

Niven's Theorem¹

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Summary. This article formalizes the proof of Niven's theorem [12] which states that if x/π and $\sin(x)$ are both rational, then the sine takes values 0, $\pm 1/2$, and ± 1 . The main part of the formalization follows the informal proof presented at ProofWiki (https://proofwiki.org/wiki/Niven's_Theorem#Source_of_Name). For this proof, we have also formalized the rational and integral root theorems setting constraints on solutions of polynomial equations with integer coefficients [8, 9].

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From now on r, t denote real numbers, i denotes an integer, k, n denote natural numbers, p denotes a polynomial over \mathbb{R}_F , e denotes an element of \mathbb{R}_F , L denotes a non empty zero structure, and z, z_0, z_1, z_2 denote elements of L .

Now we state the propositions:

- (1) Let us consider complexes a, b, c, d . If $b \neq 0$ and $\frac{a}{b} = \frac{c}{d}$, then $a = \frac{b \cdot c}{d}$.
- (2) Let us consider real numbers a, b . If $|a| = b$, then $a = b$ or $a = -b$.
- (3) If $|i| \leq 2$, then $i = -2$ or $i = -1$ or $i = 0$ or $i = 1$ or $i = 2$. The theorem is a consequence of (2).
- (4) If $n \neq 0$, then $i \mid i^n$.
- (5) If $t > 0$, then there exists i such that $t \cdot i \leq r \leq t \cdot (i + 1)$.

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PROOF: Define $\mathcal{P}[\text{integer}] \equiv t \cdot \$1 \leq r$. There exists an integer i_1 such that $\mathcal{P}[i_1]$. Set $F = \lceil \frac{r}{t} \rceil$. For every integer i_1 such that $\mathcal{P}[i_1]$ holds $i_1 \leq F$. Consider i such that $\mathcal{P}[i]$ and for every integer i_1 such that $\mathcal{P}[i_1]$ holds $i_1 \leq i$ from [15, Sch. 6]. \square

- (6) Let us consider a finite sequence p of elements of \mathbb{R}_F , and a real-valued finite sequence q . If $p = q$, then $\sum p = \sum q$.

PROOF: Define $\mathcal{P}[\text{finite sequence}] \equiv$ for every finite sequence p of elements of \mathbb{R}_F for every real-valued finite sequence q such that $p = q$ and $p = \$1$ holds $\sum p = \sum q$. $\mathcal{P}[\emptyset]$ by [16, (43)], [4, (72)]. For every finite sequence f and for every object x such that $\mathcal{P}[f]$ holds $\mathcal{P}[f \cap \langle x \rangle]$ by [2, (36), (38)], [5, (31)], [16, (41), (44)]. For every finite sequence f , $\mathcal{P}[f]$ from [2, Sch. 3]. \square

- (7) Let us consider a natural number i , and an element r of \mathbb{R}_F . Then $\text{power}_{\mathbb{R}_F}(r, i) = r^i$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{power}_{\mathbb{R}_F}(r, \$1) = r^{\$1}$. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

- (8) $\sin(\frac{5 \cdot \pi}{6}) = \frac{1}{2}$.
 (9) $\sin(\frac{5 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = \frac{1}{2}$.
 (10) $\sin(\frac{7 \cdot \pi}{6}) = -\frac{1}{2}$.
 (11) $\sin(\frac{7 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$.
 (12) $\sin(\frac{11 \cdot \pi}{6}) = -\frac{1}{2}$.
 (13) $\sin(\frac{11 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$.
 (14) $\cos(\frac{4 \cdot \pi}{3}) = -\frac{1}{2}$.
 (15) $\cos(\frac{4 \cdot \pi}{3} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$.
 (16) $\cos(\frac{5 \cdot \pi}{3}) = \frac{1}{2}$.
 (17) $\cos(\frac{5 \cdot \pi}{3} + 2 \cdot \pi \cdot i) = \frac{1}{2}$.
 (18) If $0 \leq r \leq \frac{\pi}{2}$ and $\cos r = \frac{1}{2}$, then $r = \frac{\pi}{3}$.
 (19) Let us consider an add-associative, right zeroed, right complementable, left distributive, non empty double loop structure L , and a sequence p of L . Then $\mathbf{0}.L * p = \mathbf{0}.L$.

Let us consider L , z , and n . One can verify that $\mathbf{0}.L + \cdot (n, z)$ is finite-Support as a sequence of L .

Let us consider a polynomial p over L . Now we state the propositions:

- (20) If $z \neq 0_L$, then if $p = \mathbf{0}.L + \cdot (n, z)$, then $\text{len } p = n + 1$.

PROOF: the length of p is at most $n + 1$ by [1, (13)], [3, (32)], [14, (7)]. For every natural number m such that the length of p is at most m holds $n + 1 \leq m$ by [14, (13)], [3, (31)], [1, (13)]. \square

- (21) If $z \neq 0_L$, then if $p = \mathbf{0}.L + \cdot (n, z)$, then $\deg p = n$. The theorem is a consequence of (20).

Note that $\mathbf{0}. \mathbb{R}_F$ is \mathbb{Z} -valued and $\mathbf{1}. \mathbb{R}_F$ is \mathbb{Z} -valued and there exists an element of \mathbb{R}_F which is integer.

Now we state the proposition:

- (22) $\text{rng}\langle z \rangle = \{z, 0_L\}$.

PROOF: Set $p = \langle z \rangle$. $\text{rng } p \subseteq \{z, 0_L\}$ by [11, (32)], [1, (14)]. \square

Let us consider L , z_0 , z_1 , and z_2 . The functor $\langle z_0, z_1, z_2 \rangle$ yielding a sequence of L is defined by the term

- (Def. 1) $((\mathbf{0}.L + \cdot (0, z_0)) + \cdot (1, z_1)) + \cdot (2, z_2)$.

Now we state the propositions:

- (23) $\langle z_0, z_1, z_2 \rangle(0) = z_0$.
 (24) $\langle z_0, z_1, z_2 \rangle(1) = z_1$.
 (25) $\langle z_0, z_1, z_2 \rangle(2) = z_2$.
 (26) If $3 \leq n$, then $\langle z_0, z_1, z_2 \rangle(n) = 0_L$.

Let us consider L , z_0 , z_1 , and z_2 . Let us observe that $\langle z_0, z_1, z_2 \rangle$ is finite-Support.

Now we state the propositions:

- (27) $\text{len}\langle z_0, z_1, z_2 \rangle \leq 3$. The theorem is a consequence of (26).
 (28) If $z_2 \neq 0_L$, then $\text{len}\langle z_0, z_1, z_2 \rangle = 3$. The theorem is a consequence of (25) and (26).
 (29) Let us consider a right zeroed, non empty additive loop structure L , and elements z_0, z_1 of L . Then $\langle z_0 \rangle + \langle z_1 \rangle = \langle z_0 + z_1 \rangle$.
 (30) Let us consider a right zeroed, non empty additive loop structure L , and elements z_0, z_1, z_2, z_3 of L . Then $\langle z_0, z_1 \rangle + \langle z_2, z_3 \rangle = \langle z_0 + z_2, z_1 + z_3 \rangle$.
 (31) Let us consider a right zeroed, non empty additive loop structure L , and elements $z_0, z_1, z_2, z_3, z_4, z_5$ of L . Then $\langle z_0, z_1, z_2 \rangle + \langle z_3, z_4, z_5 \rangle = \langle z_0 + z_3, z_1 + z_4, z_2 + z_5 \rangle$. The theorem is a consequence of (23), (24), (25), and (26).
 (32) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L , and an element z_0 of L . Then $-\langle z_0 \rangle = \langle -z_0 \rangle$.
 (33) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L , and elements z_0, z_1 of L . Then $-\langle z_0, z_1 \rangle = \langle -z_0, -z_1 \rangle$.
 (34) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L , and elements z_0, z_1, z_2 of L . Then

$-\langle z_0, z_1, z_2 \rangle = \langle -z_0, -z_1, -z_2 \rangle$. The theorem is a consequence of (23), (24), (25), and (26).

- (35) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L , and elements z_0, z_1 of L . Then $\langle z_0 \rangle - \langle z_1 \rangle = \langle z_0 - z_1 \rangle$. The theorem is a consequence of (32) and (29).
- (36) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L , and elements z_0, z_1, z_2, z_3 of L . Then $\langle z_0, z_1 \rangle - \langle z_2, z_3 \rangle = \langle z_0 - z_2, z_1 - z_3 \rangle$. The theorem is a consequence of (33) and (30).
- (37) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L , and elements $z_0, z_1, z_2, z_3, z_4, z_5$ of L . Then $\langle z_0, z_1, z_2 \rangle - \langle z_3, z_4, z_5 \rangle = \langle z_0 - z_3, z_1 - z_4, z_2 - z_5 \rangle$. The theorem is a consequence of (34) and (31).
- (38) Let us consider an add-associative, right zeroed, right complementable, left distributive, unital, associative, non empty double loop structure L , and elements z_0, z_1, z_2, x of L . Then $\text{eval}(\langle z_0, z_1, z_2 \rangle, x) = z_0 + z_1 \cdot x + z_2 \cdot x \cdot x$. The theorem is a consequence of (23), (24), (27), and (25).

Let a be an integer element of \mathbb{R}_F . Note that $\langle a \rangle$ is \mathbb{Z} -valued.

Let a, b be integer elements of \mathbb{R}_F . One can verify that $\langle a, b \rangle$ is \mathbb{Z} -valued.

Let a, b, c be integer elements of \mathbb{R}_F . Observe that $\langle a, b, c \rangle$ is \mathbb{Z} -valued and there exists a polynomial over \mathbb{R}_F which is monic and \mathbb{Z} -valued and there exists a finite sequence of elements of \mathbb{R}_F which is \mathbb{Z} -valued.

Let F be a \mathbb{Z} -valued finite sequence of elements of \mathbb{R}_F . One can check that $\sum F$ is integer.

Let f be a \mathbb{Z} -valued sequence of \mathbb{R}_F . Let us note that $-f$ is \mathbb{Z} -valued.

Let g be a \mathbb{Z} -valued sequence of \mathbb{R}_F . Observe that $f + g$ is \mathbb{Z} -valued and $f - g$ is \mathbb{Z} -valued and $f * g$ is \mathbb{Z} -valued.

Now we state the proposition:

- (39) Let us consider a non degenerated, non empty double loop structure L , and an element z of L . Then $\text{LC}\langle z, 1_L \rangle = 1_L$.

Let L be a non degenerated, non empty double loop structure and z be an element of L . One can check that $\langle z, 1_L \rangle$ is monic.

Now we state the proposition:

- (40) Let us consider a non degenerated, non empty double loop structure L , and elements z_1, z_2 of L . Then $\text{LC}\langle z_1, z_2, 1_L \rangle = 1_L$. The theorem is a consequence of (28) and (25).

Let L be a non degenerated, non empty double loop structure and z_1, z_2 be elements of L . Let us observe that $\langle z_1, z_2, 1_L \rangle$ is monic.

Let p be a \mathbb{Z} -valued polynomial over \mathbb{R}_F . Let us note that $\text{LC } p$ is integer.

Now we state the proposition:

- (41) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L , and a polynomial p over L . Then $\deg(-p) = \deg p$.

Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L and polynomials p, q over L . Now we state the propositions:

- (42) If $\deg p > \deg q$, then $\deg(p + q) = \deg p$.
 (43) If $\deg p > \deg q$, then $\deg(p - q) = \deg p$.
 (44) If $\deg p < \deg q$, then $\deg(p - q) = \deg q$.
 (45) Let us consider an add-associative, right zeroed, right complementable, distributive, non degenerated double loop structure L , and a polynomial p over L . Then $\text{LC } p = -\text{LC}(-p)$.
 (46) Let us consider an add-associative, right zeroed, right complementable, associative, commutative, well unital, almost left invertible, distributive, non degenerated double loop structure L , and polynomials p, q over L . Then $\text{LC}(p * q) = \text{LC } p \cdot \text{LC } q$. The theorem is a consequence of (19).

Let us consider an add-associative, right zeroed, right complementable, distributive, non degenerated double loop structure L , a monic polynomial p over L , and a polynomial q over L . Now we state the propositions:

- (47) If $\deg p > \deg q$, then $p + q$ is monic. The theorem is a consequence of (42).
 (48) If $\deg p > \deg q$, then $p - q$ is monic. The theorem is a consequence of (43).

Let L be an add-associative, right zeroed, right complementable, associative, commutative, well unital, almost left invertible, distributive, non degenerated double loop structure and p, q be monic polynomials over L . Let us note that $p * q$ is monic.

Now we state the propositions:

- (49) Let us consider an Abelian, add-associative, right zeroed, right complementable, unital, distributive, non empty double loop structure L , elements z_1, z_2 of L , and a polynomial p over L . Suppose $\text{eval}(p, z_1) = z_2$. Then $\text{eval}(p - \langle z_2 \rangle, z_1) = 0_L$.
 (50) RATIONAL ROOT THEOREM:

Let us consider a \mathbb{Z} -valued polynomial p over \mathbb{R}_F , and an element e of \mathbb{R}_F . Suppose e is a root of p . Let us consider integers k, l . Suppose $l \neq 0$ and $e = \frac{k}{l}$ and k and l are relatively prime. Then

- (i) $k \mid p(0)$, and

(ii) $l \mid \text{LC}p$.

The theorem is a consequence of (7), (6), and (4).

(51) INTEGRAL ROOT THEOREM:

Let us consider a monic, \mathbb{Z} -valued polynomial p over \mathbb{R}_F , and a rational element e of \mathbb{R}_F . If e is a root of p , then e is integer. The theorem is a consequence of (50).

(52) Suppose $1 \leq n$ and $e = 2 \cdot \cos t$. Then there exists a monic, \mathbb{Z} -valued polynomial p over \mathbb{R}_F such that

(i) $\text{eval}(p, e) = 2 \cdot \cos(n \cdot t)$, and

(ii) $\deg p = n$, and

(iii) if $n = 1$, then $p = \langle 0_{\mathbb{R}_F}, 1_{\mathbb{R}_F} \rangle$, and

(iv) if $n = 2$, then there exists an element r of \mathbb{R}_F such that $r = -2$ and $p = \langle r, 0_{\mathbb{R}_F}, 1_{\mathbb{R}_F} \rangle$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leq \$_1$, then there exists a monic, \mathbb{Z} -valued polynomial p over \mathbb{R}_F such that $\text{eval}(p, e) = 2 \cdot \cos(\$_1 \cdot t)$ and $\deg p = \$_1$ and if $\$_1 = 1$, then $p = \langle 0_{\mathbb{R}_F}, 1_{\mathbb{R}_F} \rangle$ and if $\$_1 = 2$, then there exists an element r of \mathbb{R}_F such that $r = -2$ and $p = \langle r, 0_{\mathbb{R}_F}, 1_{\mathbb{R}_F} \rangle$. $\mathcal{P}[1]$ by [11, (48), (40)]. $\mathcal{P}[2]$ by [6, (7)], (38), (28). For every non zero natural number k such that $\mathcal{P}[k]$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k+2]$ by [1, (13)], [13, (38)], (48), [10, (24)]. For every non zero natural number k , $\mathcal{P}[k]$ from [7, Sch. 1]. \square

(53) If $0 \leq r \leq \frac{\pi}{2}$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational, then $r \in \{0, \frac{\pi}{3}, \frac{\pi}{2}\}$.

The theorem is a consequence of (52), (1), (49), (48), (51), (3), and (18).

(54) Suppose $2 \cdot \pi \cdot i \leq r \leq \frac{\pi}{2} + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{2 \cdot \pi \cdot i, \frac{\pi}{3} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (53).

(55) If $\frac{\pi}{2} \leq r \leq \pi$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational, then $r \in \{\frac{\pi}{2}, \frac{2\pi}{3}, \pi\}$. The theorem is a consequence of (53).

(56) Suppose $\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r \leq \pi + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{2\pi}{3} + 2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (55).

(57) Suppose $\pi \leq r \leq \frac{3\pi}{2}$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\pi, \frac{4\pi}{3}, \frac{3\pi}{2}\}$. The theorem is a consequence of (53).

(58) Suppose $\pi + 2 \cdot \pi \cdot i \leq r \leq \frac{3\pi}{2} + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\pi + 2 \cdot \pi \cdot i, \frac{4\pi}{3} + 2 \cdot \pi \cdot i, \frac{3\pi}{2} + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (57).

(59) Suppose $\frac{3\pi}{2} \leq r \leq 2 \cdot \pi$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\frac{3\pi}{2}, \frac{5\pi}{3}, 2 \cdot \pi\}$. The theorem is a consequence of (53).

- (60) Suppose $\frac{3\pi}{2} + 2 \cdot \pi \cdot i \leq r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\frac{3\pi}{2} + 2 \cdot \pi \cdot i, \frac{5\pi}{3} + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (59).
- (61) If $\frac{r}{\pi}$ is rational and $\cos r$ is rational, then $\cos r \in \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$.
- (62) If $0 \leq r \leq \frac{\pi}{2}$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational, then $r \in \{0, \frac{\pi}{6}, \frac{\pi}{2}\}$. The theorem is a consequence of (53).
- (63) Suppose $2 \cdot \pi \cdot i \leq r \leq \frac{\pi}{2} + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{2 \cdot \pi \cdot i, \frac{\pi}{6} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (62).
- (64) If $\frac{\pi}{2} \leq r \leq \pi$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational, then $r \in \{\frac{\pi}{2}, \frac{5\pi}{6}, \pi\}$. The theorem is a consequence of (62).
- (65) Suppose $\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r \leq \pi + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{5\pi}{6} + 2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (64).
- (66) Suppose $\pi \leq r \leq \frac{3\pi}{2}$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\pi, \frac{7\pi}{6}, \frac{3\pi}{2}\}$. The theorem is a consequence of (62).
- (67) Suppose $\pi + 2 \cdot \pi \cdot i \leq r \leq \frac{3\pi}{2} + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\pi + 2 \cdot \pi \cdot i, \frac{7\pi}{6} + 2 \cdot \pi \cdot i, \frac{3\pi}{2} + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (66).
- (68) Suppose $\frac{3\pi}{2} \leq r \leq 2 \cdot \pi$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\frac{3\pi}{2}, \frac{11\pi}{6}, 2 \cdot \pi\}$. The theorem is a consequence of (62).
- (69) Suppose $\frac{3\pi}{2} + 2 \cdot \pi \cdot i \leq r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\frac{3\pi}{2} + 2 \cdot \pi \cdot i, \frac{11\pi}{6} + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (68).
- (70) NIVEN'S THEOREM:
If $\frac{r}{\pi}$ is rational and $\sin r$ is rational, then $\sin r \in \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$.

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