we state the propositions:

(8) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and a σ-measure M_2 on S_2. Then $\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2)$ is a σ-measure on $\sigma(\mathrm{MeasRect}(S_1, S_2))$. The theorem is a consequence of (7).

(9) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a set sequence F_1 of S_1, a set sequence F_2 of S_2, and a natural number n. Then $F_1(n) \times F_2(n)$ is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. The theorem is a consequence of (7).

(10) Let us consider sets X_1, X_2, a sequence F_1 of subsets of X_1, a sequence F_2 of subsets of X_2, and a natural number n. Suppose F_1 is non descending and F_2 is non descending. Then $F_1(n) \times F_2(n) \subseteq F_1(n+1) \times F_2(n+1)$.

(11) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element A of S_1, and an element B of S_2. Then $(\mathrm{ProdMeas}(M_1, M_2))(A \times B) = M_1(A) \cdot M_2(B)$.

(12) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a set sequence F_1 of S_1, a set sequence F_2 of S_2, and a natural number n. Then $(\mathrm{ProdMeas}(M_1, M_2))(F_1(n) \times F_2(n)) = M_1(F_1(n)) \cdot M_2(F_2(n))$. The theorem is a consequence of (11).

(13) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a finite sequence F_1 of elements of S_1, a finite sequence F_2 of elements of S_2, and a natural number n. Suppose $n \in \mathrm{dom}\, F_1$ and $n \in \mathrm{dom}\, F_2$. Then $(\mathrm{ProdMeas}(M_1, M_2))(F_1(n) \times F_2(n)) = M_1(F_1(n)) \cdot M_2(F_2(n))$.

(14) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and a subset E of $X_1 \times X_2$. Then (the Caratheodory measure determined by $\mathrm{ProdMeas}(M_1, M_2))(E) = \inf \mathrm{Svc}(\mathrm{ProdMeas}(M_1, M_2), E)$.

(15) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and a σ-measure M_2 on S_2. Then $\sigma(\mathrm{MeasRect}(S_1, S_2)) \subseteq \sigma$-Field(the Caratheodory measure determined by $\mathrm{ProdMeas}(M_1, M_2)$). The theorem is a consequence of (7).

(16) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element A of S_1, and an element B of S_2. Suppose $E = A \times B$. Then $(\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(E) = M_1(A) \cdot M_2(B)$. The theorem is a consequence of (15) and (11).

(17) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1,

a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a set sequence F_1 of S_1, a set sequence F_2 of S_2, and a natural number n. Then $(\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(F_1(n) \times F_2(n)) = M_1(F_1(n)) \cdot M_2(F_2(n))$. The theorem is a consequence of (9), (15), and (12).

(18) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose E_1 misses E_2. Then $(\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E_1 \cup E_2) = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E_1) + (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E_2)$. The theorem is a consequence of (8).

(19) Let us consider sets X_1, X_2, A, B, a sequence F_1 of subsets of X_1, a sequence F_2 of subsets of X_2, and a sequence F of subsets of $X_1 \times X_2$. Suppose F_1 is non descending and $\lim F_1 = A$ and F_2 is non descending and $\lim F_2 = B$ and for every natural number n, $F(n) = F_1(n) \times F_2(n)$. Then $\lim F = A \times B$. The theorem is a consequence of (10).

3. Sections

Let X be a set, Y be a non empty set, E be a subset of $X \times Y$, and x be a set. The functor $\mathrm{Xsection}(E, x)$ yielding a subset of Y is defined by the term

(Def. 4) $\{y$, where y is an element of $Y : \langle x, y\rangle \in E\}$.

Let X be a non empty set, Y be a set, and y be a set.

The functor $\mathrm{Ysection}(E, y)$ yielding a subset of X is defined by the term

(Def. 5) $\{x$, where x is an element of $X : \langle x, y\rangle \in E\}$.

Now we state the propositions:

(20) Let us consider a set X, a non empty set Y, subsets E_1, E_2 of $X \times Y$, and a set p. Suppose $E_1 \subseteq E_2$. Then $\mathrm{Xsection}(E_1, p) \subseteq \mathrm{Xsection}(E_2, p)$.

(21) Let us consider a non empty set X, a set Y, subsets E_1, E_2 of $X \times Y$, and a set p. Suppose $E_1 \subseteq E_2$. Then $\mathrm{Ysection}(E_1, p) \subseteq \mathrm{Ysection}(E_2, p)$.

(22) Let us consider non empty sets X, Y, a subset A of X, a subset B of Y, and a set p. Then

(i) if $p \in A$, then $\mathrm{Xsection}(A \times B, p) = B$, and

(ii) if $p \notin A$, then $\mathrm{Xsection}(A \times B, p) = \emptyset$, and

(iii) if $p \in B$, then $\mathrm{Ysection}(A \times B, p) = A$, and

(iv) if $p \notin B$, then $\mathrm{Ysection}(A \times B, p) = \emptyset$.

(23) Let us consider non empty sets X, Y, a subset E of $X \times Y$, and a set p. Then

(i) if $p \notin X$, then $\mathrm{Xsection}(E, p) = \emptyset$, and

(ii) if $p \notin Y$, then $\mathrm{Ysection}(E, p) = \emptyset$.

(24) Let us consider non empty sets X, Y, and a set p. Then

(i) $\mathrm{Xsection}(\emptyset_{X \times Y}, p) = \emptyset$, and

(ii) $\mathrm{Ysection}(\emptyset_{X \times Y}, p) = \emptyset$, and

(iii) if $p \in X$, then $\mathrm{Xsection}(\Omega_{X \times Y}, p) = Y$, and

(iv) if $p \in Y$, then $\mathrm{Ysection}(\Omega_{X \times Y}, p) = X$.

The theorem is a consequence of (22).

(25) Let us consider non empty sets X, Y, a subset E of $X \times Y$, and a set p. Then

(i) if $p \in X$, then $\mathrm{Xsection}(X \times Y \setminus E, p) = Y \setminus \mathrm{Xsection}(E, p)$, and

(ii) if $p \in Y$, then $\mathrm{Ysection}(X \times Y \setminus E, p) = X \setminus \mathrm{Ysection}(E, p)$.

Let us consider non empty sets X, Y, subsets E_1, E_2 of $X \times Y$, and a set p.

(26) (i) $\mathrm{Xsection}(E_1 \cup E_2, p) = \mathrm{Xsection}(E_1, p) \cup \mathrm{Xsection}(E_2, p)$, and

(ii) $\mathrm{Ysection}(E_1 \cup E_2, p) = \mathrm{Ysection}(E_1, p) \cup \mathrm{Ysection}(E_2, p)$.

(27) (i) $\mathrm{Xsection}(E_1 \cap E_2, p) = \mathrm{Xsection}(E_1, p) \cap \mathrm{Xsection}(E_2, p)$, and

(ii) $\mathrm{Ysection}(E_1 \cap E_2, p) = \mathrm{Ysection}(E_1, p) \cap \mathrm{Ysection}(E_2, p)$.

Now we state the propositions:

(28) Let us consider a set X, a non empty set Y, a finite sequence F of elements of $2^{X \times Y}$, a finite sequence F_4 of elements of 2^Y, and a set p. Suppose $\mathrm{dom}\, F = \mathrm{dom}\, F_4$ and for every natural number n such that $n \in \mathrm{dom}\, F_4$ holds $F_4(n) = \mathrm{Xsection}(F(n), p)$. Then $\mathrm{Xsection}(\bigcup \mathrm{rng}\, F, p) = \bigcup \mathrm{rng}\, F_4$.

(29) Let us consider a non empty set X, a set Y, a finite sequence F of elements of $2^{X \times Y}$, a finite sequence F_3 of elements of 2^X, and a set p. Suppose $\mathrm{dom}\, F = \mathrm{dom}\, F_3$ and for every natural number n such that $n \in \mathrm{dom}\, F_3$ holds $F_3(n) = \mathrm{Ysection}(F(n), p)$. Then $\mathrm{Ysection}(\bigcup \mathrm{rng}\, F, p) = \bigcup \mathrm{rng}\, F_3$.

Let us consider a set X, a non empty set Y, a set p, a sequence F of subsets of $X \times Y$, and a sequence F_4 of subsets of Y. Now we state the propositions:

(30) If for every natural number n, $F_4(n) = \mathrm{Xsection}(F(n), p)$, then $\mathrm{Xsection}(\bigcup \mathrm{rng}\, F, p) = \bigcup \mathrm{rng}\, F_4$.

(31) If for every natural number n, $F_4(n) = \mathrm{Xsection}(F(n), p)$, then $\mathrm{Xsection}(\bigcap \mathrm{rng}\, F, p) = \bigcap \mathrm{rng}\, F_4$.

Let us consider a non empty set X, a set Y, a set p, a sequence F of subsets of $X \times Y$, and a sequence F_3 of subsets of X. Now we state the propositions:

(32) If for every natural number n, $F_3(n) = \mathrm{Ysection}(F(n), p)$, then $\mathrm{Ysection}(\bigcup \mathrm{rng}\, F, p) = \bigcup \mathrm{rng}\, F_3$.

(33) If for every natural number n, $F_3(n) = \mathrm{Ysection}(F(n), p)$, then $\mathrm{Ysection}(\bigcap \mathrm{rng}\, F, p) = \bigcap \mathrm{rng}\, F_3$.

(34) Let us consider non empty sets X, Y, sets x, y, and a subset E of $X \times Y$. Then

(i) $\chi_{E, X \times Y}(x, y) = \chi_{\mathrm{Xsection}(E,x), Y}(y)$, and

(ii) $\chi_{E, X \times Y}(x, y) = \chi_{\mathrm{Ysection}(E,y), X}(x)$.

(35) Let us consider non empty sets X, Y, subsets E_1, E_2 of $X \times Y$, and a set p. Suppose E_1 misses E_2. Then

(i) $\mathrm{Xsection}(E_1, p)$ misses $\mathrm{Xsection}(E_2, p)$, and

(ii) $\mathrm{Ysection}(E_1, p)$ misses $\mathrm{Ysection}(E_2, p)$.

(36) Let us consider non empty sets X, Y, a disjoint valued finite sequence F of elements of $2^{X \times Y}$, and a set p. Then

(i) there exists a disjoint valued finite sequence F_4 of elements of 2^X such that $\mathrm{dom}\, F = \mathrm{dom}\, F_4$ and for every natural number n such that $n \in \mathrm{dom}\, F_4$ holds $F_4(n) = \mathrm{Ysection}(F(n), p)$, and

(ii) there exists a disjoint valued finite sequence F_3 of elements of 2^Y such that $\mathrm{dom}\, F = \mathrm{dom}\, F_3$ and for every natural number n such that $n \in \mathrm{dom}\, F_3$ holds $F_3(n) = \mathrm{Xsection}(F(n), p)$.

Proof: There exists a disjoint valued finite sequence F_4 of elements of 2^X such that $\mathrm{dom}\, F = \mathrm{dom}\, F_4$ and for every natural number n such that $n \in \mathrm{dom}\, F_4$ holds $F_4(n) = \mathrm{Ysection}(F(n), p)$ by (35), [19, (29)]. There exists a disjoint valued finite sequence F_3 of elements of 2^Y such that $\mathrm{dom}\, F = \mathrm{dom}\, F_3$ and for every natural number n such that $n \in \mathrm{dom}\, F_3$ holds $F_3(n) = \mathrm{Xsection}(F(n), p)$ by (35), [19, (29)]. □

(37) Let us consider non empty sets X, Y, a disjoint valued sequence F of subsets of $X \times Y$, and a set p. Then

(i) there exists a disjoint valued sequence F_4 of subsets of X such that for every natural number n, $F_4(n) = \mathrm{Ysection}(F(n), p)$, and

(ii) there exists a disjoint valued sequence F_3 of subsets of Y such that for every natural number n, $F_3(n) = \mathrm{Xsection}(F(n), p)$.

Proof: There exists a disjoint valued sequence F_4 of subsets of X such that for every natural number n, $F_4(n) = \mathrm{Ysection}(F(n), p)$. Define $\mathcal{A}$(natural number) $= \mathrm{Xsection}(F(\$_1), p)$. Consider F_3 being a sequence of subsets of Y such that for every element n of $\mathbb{N}$, $F_3(n) = \mathcal{A}(n)$ from [11, Sch. 4]. □

(38) Let us consider non empty sets X, Y, sets x, y, and subsets E_1, E_2 of $X \times Y$. Suppose E_1 misses E_2. Then

(i) $\chi_{E_1 \cup E_2, X \times Y}(x, y) = \chi_{\mathrm{Xsection}(E_1, x), Y}(y) + \chi_{\mathrm{Xsection}(E_2, x), Y}(y)$, and

(ii) $\chi_{E_1 \cup E_2, X \times Y}(x, y) = \chi_{\mathrm{Ysection}(E_1, y), X}(x) + \chi_{\mathrm{Ysection}(E_2, y), X}(x)$.

The theorem is a consequence of (35), (34), and (26).

(39) Let us consider a set X, a non empty set Y, a set x, a sequence E of subsets of $X \times Y$, and a sequence G of subsets of Y. Suppose E is non descending and for every natural number n, $G(n) = \mathrm{Xsection}(E(n), x)$. Then G is non descending. The theorem is a consequence of (20).

(40) Let us consider a non empty set X, a set Y, a set x, a sequence E of subsets of $X \times Y$, and a sequence G of subsets of X. Suppose E is non descending and for every natural number n, $G(n) = \mathrm{Ysection}(E(n), x)$. Then G is non descending. The theorem is a consequence of (21).

(41) Let us consider a set X, a non empty set Y, a set x, a sequence E of subsets of $X \times Y$, and a sequence G of subsets of Y. Suppose E is non ascending and for every natural number n, $G(n) = \mathrm{Xsection}(E(n), x)$. Then G is non ascending. The theorem is a consequence of (20).

(42) Let us consider a non empty set X, a set Y, a set x, a sequence E of subsets of $X \times Y$, and a sequence G of subsets of X. Suppose E is non ascending and for every natural number n, $G(n) = \mathrm{Ysection}(E(n), x)$. Then G is non ascending. The theorem is a consequence of (21).

(43) Let us consider a set X, a non empty set Y, a sequence E of subsets of $X \times Y$, and a set x. Suppose E is non descending. Then there exists a sequence G of subsets of Y such that

(i) G is non descending, and

(ii) for every natural number n, $G(n) = \mathrm{Xsection}(E(n), x)$.

Proof: Define $\mathcal{F}(\text{natural number}) = \mathrm{Xsection}(E(\$_1), x)$. Consider G being a function from $\mathbb{N}$ into 2^Y such that for every element n of $\mathbb{N}$, $G(n) = \mathcal{F}(n)$ from [11, Sch. 4]. For every natural number n, $G(n) = \mathrm{Xsection}(E(n), x)$. $\square$

(44) Let us consider a non empty set X, a set Y, a sequence E of subsets of $X \times Y$, and a set x. Suppose E is non descending. Then there exists a sequence G of subsets of X such that

(i) G is non descending, and

(ii) for every natural number n, $G(n) = \mathrm{Ysection}(E(n), x)$.

Proof: Define $\mathcal{F}(\text{natural number}) = \mathrm{Ysection}(E(\$_1), x)$. Consider G being a function from $\mathbb{N}$ into 2^X such that for every element n of $\mathbb{N}$, $G(n) = \mathcal{F}(n)$ from [11, Sch. 4]. For every natural number n, $G(n) = \mathrm{Ysection}(E(n), x)$. $\square$

(45) Let us consider a set X, a non empty set Y, a sequence E of subsets of $X \times Y$, and a set x. Suppose E is non ascending. Then there exists a sequence G of subsets of Y such that

(i) G is non ascending, and

(ii) for every natural number n, $G(n) = \mathrm{Xsection}(E(n), x)$.

PROOF: Define $\mathcal{F}(\text{natural number}) = \mathrm{Xsection}(E(\$_1), x)$. Consider G being a function from $\mathbb{N}$ into 2^Y such that for every element n of $\mathbb{N}$, $G(n) = \mathcal{F}(n)$ from [11, Sch. 4]. For every natural number n, $G(n) = \mathrm{Xsection}(E(n), x)$. $\square$

(46) Let us consider a non empty set X, a set Y, a sequence E of subsets of $X \times Y$, and a set x. Suppose E is non ascending. Then there exists a sequence G of subsets of X such that

(i) G is non ascending, and

(ii) for every natural number n, $G(n) = \mathrm{Ysection}(E(n), x)$.

PROOF: Define $\mathcal{F}(\text{natural number}) = \mathrm{Ysection}(E(\$_1), x)$. Consider G being a function from $\mathbb{N}$ into 2^X such that for every element n of $\mathbb{N}$, $G(n) = \mathcal{F}(n)$ from [11, Sch. 4]. For every natural number n, $G(n) = \mathrm{Ysection}(E(n), x)$. $\square$

4. Measurable Sections

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a set K. Now we state the propositions:

(47) Suppose $K = \{C$, where C is a subset of $X_1 \times X_2 :$ for every set p, $\mathrm{Xsection}(C, p) \in S_2\}$. Then

(i) the field generated by $\mathrm{MeasRect}(S_1, S_2) \subseteq K$, and

(ii) K is a σ-field of subsets of $X_1 \times X_2$.

PROOF: For every set x, $\mathrm{Xsection}(\emptyset_{X_1 \times X_2}, x) \in S_2$ by (24), [5, (7)]. For every subset C of $X_1 \times X_2$ such that $C \in K$ holds $C^{\mathrm{c}} \in K$ by [17, (5), (6)], (25), (23). $\square$

(48) Suppose $K = \{C$, where C is a subset of $X_1 \times X_2 :$ for every set p, $\mathrm{Ysection}(C, p) \in S_1\}$. Then

(i) the field generated by $\mathrm{MeasRect}(S_1, S_2) \subseteq K$, and

(ii) K is a σ-field of subsets of $X_1 \times X_2$.

PROOF: For every set y, $\mathrm{Ysection}(\emptyset_{X_1 \times X_2}, y) \in S_1$ by (24), [5, (7)]. For every subset C of $X_1 \times X_2$ such that $C \in K$ holds $C^{\mathrm{c}} \in K$ by [17, (5), (6)], (25), (23). $\square$

(49) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Then

(i) for every set p, $\mathrm{Xsection}(E, p) \in S_2$, and

(ii) for every set p, $\mathrm{Ysection}(E, p) \in S_1$.

The theorem is a consequence of (47) and (48).

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, S_2 be a σ-field of subsets of X_2, E be an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and x be a set. The functor $\mathrm{MeasurableXsection}(E, x)$ yielding an element of S_2 is defined by the term

(Def. 6) $\mathrm{Xsection}(E, x)$.

Let y be a set. The functor $\mathrm{MeasurableYsection}(E, y)$ yielding an element of S_1 is defined by the term

(Def. 7) $\mathrm{Ysection}(E, y)$.

Now we state the propositions:

(50) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a finite sequence F of elements of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, a finite sequence F_4 of elements of S_2, and a set p. Suppose $\mathrm{dom}\, F = \mathrm{dom}\, F_4$ and for every natural number n such that $n \in \mathrm{dom}\, F_4$ holds $F_4(n) = \mathrm{MeasurableXsection}(F(n), p)$. Then $\mathrm{MeasurableXsection}(\bigcup F, p) = \bigcup F_4$. The theorem is a consequence of (28).

(51) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a finite sequence F of elements of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, a finite sequence F_3 of elements of S_1, and a set p. Suppose $\mathrm{dom}\, F = \mathrm{dom}\, F_3$ and for every natural number n such that $n \in \mathrm{dom}\, F_3$ holds $F_3(n) = \mathrm{MeasurableYsection}(F(n), p)$. Then $\mathrm{MeasurableYsection}(\bigcup F, p) = \bigcup F_3$. The theorem is a consequence of (29).

(52) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, an element A of S_1, an element B of S_2, and an element x of X_1. Then $M_2(B) \cdot \chi_{A, X_1}(x) = \int \mathrm{curry}(\chi_{A \times B, X_1 \times X_2}, x)\, \mathrm{d}M_2$.

PROOF: For every element y of X_2, $(\mathrm{curry}(\chi_{A \times B, X_1 \times X_2}, x))(y) = \chi_{A, X_1}(x) \cdot \chi_{B, X_2}(y)$. $\square$

(53) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, an element E of

$\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element A of S_1, an element B of S_2, and an element x of X_1. Suppose $E = A \times B$. Then $M_2(\mathrm{MeasurableXsection}(E, x)) = M_2(B) \cdot \chi_{A,X_1}(x)$. The theorem is a consequence of (22).

(54) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, an element A of S_1, an element B of S_2, and an element y of X_2. Then $M_1(A) \cdot \chi_{B,X_2}(y) = \int \mathrm{curry}'(\chi_{A \times B, X_1 \times X_2}, y)\, \mathrm{d}M_1$.
PROOF: For every element x of X_1, $(\mathrm{curry}'(\chi_{A \times B, X_1 \times X_2}, y))(x) = \chi_{A,X_1}(x) \cdot \chi_{B,X_2}(y)$. $\square$

(55) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element A of S_1, an element B of S_2, and an element y of X_2. Suppose $E = A \times B$. Then $M_1(\mathrm{MeasurableYsection}(E, y)) = M_1(A) \cdot \chi_{B,X_2}(y)$. The theorem is a consequence of (22).

5. Finite Sequence of Functions

Let X, Y be non empty sets, G be a non empty set of functions from X to Y, F be a finite sequence of elements of G, and n be a natural number. Observe that the functor F_n yields an element of G. Let X be a set and F be a finite sequence of elements of $\overline{\mathbb{R}}^X$. We say that F is (without $+\infty$)-valued if and only if

(Def. 8) for every natural number n such that $n \in \mathrm{dom}\, F$ holds $F(n)$ is without $+\infty$.

We say that F is (without $-\infty$)-valued if and only if

(Def. 9) for every natural number n such that $n \in \mathrm{dom}\, F$ holds $F(n)$ is without $-\infty$.

Now we state the proposition:

(56) Let us consider a non empty set X. Then
   (i) $\langle X \longmapsto 0 \rangle$ is a finite sequence of elements of $\overline{\mathbb{R}}^X$, and
   (ii) for every natural number n such that $n \in \mathrm{dom}\langle X \longmapsto 0 \rangle$ holds $\langle X \longmapsto 0 \rangle(n)$ is without $+\infty$, and
   (iii) for every natural number n such that $n \in \mathrm{dom}\langle X \longmapsto 0 \rangle$ holds $\langle X \longmapsto 0 \rangle(n)$ is without $-\infty$.

Let X be a non empty set. One can verify that there exists a finite sequence of elements of $\overline{\mathbb{R}}^X$ which is (without $+\infty$)-valued and (without $-\infty$)-valued.

(57) Let us consider a non empty set X, a (without $+\infty$)-valued finite sequence F of elements of $\overline{\mathbb{R}}^X$, and a natural number n. If $n \in \operatorname{dom} F$, then $(F_n)^{-1}(\{+\infty\}) = \emptyset$.

(58) Let us consider a non empty set X, a (without $-\infty$)-valued finite sequence F of elements of $\overline{\mathbb{R}}^X$, and a natural number n. If $n \in \operatorname{dom} F$, then $(F_n)^{-1}(\{-\infty\}) = \emptyset$.

(59) Let us consider a non empty set X, and a finite sequence F of elements of $\overline{\mathbb{R}}^X$. Suppose F is (without $+\infty$)-valued or (without $-\infty$)-valued. Let us consider natural numbers n, m. If $n, m \in \operatorname{dom} F$, then $\operatorname{dom}(F_n + F_m) = X$. The theorem is a consequence of (57) and (58).

Let X be a non empty set and F be a finite sequence of elements of $\overline{\mathbb{R}}^X$. We say that F is summable if and only if

(Def. 10) F is (without $+\infty$)-valued or (without $-\infty$)-valued.

Observe that there exists a finite sequence of elements of $\overline{\mathbb{R}}^X$ which is summable.

Let F be a summable finite sequence of elements of $\overline{\mathbb{R}}^X$. The functor $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ yielding a finite sequence of elements of $\overline{\mathbb{R}}^X$ is defined by

(Def. 11) $\operatorname{len} F = \operatorname{len} \mathit{it}$ and $F(1) = \mathit{it}(1)$ and for every natural number n such that $1 \leqslant n < \operatorname{len} F$ holds $\mathit{it}(n+1) = \mathit{it}_n + F_{n+1}$.

One can check that every finite sequence of elements of $\overline{\mathbb{R}}^X$ which is (without $+\infty$)-valued is also summable and every finite sequence of elements of $\overline{\mathbb{R}}^X$ which is (without $-\infty$)-valued is also summable.

Now we state the propositions:

(60) Let us consider a non empty set X, and a (without $+\infty$)-valued finite sequence F of elements of $\overline{\mathbb{R}}^X$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ is (without $+\infty$)-valued.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$ is without $+\infty$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [19, (29)], [2, (14)], [19, (25)], [2, (13)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

(61) Let us consider a non empty set X, and a (without $-\infty$)-valued finite sequence F of elements of $\overline{\mathbb{R}}^X$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ is (without $-\infty$)-valued.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \in \operatorname{dom}(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$, then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$ is without $-\infty$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [19, (29)], [2, (14)], [19, (25)], [2, (13)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. □

(62) Let us consider a non empty set X, a set A, an extended real e, and a function f from X into $\overline{\mathbb{R}}$. Suppose for every element x of X, $f(x) = e \cdot \chi_{A,X}(x)$. Then

(i) if $e = +\infty$, then $f = \overline{\chi}_{A,X}$, and

(ii) if $e = -\infty$, then $f = -\overline{\chi}_{A,X}$, and

(iii) if $e \neq +\infty$ and $e \neq -\infty$, then there exists a real number r such that $r = e$ and $f = r \cdot \chi_{A,X}$.

(63) Let us consider a non empty set X, a σ-field S of subsets of X, a partial function f from X to $\overline{\mathbb{R}}$, and an element A of S. Suppose f is measurable on A and $A \subseteq \operatorname{dom} f$. Then $-f$ is measurable on A.

Let X be a non empty set and f be a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$. Observe that $-f$ is without $+\infty$.

Let f be a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$. One can check that $-f$ is without $-\infty$.

Let f_1, f_2 be without $+\infty$ partial functions from X to $\overline{\mathbb{R}}$. Let us note that the functor $f_1 + f_2$ yields a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$. Let f_1, f_2 be without $-\infty$ partial functions from X to $\overline{\mathbb{R}}$. Note that the functor $f_1 + f_2$ yields a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$. Let f_1 be a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$ and f_2 be a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$. One can verify that the functor $f_1 - f_2$ yields a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$. Let f_1 be a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$ and f_2 be a without $+\infty$ partial function from X to $\overline{\mathbb{R}}$. Observe that the functor $f_1 - f_2$ yields a without $-\infty$ partial function from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(64) Let us consider a non empty set X, and partial functions f, g from X to $\overline{\mathbb{R}}$. Then

(i) $-(f + g) = -f + -g$, and

(ii) $-(f - g) = -f + g$, and

(iii) $-(f - g) = g - f$, and

(iv) $-(-f + g) = f - g$, and

(v) $-(-f + g) = f + -g$.

(65) Let us consider a non empty set X, a σ-field S of subsets of X, without $+\infty$ partial functions f, g from X to $\overline{\mathbb{R}}$, and an element A of S. Suppose f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom}(f + g)$. Then $f + g$ is measurable on A. The theorem is a consequence of (63) and (64).

(66) Let us consider a non empty set X, a σ-field S of subsets of X, an element A of S, a without $+\infty$ partial function f from X to $\overline{\mathbb{R}}$, and a without $-\infty$

partial function g from X to $\overline{\mathbb{R}}$. Suppose f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom}(f-g)$. Then $f-g$ is measurable on A. The theorem is a consequence of (63) and (64).

(67) Let us consider a non empty set X, a σ-field S of subsets of X, an element A of S, a without $-\infty$ partial function f from X to $\overline{\mathbb{R}}$, and a without $+\infty$ partial function g from X to $\overline{\mathbb{R}}$. Suppose f is measurable on A and g is measurable on A and $A \subseteq \operatorname{dom}(f-g)$. Then $f-g$ is measurable on A. The theorem is a consequence of (64), (63), and (65).

(68) Let us consider a non empty set X, a σ-field S of subsets of X, an element P of S, and a summable finite sequence F of elements of $\overline{\mathbb{R}}^X$. Suppose for every natural number n such that $n \in \operatorname{dom} F$ holds F_n is measurable on P. Let us consider a natural number n. Suppose $n \in \operatorname{dom} F$. Then $((\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})_n$ is measurable on P. The theorem is a consequence of (60), (65), and (61).

6. Some Properties of Integral

Now we state the propositions:

(69) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1, an element B of S_2, an element x of X_1, and an element y of X_2. Suppose $E = A \times B$. Then

(i) $\int \operatorname{curry}(\chi_{E,X_1\times X_2}, x)\,\mathrm{d}M_2 = M_2(\text{MeasurableXsection}(E,x))\cdot\chi_{A,X_1}(x)$, and

(ii) $\int \operatorname{curry}'(\chi_{E,X_1\times X_2}, y)\,\mathrm{d}M_1 = M_1(\text{MeasurableYsection}(E,y))\cdot\chi_{B,X_2}(y)$.

The theorem is a consequence of (52), (53), (54), and (55).

(70) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by MeasRect(S_1, S_2). Then there exists a disjoint valued finite sequence f of elements of MeasRect(S_1, S_2) and there exists a finite sequence A of elements of S_1.

There exists a finite sequence B of elements of S_2 such that $\operatorname{len} f = \operatorname{len} A$ and $\operatorname{len} f = \operatorname{len} B$ and $E = \bigcup f$ and for every natural number n such that $n \in \operatorname{dom} f$ holds $\pi_1(f(n)) = A(n)$ and $\pi_2(f(n)) = B(n)$ and for every natural number n and for every sets x, y such that $n \in \operatorname{dom} f$ and $x \in X_1$ and $y \in X_2$ holds $\chi_{f(n),X_1\times X_2}(x,y) = \chi_{A(n),X_1}(x) \cdot \chi_{B(n),X_2}(y)$.

Proof: Consider E_1 being a subset of $X_1 \times X_2$ such that $E = E_1$ and there exists a disjoint valued finite sequence f of elements of $\mathrm{MeasRect}(S_1, S_2)$ such that $E_1 = \bigcup f$. Consider f being a disjoint valued finite sequence of elements of $\mathrm{MeasRect}(S_1, S_2)$ such that $E_1 = \bigcup f$. Define $\mathcal{S}[\text{natural number}, \text{object}] \equiv \$_2 = \pi_1(f(\$_1))$. For every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, f$ there exists an element A_1 of S_1 such that $\mathcal{S}[i, A_1]$ by [12, (4)], [1, (9)], [5, (7)]. Consider A being a finite sequence of elements of S_1 such that $\mathrm{dom}\, A = \mathrm{Seg}\,\mathrm{len}\, f$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, f$ holds $\mathcal{S}[i, A(i)]$ from [3, Sch. 5]. Define $\mathcal{T}[\text{natural number}, \text{object}] \equiv \$_2 = \pi_2(f(\$_1))$. For every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, f$ there exists an element B_1 of S_2 such that $\mathcal{T}[i, B_1]$ by [12, (4)], [1, (9)], [5, (7)]. Consider B being a finite sequence of elements of S_2 such that $\mathrm{dom}\, B = \mathrm{Seg}\,\mathrm{len}\, f$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\, f$ holds $\mathcal{T}[i, B(i)]$ from [3, Sch. 5]. For every natural number n such that $n \in \mathrm{dom}\, f$ holds $\pi_1(f(n)) = A(n)$ and $\pi_2(f(n)) = B(n)$. Consider A_2 being an element of S_1, B_2 being an element of S_2 such that $f(n) = A_2 \times B_2$. $\square$

(71) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element x of X_1, an element y of X_2, an element U of S_1, and an element V of S_2. Then

(i) $M_1(\mathrm{MeasurableYsection}(E, y) \cap U) = \int \mathrm{curry}'(\chi_{E \cap (U \times X_2), X_1 \times X_2}, y)\, \mathrm{d}M_1$, and

(ii) $M_2(\mathrm{MeasurableXsection}(E, x) \cap V) = \int \mathrm{curry}(\chi_{E \cap (X_1 \times V), X_1 \times X_2}, x)\, \mathrm{d}M_2$.

The theorem is a consequence of (34), (27), and (22).

(72) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element x of X_1, and an element y of X_2. Then

(i) $M_1(\mathrm{MeasurableYsection}(E, y)) = \int \mathrm{curry}'(\chi_{E, X_1 \times X_2}, y)\, \mathrm{d}M_1$, and

(ii) $M_2(\mathrm{MeasurableXsection}(E, x)) = \int \mathrm{curry}(\chi_{E, X_1 \times X_2}, x)\, \mathrm{d}M_2$.

The theorem is a consequence of (71).

(73) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, a disjoint valued finite sequence f of elements of $\mathrm{MeasRect}(S_1, S_2)$, an element x of X_1, a natural number n, an element E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element A_2 of S_1, and an element B_2 of S_2. Suppose $n \in \mathrm{dom}\, f$ and $f(n) = E_2$ and $E_2 = A_2 \times$

B_2. Then $\int \mathrm{curry}(\chi_{f(n),X_1 \times X_2}, x)\, \mathrm{d}M_2 = M_2(\mathrm{MeasurableXsection}(E_2, x)) \cdot \chi_{A_2,X_1}(x)$.

(74) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\mathrm{MeasRect}(S_1, S_2)$ and $E \neq \emptyset$. Then there exists a disjoint valued finite sequence f of elements of $\mathrm{MeasRect}(S_1, S_2)$ and there exists a finite sequence A of elements of S_1 and there exists a finite sequence B of elements of S_2.

There exists a summable finite sequence X_3 of elements of $\overline{\mathbb{R}}^{X_1 \times X_2}$ such that $E = \bigcup f$ and $\mathrm{len}\, f \in \mathrm{dom}\, f$ and $\mathrm{len}\, f = \mathrm{len}\, A$ and $\mathrm{len}\, f = \mathrm{len}\, B$ and $\mathrm{len}\, f = \mathrm{len}\, X_3$ and for every natural number n such that $n \in \mathrm{dom}\, f$ holds $f(n) = A(n) \times B(n)$ and for every natural number n such that $n \in \mathrm{dom}\, X_3$ holds $X_3(n) = \chi_{f(n),X_1 \times X_2}$ and $(\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}}(\mathrm{len}\, X_3) = \chi_{E,X_1 \times X_2}$ and for every natural number n and for every sets x, y such that $n \in \mathrm{dom}\, X_3$ and $x \in X_1$ and $y \in X_2$ holds $X_3(n)(x, y) = \chi_{A(n),X_1}(x) \cdot \chi_{B(n),X_2}(y)$.

For every element x of X_1, $\mathrm{curry}(\chi_{E,X_1 \times X_2}, x) =$
$\mathrm{curry}(((\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}})_{\mathrm{len}\, X_3}, x)$ and for every element y of X_2,
$\mathrm{curry}'(\chi_{E,X_1 \times X_2}, y) = \mathrm{curry}'(((\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}})_{\mathrm{len}\, X_3}, y)$.

PROOF: Consider f being a disjoint valued finite sequence of elements of $\mathrm{MeasRect}(S_1, S_2)$, A being a finite sequence of elements of S_1, B being a finite sequence of elements of S_2 such that $\mathrm{len}\, f = \mathrm{len}\, A$ and $\mathrm{len}\, f = \mathrm{len}\, B$ and $E = \bigcup f$ and for every natural number n such that $n \in \mathrm{dom}\, f$ holds $\pi_1(f(n)) = A(n)$ and $\pi_2(f(n)) = B(n)$ and for every natural number n and for every sets x, y such that $n \in \mathrm{dom}\, f$ and $x \in X_1$ and $y \in X_2$ holds $\chi_{f(n),X_1 \times X_2}(x, y) = \chi_{A(n),X_1}(x) \cdot \chi_{B(n),X_2}(y)$. Define $\mathcal{F}(\text{set}) = \chi_{f(\$_1),X_1 \times X_2}$. Consider X_3 being a finite sequence such that $\mathrm{len}\, X_3 = \mathrm{len}\, f$ and for every natural number n such that $n \in \mathrm{dom}\, X_3$ holds $X_3(n) = \mathcal{F}(n)$ from [3, Sch. 2]. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \in \mathrm{dom}\, f$, then $(\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}}(\$_1) = \chi_{\bigcup (f \restriction \$_1),X_1 \times X_2}$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [9, (20)], [3, (39)], [13, (25)], [2, (14)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. For every natural number n such that $n \in \mathrm{dom}\, f$ holds $f(n) = A(n) \times B(n)$ by [12, (4)], [13, (90)], [1, (9)]. For every natural number n and for every sets x, y such that $n \in \mathrm{dom}\, X_3$ and $x \in X_1$ and $y \in X_2$ holds $X_3(n)(x, y) = \chi_{A(n),X_1}(x) \cdot \chi_{B(n),X_2}(y)$. For every element x of X_1, $\mathrm{curry}(\chi_{E,X_1 \times X_2}, x) = \mathrm{curry}(((\sum_{\alpha=0}^{\kappa} X_3(\alpha))_{\kappa \in \mathbb{N}})_{\mathrm{len}\, X_3}, x)$. $\square$

(75) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, and a finite sequence F of elements of $\mathrm{MeasRect}(S_1, S_2)$. Then $\bigcup F \in \sigma(\mathrm{MeasRect}(S_1, S_2))$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} F$, then $\bigcup \operatorname{rng}(F \upharpoonright \$_1) \in \sigma(\operatorname{MeasRect}(S_1, S_2))$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (11)], [19, (25)], [8, (11)], [3, (59)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. $\square$

(76) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\operatorname{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\operatorname{MeasRect}(S_1, S_2)$ and $E \neq \emptyset$.

Then there exists a disjoint valued finite sequence F of elements of MeasRect (S_1, S_2) and there exists a finite sequence A of elements of S_1 and there exists a finite sequence B of elements of S_2 and there exists a summable finite sequence C of elements of $\overline{\mathbb{R}}^{X_1 \times X_2}$ and there exists a summable finite sequence I of elements of $\overline{\mathbb{R}}^{X_1}$ and there exists a summable finite sequence J of elements of $\overline{\mathbb{R}}^{X_2}$ such that $E = \bigcup F$ and $\operatorname{len} F \in \operatorname{dom} F$ and $\operatorname{len} F = \operatorname{len} A$ and $\operatorname{len} F = \operatorname{len} B$ and $\operatorname{len} F = \operatorname{len} C$ and $\operatorname{len} F = \operatorname{len} I$ and $\operatorname{len} F = \operatorname{len} J$ and for every natural number n such that $n \in \operatorname{dom} C$ holds $C(n) = \chi_{F(n), X_1 \times X_2}$ and $((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} C} = \chi_{E, X_1 \times X_2}$.

For every element x of X_1 and for every natural number n such that $n \in \operatorname{dom} I$ holds $I(n)(x) = \int \operatorname{curry}(C_n, x) \, \mathrm{d}M_2$ and for every natural number n and for every element P of S_1 such that $n \in \operatorname{dom} I$ holds I_n is measurable on P and for every element x of X_1, $\int \operatorname{curry}(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} C}, x) \, \mathrm{d}M_2 = ((\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} I}(x)$ and for every element y of X_2 and for every natural number n such that $n \in \operatorname{dom} J$ holds $J(n)(y) = \int \operatorname{curry}'(C_n, y) \, \mathrm{d}M_1$ and for every natural number n and for every element P of S_2 such that $n \in \operatorname{dom} J$ holds J_n is measurable on P and for every element y of X_2, $\int \operatorname{curry}'(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} C}, y) \, \mathrm{d}M_1 = ((\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} J}(y)$.

PROOF: Consider F being a disjoint valued finite sequence of elements of $\operatorname{MeasRect}(S_1, S_2)$, A being a finite sequence of elements of S_1, B being a finite sequence of elements of S_2, C being a summable finite sequence of elements of $\overline{\mathbb{R}}^{X_1 \times X_2}$ such that $E = \bigcup F$ and $\operatorname{len} F \in \operatorname{dom} F$ and $\operatorname{len} F = \operatorname{len} A$ and $\operatorname{len} F = \operatorname{len} B$ and $\operatorname{len} F = \operatorname{len} C$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = A(n) \times B(n)$ and for every natural number n such that $n \in \operatorname{dom} C$ holds $C(n) = \chi_{F(n), X_1 \times X_2}$ and $(\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}}(\operatorname{len} C) = \chi_{E, X_1 \times X_2}$ and for every natural number n and for every sets x, y such that $n \in \operatorname{dom} C$ and $x \in X_1$ and $y \in X_2$ holds $C(n)(x, y) = \chi_{A(n), X_1}(x) \cdot \chi_{B(n), X_2}(y)$ and for every element x of X_1, $\operatorname{curry}(\chi_{E, X_1 \times X_2}, x) = \operatorname{curry}(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} C}, x)$ and for every element y of X_2, $\operatorname{curry}'(\chi_{E, X_1 \times X_2}, y) = \operatorname{curry}'(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} C}, y)$. Define $\mathcal{S}[\text{natural number}, \text{object}] \equiv$ there exists a function f from X_1 into $\overline{\mathbb{R}}$ such that $f = \$_2$ and for every element x of X_1, $f(x) = \int \operatorname{curry}(C_{\$_1}, x) \, \mathrm{d}M_2$.

For every natural number n such that $n \in \operatorname{Seg} \operatorname{len} F$ there exists an object z such that $\mathcal{S}[n, z]$. Consider I being a finite sequence such that $\operatorname{dom} I = \operatorname{Seg} \operatorname{len} F$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{S}[n, I(n)]$ from [3, Sch. 1]. For every element x of X_1 and for every natural number n such that $n \in \operatorname{dom} I$ holds $I(n)(x) = \int \operatorname{curry}(C_n, x)\, \mathrm{d}M_2$ by [12, (4)]. Define $\mathcal{T}[\text{natural number}, \text{object}] \equiv$ there exists a function f from X_2 into $\overline{\mathbb{R}}$ such that $f = \$_2$ and for every element x of X_2, $f(x) = \int \operatorname{curry}'(C_{\$_1}, x)\, \mathrm{d}M_1$. For every natural number n such that $n \in \operatorname{Seg} \operatorname{len} F$ there exists an object z such that $\mathcal{T}[n, z]$. Consider J being a finite sequence such that $\operatorname{dom} J = \operatorname{Seg} \operatorname{len} F$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} F$ holds $\mathcal{T}[n, J(n)]$ from [3, Sch. 1]. For every element x of X_2 and for every natural number n such that $n \in \operatorname{dom} J$ holds $J(n)(x) = \int \operatorname{curry}'(C_n, x)\, \mathrm{d}M_1$ by [12, (4)]. For every natural number n and for every element P of S_1 such that $n \in \operatorname{dom} I$ holds I_n is measurable on P by [12, (4)], (69), (22). For every element x of X_1, $\int \operatorname{curry}(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} C}, x)\, \mathrm{d}M_2 = ((\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} I}(x)$ by [19, (24), (25)], [2, (13)], [9, (20)]. For every natural number n and for every element P of S_2 such that $n \in \operatorname{dom} J$ holds J_n is measurable on P by [12, (4)], (69), (22). For every element x of X_2, $\int \operatorname{curry}'(((\sum_{\alpha=0}^{\kappa} C(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} C}, x)$ $\mathrm{d}M_1 = ((\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}})_{\operatorname{len} J}(x)$ by [19, (24), (25)], [2, (13)], [9, (20)]. $\square$

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, S_2 be a σ-field of subsets of X_2, F be a set sequence of $\sigma(\operatorname{MeasRect}(S_1, S_2))$, and n be a natural number. One can verify that the functor $F(n)$ yields an element of $\sigma(\operatorname{MeasRect}(S_1, S_2))$. Let F be a function from $\mathbb{N} \times \sigma(\operatorname{MeasRect}(S_1, S_2))$ into $\sigma(\operatorname{MeasRect}(S_1, S_2))$, n be an element of $\mathbb{N}$, and E be an element of

$\sigma(\operatorname{MeasRect}(S_1, S_2))$. Let us observe that the functor $F(n, E)$ yields an element of $\sigma(\operatorname{MeasRect}(S_1, S_2))$. Now we state the propositions:

(77) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\operatorname{MeasRect}(S_1, S_2))$, and an element V of S_2. Suppose $E \in$ the field generated by $\operatorname{MeasRect}(S_1, S_2)$. Then there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that

(i) for every element x of X_1, $F(x) = M_2(\operatorname{MeasurableXsection}(E, x) \cap V)$, and

(ii) for every element P of S_1, F is measurable on P.

The theorem is a consequence of (22), (27), (24), (76), (71), and (68).

(78) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\operatorname{MeasRect}(S_1, S_2))$, and an element V of S_1.

Suppose $E \in$ the field generated by $\operatorname{MeasRect}(S_1, S_2)$. Then there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that

(i) for every element x of X_2, $F(x) = M_1(\operatorname{MeasurableYsection}(E, x) \cap V)$, and

(ii) for every element P of S_2, F is measurable on P.

The theorem is a consequence of (22), (27), (24), (76), (71), and (68).

(79) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, and an element E of $\sigma(\operatorname{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\operatorname{MeasRect}(S_1, S_2)$. Let us consider an element B of S_2. Then $E \in \{E$, where E is an element of $\sigma(\operatorname{MeasRect}(S_1, S_2))$: there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that for every element x of $X_1, F(x) = M_2(\operatorname{MeasurableXsection}(E, x) \cap B)$ and for every element V of S_1, F is measurable on $V\}$. The theorem is a consequence of (77).

(80) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and an element E of $\sigma(\operatorname{MeasRect}(S_1, S_2))$. Suppose $E \in$ the field generated by $\operatorname{MeasRect}(S_1, S_2)$. Let us consider an element B of S_1. Then $E \in \{E$, where E is an element of $\sigma(\operatorname{MeasRect}(S_1, S_2))$: there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that for every element x of $X_2, F(x) = M_1(\operatorname{MeasurableYsection}(E, x) \cap B)$ and for every element V of S_2, F is measurable on $V\}$. The theorem is a consequence of (78).

(81) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, and an element B of S_2. Then the field generated by $\operatorname{MeasRect}(S_1, S_2) \subseteq \{E$, where E is an element of $\sigma(\operatorname{MeasRect}(S_1, S_2))$: there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that for every element x of $X_1, F(x) = M_2(\operatorname{MeasurableXsection}(E, x) \cap B)$ and for every element V of S_1, F is measurable on $V\}$. The theorem is a consequence of (7) and (79).

(82) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and an element B of S_1. Then the field generated by $\operatorname{MeasRect}(S_1, S_2) \subseteq \{E$, where E is an element of $\sigma(\operatorname{MeasRect}(S_1, S_2))$: there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that for every element y of $X_2, F(y) = M_1(\operatorname{MeasurableYsection}(E, y) \cap B)$ and for every element V of S_2, F is measurable on $V\}$. The theorem is a consequence of (7) and (80).

7. σ-finite Measure

Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. We say that M is σ-finite if and only if

(Def. 12) there exists a set sequence E of S such that for every natural number n, $M(E(n)) < +\infty$ and $\bigcup E = X$.

Now we state the propositions:

(83) Let us consider a non empty set X, a σ-field S of subsets of X, and a σ-measure M on S. Then M is σ-finite if and only if there exists a set sequence F of S such that F is non descending and for every natural number n, $M(F(n)) < +\infty$ and $\lim F = X$.

(84) Let us consider a set X, a semialgebra S of sets of X, a pre-measure P of S, and an induced measure M of S and P. Then $M =$ (the Caratheodory measure determined by M)$\restriction$(the field generated by S).

8. Fubini's Theorem on Measure

Now we state the propositions:

(85) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, and an element B of S_2. Suppose $M_2(B) < +\infty$. Then $\{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$: there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that for every element x of X_1, $F(x) = M_2(\mathrm{MeasurableXsection}(E, x) \cap B)$ and for every element V of S_1, F is measurable on $V\}$ is a monotone class of $X_1 \times X_2$.

Proof: Set $Z = \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$: there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that for every element x of $X_1, F(x) = M_2(\mathrm{MeasurableXsection}(E, x) \cap B)$ and for every element V of S_1, F is measurable on $V\}$. For every sequence A_1 of subsets of $X_1 \times X_2$ such that A_1 is monotone and $\mathrm{rng}\, A_1 \subseteq Z$ holds $\lim A_1 \in Z$ by [10, (3)], [5, (35)], [21, (63)], [12, (45)]. □

(86) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and an element B of S_1. Suppose $M_1(B) < +\infty$. Then $\{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$: there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that for every element y of X_2, $F(y) = M_1(\mathrm{MeasurableYsection}(E, y) \cap B)$ and for every element V of S_2, F is measurable on $V\}$ is a monotone class of $X_1 \times X_2$.

Proof: Set $Z = \{E$, where E is an element of $\sigma(\text{MeasRect}(S_1, S_2))$: there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that for every element y of $X_2, F(y) = M_1(\text{MeasurableYsection}(E, y) \cap B)$ and for every element V of S_2, F is measurable on $V\}$. For every sequence A_1 of subsets of $X_1 \times X_2$ such that A_1 is monotone and $\operatorname{rng} A_1 \subseteq Z$ holds $\lim A_1 \in Z$ by [10, (3)], [5, (35)], [21, (63)], [12, (45)]. □

(87) Let us consider a non empty set X, a field F of subsets of X, and a sequence L of subsets of X. Suppose $\operatorname{rng} L$ is a monotone class of X and $F \subseteq \operatorname{rng} L$. Then

(i) $\sigma(F) = \text{monotone-class}(F)$, and

(ii) $\sigma(F) \subseteq \operatorname{rng} L$.

(88) Let us consider a non empty set X, a field F of subsets of X, and a family K of subsets of X. Suppose K is a monotone class of X and $F \subseteq K$. Then

(i) $\sigma(F) = \text{monotone-class}(F)$, and

(ii) $\sigma(F) \subseteq K$.

(89) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, and an element B of S_2. Suppose $M_2(B) < +\infty$. Then $\sigma(\text{MeasRect}(S_1, S_2)) \subseteq \{E$, where E is an element of $\sigma(\text{MeasRect}(S_1, S_2))$: there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that for every element x of $X_1, F(x) =$ $M_2(\text{MeasurableXsection}(E, x) \cap B)$ and for every element V of S_1, F is measurable on $V\}$. The theorem is a consequence of (85), (81), (7), and (88).

(90) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and an element B of S_1. Suppose $M_1(B) < +\infty$. Then $\sigma(\text{MeasRect}(S_1, S_2)) \subseteq \{E$, where E is an element of $\sigma(\text{MeasRect}(S_1, S_2))$: there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that for every element y of $X_2, F(y) =$ $M_1(\text{MeasurableYsection}(E, y) \cap B)$ and for every element V of S_2, F is measurable on $V\}$. The theorem is a consequence of (86), (82), (7), and (88).

(91) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_2 is σ-finite. Then there exists a function F from X_1 into $\overline{\mathbb{R}}$ such that

(i) for every element x of X_1, $F(x) = M_2(\text{MeasurableXsection}(E, x))$, and

(ii) for every element V of S_1, F is measurable on V.

PROOF: Consider B being a set sequence of S_2 such that B is non descending and for every natural number n, $M_2(B(n)) < +\infty$ and $\lim B = X_2$. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv$ there exists a function f_1 from X_1 into $\overline{\mathbb{R}}$ such that $\$_2 = f_1$ and for every element x of X_1, $f_1(x) = M_2(\text{MeasurableXsection}(E, x) \cap B(\$_1))$ and for every element V of S_1, f_1 is measurable on V. For every element n of $\mathbb{N}$, there exists an element f of $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, f]$ by (89), [12, (45)]. Consider f being a function from $\mathbb{N}$ into $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, f(n)]$ from [11, Sch. 3]. For every natural number n, $f(n)$ is a function from X_1 into $\overline{\mathbb{R}}$ and for every element x of X_1, $f(n)(x) = M_2(\text{MeasurableXsection}(E, x) \cap B(n))$ and for every element V of S_1, $f(n)$ is measurable on V. For every natural numbers n, m, $\text{dom}(f(n)) = \text{dom}(f(m))$. For every element x of X_1 such that $x \in X_1$ holds $f\#x$ is convergent by [5, (11), (31)], [20, (7), (37)]. Reconsider $F = \lim f$ as a function from X_1 into $\overline{\mathbb{R}}$. For every element x of X_1, $F(x) = M_2(\text{MeasurableXsection}(E, x))$ by [21, (80)], [22, (92)], (49), [5, (11)]. □

(92) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and an element E of $\sigma(\text{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite. Then there exists a function F from X_2 into $\overline{\mathbb{R}}$ such that

(i) for every element y of X_2, $F(y) = M_1(\text{MeasurableYsection}(E, y))$, and

(ii) for every element V of S_2, F is measurable on V.

PROOF: Consider B being a set sequence of S_1 such that B is non descending and for every natural number n, $M_1(B(n)) < +\infty$ and $\lim B = X_1$. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv$ there exists a function f_1 from X_2 into $\overline{\mathbb{R}}$ such that $\$_2 = f_1$ and for every element y of X_2, $f_1(y) = M_1(\text{MeasurableYsection}(E, y) \cap B(\$_1))$ and for every element V of S_2, f_1 is measurable on V. For every element n of $\mathbb{N}$, there exists an element f of $X_2 \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, f]$ by (90), [12, (45)]. Consider f being a function from $\mathbb{N}$ into $X_2 \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, f(n)]$ from [11, Sch. 3]. For every natural number n, $f(n)$ is a function from X_2 into $\overline{\mathbb{R}}$ and for every element y of X_2, $f(n)(y) = M_1(\text{MeasurableYsection}(E, y) \cap B(n))$ and for every element V of S_2, $f(n)$ is measurable on V. For every natural numbers n, m, $\text{dom}(f(n)) = \text{dom}(f(m))$. For every element y of X_2 such that $y \in X_2$ holds $f\#y$ is convergent by [5, (11), (31)], [20, (7), (37)]. Reconsider $F = \lim f$ as a function from X_2 into $\overline{\mathbb{R}}$. For every element y of X_2, $F(y) = M_1(\text{MeasurableYsection}(E, y))$ by [21, (80)], [22, (92)], (49), [5, (11)]. □

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, S_2 be a σ-field of subsets of X_2, M_2 be a σ-measure on S_2, and E be an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Assume M_2 is σ-finite. The functor $\mathrm{Yvol}(E, M_2)$ yielding a non-negative function from X_1 into $\overline{\mathbb{R}}$ is defined by

(Def. 13) for every element x of X_1, $it(x) = M_2(\mathrm{MeasurableXsection}(E, x))$ and for every element V of S_1, it is measurable on V.

Let M_1 be a σ-measure on S_1. Assume M_1 is σ-finite. The functor $\mathrm{Xvol}(E, M_1)$ yielding a non-negative function from X_2 into $\overline{\mathbb{R}}$ is defined by

(Def. 14) for every element y of X_2, $it(y) = M_1(\mathrm{MeasurableYsection}(E, y))$ and for every element V of S_2, it is measurable on V.

Now we state the propositions:

(93) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, and elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_2 is σ-finite and E_1 misses E_2. Then $\mathrm{Yvol}(E_1 \cup E_2, M_2) = \mathrm{Yvol}(E_1, M_2) + \mathrm{Yvol}(E_2, M_2)$.

PROOF: For every element x of X_1 such that $x \in \mathrm{dom}\,\mathrm{Yvol}(E_1 \cup E_2, M_2)$ holds $(\mathrm{Yvol}(E_1 \cup E_2, M_2))(x) = (\mathrm{Yvol}(E_1, M_2) + \mathrm{Yvol}(E_2, M_2))(x)$ by (26), (35), [5, (30)]. $\square$

(94) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite and E_1 misses E_2. Then $\mathrm{Xvol}(E_1 \cup E_2, M_1) = \mathrm{Xvol}(E_1, M_1) + \mathrm{Xvol}(E_2, M_1)$.

PROOF: For every element x of X_2 such that $x \in \mathrm{dom}\,\mathrm{Xvol}(E_1 \cup E_2, M_1)$ holds $(\mathrm{Xvol}(E_1 \cup E_2, M_1))(x) = (\mathrm{Xvol}(E_1, M_1) + \mathrm{Xvol}(E_2, M_1))(x)$ by (26), (35), [5, (30)]. $\square$

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and elements E_1, E_2 of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Now we state the propositions:

(95) Suppose M_2 is σ-finite and E_1 misses E_2. Then $\int \mathrm{Yvol}(E_1 \cup E_2, M_2)\,\mathrm{d}M_1 = \int \mathrm{Yvol}(E_1, M_2)\,\mathrm{d}M_1 + \int \mathrm{Yvol}(E_2, M_2)\,\mathrm{d}M_1$. The theorem is a consequence of (93).

(96) Suppose M_1 is σ-finite and E_1 misses E_2. Then $\int \mathrm{Xvol}(E_1 \cup E_2, M_1)\,\mathrm{d}M_2 = \int \mathrm{Xvol}(E_1, M_1)\,\mathrm{d}M_2 + \int \mathrm{Xvol}(E_2, M_1)\,\mathrm{d}M_2$. The theorem is a consequence of (94).

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element A of S_1, and an element B of S_2. Now we state the propositions:

(97) Suppose $E = A \times B$ and M_2 is σ-finite. Then

(i) if $M_2(B) = +\infty$, then $\mathrm{Yvol}(E, M_2) = \overline{\chi}_{A,X_1}$, and

(ii) if $M_2(B) \neq +\infty$, then there exists a real number r such that $r = M_2(B)$ and $\mathrm{Yvol}(E, M_2) = r \cdot \chi_{A,X_1}$.

The theorem is a consequence of (53).

(98) Suppose $E = A \times B$ and M_1 is σ-finite. Then

(i) if $M_1(A) = +\infty$, then $\mathrm{Xvol}(E, M_1) = \overline{\chi}_{B,X_2}$, and

(ii) if $M_1(A) \neq +\infty$, then there exists a real number r such that $r = M_1(A)$ and $\mathrm{Xvol}(E, M_1) = r \cdot \chi_{B,X_2}$.

The theorem is a consequence of (55).

(99) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element A of S, and a real number r. If $r \geqslant 0$, then $\int r \cdot \chi_{A,X} \, \mathrm{d}M = r \cdot M(A)$.

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a finite sequence F of elements of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a natural number n. Now we state the propositions:

(100) Suppose M_2 is σ-finite and F is a finite sequence of elements of MeasRect (S_1, S_2). Then $(\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(F(n)) = \int \mathrm{Yvol}(F(n), M_2) \, \mathrm{d}M_1$. The theorem is a consequence of (16), (97), and (99).

(101) Suppose M_1 is σ-finite and F is a finite sequence of elements of MeasRect (S_1, S_2). Then $(\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(F(n)) = \int \mathrm{Xvol}(F(n), M_1) \, \mathrm{d}M_2$. The theorem is a consequence of (16), (98), and (99).

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a disjoint valued finite sequence F of elements of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and a natural number n. Now we state the propositions:

(102) Suppose M_2 is σ-finite and F is a finite sequence of elements of MeasRect (S_1, S_2). Then $(\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(\bigcup F) = \int \mathrm{Yvol}(\bigcup F, M_2) \, \mathrm{d}M_1$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(\bigcup(F \restriction \$_1)) = \int \mathrm{Yvol}(\bigcup(F \restriction \$_1), M_2) \, \mathrm{d}M_1$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (13)], [3, (59)], [19, (55)], [3, (82)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. $\square$

(103) Suppose M_1 is σ-finite and F is a finite sequence of elements of MeasRect (S_1, S_2). Then $(\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(\bigcup F) = \int \mathrm{Xvol}(\bigcup F, M_1) \, \mathrm{d}M_2$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv (\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(\bigcup(F \restriction \$_1)) = \int \mathrm{Xvol}(\bigcup(F \restriction \$_1), M_1) \, \mathrm{d}M_2$. $\mathcal{P}[0]$. For every natural number k such that

$\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (13)], [3, (59)], [19, (55)], [3, (82)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element V of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element A of S_1, and an element B of S_2. Now we state the propositions:

(104) Suppose $E \in$ the field generated by $\mathrm{MeasRect}(S_1, S_2)$ and M_2 is σ-finite. Then suppose $V = A \times B$. Then $E \in \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2)) : \int \mathrm{Yvol}(E \cap V, M_2)\,\mathrm{d}M_1 = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (102).

(105) Suppose $E \in$ the field generated by $\mathrm{MeasRect}(S_1, S_2)$ and M_1 is σ-finite. Then suppose $V = A \times B$. Then $E \in \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2)) : \int \mathrm{Xvol}(E \cap V, M_1)\,\mathrm{d}M_2 = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (103).

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element V of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element A of S_1, and an element B of S_2. Now we state the propositions:

(106) Suppose M_2 is σ-finite and $V = A \times B$. Then the field generated by $\mathrm{MeasRect}(S_1, S_2) \subseteq \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2)) : \int \mathrm{Yvol}(E \cap V, M_2)\,\mathrm{d}M_1 = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (7) and (104).

(107) Suppose M_1 is σ-finite and $V = A \times B$. Then the field generated by $\mathrm{MeasRect}(S_1, S_2) \subseteq \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2)) : \int \mathrm{Xvol}(E \cap V, M_1)\,\mathrm{d}M_2 = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (7) and (105).

(108) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, elements E, V of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, a set sequence P of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element x of X_1. Suppose P is non descending and $\lim P = E$. Then there exists a sequence K of subsets of S_2 such that

  (i) K is non descending, and

  (ii) for every natural number n, $K(n) = \mathrm{MeasurableXsection}(P(n), x) \cap \mathrm{MeasurableXsection}(V, x)$, and

  (iii) $\lim K = \mathrm{MeasurableXsection}(E, x) \cap \mathrm{MeasurableXsection}(V, x)$.

The theorem is a consequence of (43), (49), and (30).

(109) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, elements E, V

of $\sigma(\text{MeasRect}(S_1, S_2))$, a set sequence P of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element y of X_2. Suppose P is non descending and $\lim P = E$. Then there exists a sequence K of subsets of S_1 such that

(i) K is non descending, and

(ii) for every natural number n, $K(n) = \text{MeasurableYsection}(P(n), y) \cap \text{MeasurableYsection}(V, y)$, and

(iii) $\lim K = \text{MeasurableYsection}(E, y) \cap \text{MeasurableYsection}(V, y)$.

The theorem is a consequence of (44), (49), and (32).

(110) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, elements E, V of $\sigma(\text{MeasRect}(S_1, S_2))$, a set sequence P of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element x of X_1. Suppose P is non ascending and $\lim P = E$. Then there exists a sequence K of subsets of S_2 such that

(i) K is non ascending, and

(ii) for every natural number n, $K(n) = \text{MeasurableXsection}(P(n), x) \cap \text{MeasurableXsection}(V, x)$, and

(iii) $\lim K = \text{MeasurableXsection}(E, x) \cap \text{MeasurableXsection}(V, x)$.

The theorem is a consequence of (45), (49), and (31).

(111) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, elements E, V of $\sigma(\text{MeasRect}(S_1, S_2))$, a set sequence P of $\sigma(\text{MeasRect}(S_1, S_2))$, and an element y of X_2. Suppose P is non ascending and $\lim P = E$. Then there exists a sequence K of subsets of S_1 such that

(i) K is non ascending, and

(ii) for every natural number n, $K(n) = \text{MeasurableYsection}(P(n), y) \cap \text{MeasurableYsection}(V, y)$, and

(iii) $\lim K = \text{MeasurableYsection}(E, y) \cap \text{MeasurableYsection}(V, y)$.

The theorem is a consequence of (46), (49), and (33).

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, an element V of $\sigma(\text{MeasRect}(S_1, S_2))$, an element A of S_1, and an element B of S_2. Now we state the propositions:

(112) Suppose M_2 is σ-finite and $V = A \times B$ and $(\text{Prod}\,\sigma\text{-Meas}(M_1, M_2))(V) < +\infty$ and $M_2(B) < +\infty$. Then $\{E$, where E is an element of $\sigma(\text{MeasRect}(S_1, S_2)) : \int \text{Yvol}(E \cap V, M_2)\, \mathrm{d}M_1 = (\text{Prod}\,\sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$ is a monotone class of $X_1 \times X_2$.

Proof: Set $Z = \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$: $\int \mathrm{Yvol}(E \cap V, M_2)\,\mathrm{d}M_1 = (\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. For every sequence A_1 of subsets of $X_1 \times X_2$ such that A_1 is monotone and $\mathrm{rng}\, A_1 \subseteq Z$ holds $\lim A_1 \in Z$ by [10, (3)], [5, (35)], [21, (63)], [12, (45)]. □

(113) Suppose M_1 is σ-finite and $V = A \times B$ and $(\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(V) < +\infty$ and $M_1(A) < +\infty$. Then $\{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$: $\int \mathrm{Xvol}(E \cap V, M_1)\,\mathrm{d}M_2 = (\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$ is a monotone class of $X_1 \times X_2$.

Proof: Set $Z = \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$: $\int \mathrm{Xvol}(E \cap V, M_1)\,\mathrm{d}M_2 = (\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. For every sequence A_1 of subsets of $X_1 \times X_2$ such that A_1 is monotone and $\mathrm{rng}\, A_1 \subseteq Z$ holds $\lim A_1 \in Z$ by [10, (3)], [5, (35)], [21, (63)], [12, (45)]. □

(114) Suppose M_2 is σ-finite and $V = A \times B$ and $(\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(V) < +\infty$ and $M_2(B) < +\infty$. Then $\sigma(\mathrm{MeasRect}(S_1, S_2)) \subseteq \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$: $\int \mathrm{Yvol}(E \cap V, M_2)\,\mathrm{d}M_1 = (\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (112), (106), (7), and (88).

(115) Suppose M_1 is σ-finite and $V = A \times B$ and $(\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(V) < +\infty$ and $M_1(A) < +\infty$. Then $\sigma(\mathrm{MeasRect}(S_1, S_2)) \subseteq \{E$, where E is an element of $\sigma(\mathrm{MeasRect}(S_1, S_2))$: $\int \mathrm{Xvol}(E \cap V, M_1)\,\mathrm{d}M_2 = (\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(E \cap V)\}$. The theorem is a consequence of (113), (107), (7), and (88).

(116) Let us consider sets X, Y, a sequence A of subsets of X, a sequence B of subsets of Y, and a sequence C of subsets of $X \times Y$. Suppose A is non descending and B is non descending and for every natural number n, $C(n) = A(n) \times B(n)$. Then

(i) C is non descending and convergent, and

(ii) $\bigcup C = \bigcup A \times \bigcup B$.

Proof: For every natural numbers n, m such that $n \leqslant m$ holds $C(n) \subseteq C(m)$ by [13, (96)]. □

(117) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite and M_2 is σ-finite. Then $\int \mathrm{Yvol}(E, M_2)\,\mathrm{d}M_1 = (\mathrm{Prod}\,\sigma\text{-Meas}(M_1, M_2))(E)$.

Proof: Consider A being a set sequence of S_1 such that A is non descending and for every natural number n, $M_1(A(n)) < +\infty$ and $\lim A = X_1$. Consider B being a set sequence of S_2 such that B is non descending and for every natural number n, $M_2(B(n)) < +\infty$ and $\lim B =$

X_2. Define $\mathcal{C}$(element of $\mathbb{N}$) $= A(\$_1) \times B(\$_1)$. Consider C being a function from $\mathbb{N}$ into $2^{X_1 \times X_2}$ such that for every element n of $\mathbb{N}$, $C(n) = \mathcal{C}(n)$ from [11, Sch. 4]. For every natural number n, $C(n) = A(n) \times B(n)$. For every natural number n, $C(n) \in \sigma(\mathrm{MeasRect}(S_1, S_2))$. For every natural numbers n, m such that $n \leqslant m$ holds $C(n) \subseteq C(m)$ by [13, (96)]. For every natural number n, $(\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(C(n)) < +\infty$ by (16), [6, (51)]. Set $C_1 = E \cap C$. For every object n such that $n \in \mathbb{N}$ holds $C_1(n) \in \sigma(\mathrm{MeasRect}(S_1, S_2))$. For every natural number n, $\int \mathrm{Yvol}(E \cap C(n), M_2)\, \mathrm{d}M_1 = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E \cap C(n))$. Define $\mathcal{P}$[element of $\mathbb{N}$, object] $\equiv \$_2 = \mathrm{Yvol}(E \cap C(\$_1), M_2)$. For every element n of $\mathbb{N}$, there exists an element f of $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, f]$ by [12, (45)]. Consider F being a function from $\mathbb{N}$ into $X_1 \dot{\rightarrow} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$ from [11, Sch. 3]. For every natural number n, $F(n) = \mathrm{Yvol}(E \cap C(n), M_2)$. Reconsider $X_3 = X_1$ as an element of S_1. For every natural number n and for every element x of X_1, $(F\#x)(n) = (\mathrm{Yvol}(E \cap C(n), M_2))(x)$. For every natural numbers n, m, $\mathrm{dom}(F(n)) = \mathrm{dom}(F(m))$. For every natural number n, $F(n)$ is measurable on X_3. For every natural numbers n, m such that $n \leqslant m$ for every element x of X_1 such that $x \in X_3$ holds $F(n)(x) \leqslant F(m)(x)$ by (20), [5, (31)]. For every element x of X_1 such that $x \in X_3$ holds $F\#x$ is convergent by [20, (7), (37)]. Consider I being a sequence of extended reals such that for every natural number n, $I(n) = \int F(n)\, \mathrm{d}M_1$ and I is convergent and $\int \lim F\, \mathrm{d}M_1 = \lim I$. For every element x of X_1 such that $x \in \mathrm{dom} \lim F$ holds $(\lim F)(x) = (\mathrm{Yvol}(E, M_2))(x)$ by (116), (108), (27), [10, (13)]. Set $J = E \cap C$. For every object n such that $n \in \mathbb{N}$ holds $J(n) \in \sigma(\mathrm{MeasRect}(S_1, S_2))$. $\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2)$ is a σ-measure on $\sigma(\mathrm{MeasRect}(S_1, S_2))$. For every element n of $\mathbb{N}$, $I(n) = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2)_* J)(n)$ by [10, (13)]. □

(118) Fubini's theorem:

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$. Suppose M_1 is σ-finite and M_2 is σ-finite. Then $\int \mathrm{Xvol}(E, M_1)\, \mathrm{d}M_2 = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E)$.

Proof: Consider A being a set sequence of S_1 such that A is non descending and for every natural number n, $M_1(A(n)) < +\infty$ and $\lim A = X_1$. Consider B being a set sequence of S_2 such that B is non descending and for every natural number n, $M_2(B(n)) < +\infty$ and $\lim B = X_2$. Define $\mathcal{C}$(element of $\mathbb{N}$) $= A(\$_1) \times B(\$_1)$. Consider C being a function from $\mathbb{N}$ into $2^{X_1 \times X_2}$ such that for every element n of $\mathbb{N}$, $C(n) = \mathcal{C}(n)$ from [11, Sch. 4]. For every natural number n, $C(n) = A(n) \times$

$B(n)$. For every natural number n, $C(n) \in \sigma(\mathrm{MeasRect}(S_1, S_2))$. For every natural numbers n, m such that $n \leqslant m$ holds $C(n) \subseteq C(m)$ by [13, (96)]. For every natural number n, $(\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(C(n)) < +\infty$ by (16), [6, (51)]. Set $C_1 = E \cap C$. For every object n such that $n \in \mathbb{N}$ holds $C_1(n) \in \sigma(\mathrm{MeasRect}(S_1, S_2))$. For every natural number n, $\int \mathrm{Xvol}(E \cap C(n), M_1)\,\mathrm{d}M_2 = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2))(E \cap C(n))$. Define $\mathcal{P}[\text{element of } \mathbb{N}, \text{object}] \equiv \$_2 = \mathrm{Xvol}(E \cap C(\$_1), M_1)$. For every element n of $\mathbb{N}$, there exists an element f of $X_2 \dot{\to} \overline{\mathbb{R}}$ such that $\mathcal{P}[n, f]$ by [12, (45)]. Consider F being a function from $\mathbb{N}$ into $X_2 \dot{\to} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, F(n)]$ from [11, Sch. 3]. For every natural number n, $F(n) = \mathrm{Xvol}(E \cap C(n), M_1)$. Reconsider $X_3 = X_2$ as an element of S_2. For every natural number n and for every element x of X_2, $(F\#x)(n) = (\mathrm{Xvol}(E \cap C(n), M_1))(x)$. For every natural numbers n, m, $\mathrm{dom}(F(n)) = \mathrm{dom}(F(m))$. For every natural number n, $F(n)$ is measurable on X_3. For every natural numbers n, m such that $n \leqslant m$ for every element x of X_2 such that $x \in X_3$ holds $F(n)(x) \leqslant F(m)(x)$ by (21), [5, (31)]. For every element x of X_2 such that $x \in X_3$ holds $F\#x$ is convergent by [20, (7), (37)]. Consider I being a sequence of extended reals such that for every natural number n, $I(n) = \int F(n)\,\mathrm{d}M_2$ and I is convergent and $\int \lim F\,\mathrm{d}M_2 = \lim I$. For every element x of X_2 such that $x \in \mathrm{dom} \lim F$ holds $(\lim F)(x) = (\mathrm{Xvol}(E, M_1))(x)$ by (116), (109), (27), [10, (13)]. Set $J = E \cap C$. For every object n such that $n \in \mathbb{N}$ holds $J(n) \in \sigma(\mathrm{MeasRect}(S_1, S_2))$. $\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2)$ is a σ-measure on $\sigma(\mathrm{MeasRect}(S_1, S_2))$. For every element n of $\mathbb{N}$, $I(n) = (\mathrm{Prod}\,\sigma\text{-}\mathrm{Meas}(M_1, M_2)_* J)(n)$ by [10, (13)]. $\square$

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