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Homography in $\mathbb{RP}^2$

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Summary. The real projective plane has been formalized in Isabelle/HOL by Timothy Makarios [13] and in Coq by Nicolas Magaud, Julien Narboux and Pascal Schreck [12].

Some definitions on the real projective spaces were introduced early in the Mizar Mathematical Library by Wojciech Leonczuk [9], Krzysztof Prazmowski [10] and by Wojciech Skaba [18].

In this article, we check with the Mizar system [4], some properties on the determinants and the Grassmann-Plücker relation in rank 3 [2], [1], [7], [16], [17].

Then we show that the projective space induced (in the sense defined in [9]) by $\mathbb{R}^3$ is a projective plane (in the sense defined in [10]).

Finally, in the real projective plane, we define the homography induced by a 3-by-3 invertible matrix and we show that the images of 3 collinear points are themselves collinear.

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1. Preliminaries

From now on a, b, c, d, e, f denote real numbers, k, m denote natural numbers, D denotes a non empty set, V denotes a non trivial real linear space, u, v, w denote elements of V, and p, q, r denote elements of the projective space over V.

Now we state the propositions:

(1) $\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle \in \mathrm{Seg}\, 3 \times \mathrm{Seg}\, 3$.

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(2) $\langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 1\rangle \in \operatorname{Seg} 3 \times \operatorname{Seg} 1$.

(3) $\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle \in \operatorname{Seg} 1 \times \operatorname{Seg} 3$.

(4) $\langle\langle a\rangle, \langle b\rangle, \langle c\rangle\rangle$ is a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3×1.

(5) Let us consider a matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3×1. Suppose $N = \langle\langle a\rangle, \langle b\rangle, \langle c\rangle\rangle$. Then $N_{\square,1} = \langle a, b, c\rangle$. The theorem is a consequence of (2).

(6) Let us consider a non empty multiplicative magma K, and elements a_1, a_2, a_3, b_1, b_2, b_3 of K. Then $\langle a_1, a_2, a_3\rangle \bullet \langle b_1, b_2, b_3\rangle = \langle a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3\rangle$.

(7) Let us consider a commutative, associative, left unital, Abelian, add-associative, right zeroed, right complementable, non empty double loop structure K, and elements a_1, a_2, a_3, b_1, b_2, b_3 of K. Then $\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$. The theorem is a consequence of (6).

(8) Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and a matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3×1. Suppose $N = \langle\langle 0\rangle, \langle 0\rangle, \langle 0\rangle\rangle$. Then $M \cdot N = \langle\langle 0\rangle, \langle 0\rangle, \langle 0\rangle\rangle$. The theorem is a consequence of (7), (5), and (2).

(9) u, v and w are lineary dependent if and only if $u = v$ or $u = w$ or $v = w$ or $\{u, v, w\}$ is linearly dependent.

(10) p, q and r are collinear if and only if there exists u and there exists v and there exists w such that $p =$ the direction of u and $q =$ the direction of v and $r =$ the direction of w and u is not zero and v is not zero and w is not zero and ($u = v$ or $u = w$ or $v = w$ or $\{u, v, w\}$ is linearly dependent). The theorem is a consequence of (9).

(11) p, q and r are collinear if and only if there exists u and there exists v and there exists w such that $p =$ the direction of u and $q =$ the direction of v and $r =$ the direction of w and u is not zero and v is not zero and w is not zero and there exists a and there exists b and there exists c such that $a \cdot u + b \cdot v + c \cdot w = 0_V$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$).

(12) Let us consider elements u, v, w of V. Suppose $a \neq 0$ and $a\cdot u+b\cdot v+c\cdot w = 0_V$. Then $u = (\frac{-b}{a}) \cdot v + (\frac{-c}{a}) \cdot w$.

(13) If $a \neq 0$ and $a \cdot b + c \cdot d + e \cdot f = 0$, then $b = -(\frac{c}{a}) \cdot d - (\frac{e}{a}) \cdot f$.

(14) Let us consider points u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose there exists a and there exists b and there exists c such that $a \cdot u + b \cdot v + c \cdot w = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and $a \neq 0$. Then $\langle|u, v, w|\rangle = 0$. The theorem is a consequence of (12).

(15) Let us consider a natural number n. Then $\operatorname{dom} 1_{\mathbb{R}} \operatorname{matrix}(n) = \operatorname{Seg} n$.

(16) Let us consider a matrix A over $\mathbb{R}_{\mathrm{F}}$. Then $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})(\mathbb{R}_{\mathrm{F}} \to \mathbb{R})A = A$.

(17) Let us consider matrices A, B over $\mathbb{R}_{\mathrm{F}}$, and matrices R_1, R_2 over $\mathbb{R}$. If $A = R_1$ and $B = R_2$, then $A \cdot B = R_1 \cdot R_2$. The theorem is a consequence of (16).

(18) Let us consider a natural number n, a square matrix M over $\mathbb{R}$ of dimension n, and a square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension n. If $M = N$, then M is invertible iff N is invertible. The theorem is a consequence of (17).

From now on o, p, q, r, s, t denote points of $\mathcal{E}^3_{\mathrm{T}}$ and M denotes a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

Let us consider real numbers p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3. Now we state the propositions:

(19) $\langle\langle p_1, p_2, p_3\rangle, \langle q_1, q_2, q_3\rangle, \langle r_1, r_2, r_3\rangle\rangle$ is a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3.

(20) Suppose $M = \langle\langle p_1, q_1, r_1\rangle, \langle p_2, q_2, r_2\rangle, \langle p_3, q_3, r_3\rangle\rangle$. Then

- (i) $M_{1,1} = p_1$, and
- (ii) $M_{1,2} = q_1$, and
- (iii) $M_{1,3} = r_1$, and
- (iv) $M_{2,1} = p_2$, and
- (v) $M_{2,2} = q_2$, and
- (vi) $M_{2,3} = r_2$, and
- (vii) $M_{3,1} = p_3$, and
- (viii) $M_{3,2} = q_3$, and
- (ix) $M_{3,3} = r_3$.

The theorem is a consequence of (1).

(21) Suppose $M = \langle p, q, r\rangle$. Then

- (i) $M_{1,1} = (p)_{\mathbf{1}}$, and
- (ii) $M_{1,2} = (p)_{\mathbf{2}}$, and
- (iii) $M_{1,3} = (p)_{\mathbf{3}}$, and
- (iv) $M_{2,1} = (q)_{\mathbf{1}}$, and
- (v) $M_{2,2} = (q)_{\mathbf{2}}$, and
- (vi) $M_{2,3} = (q)_{\mathbf{3}}$, and
- (vii) $M_{3,1} = (r)_{\mathbf{1}}$, and
- (viii) $M_{3,2} = (r)_{\mathbf{2}}$, and
- (ix) $M_{3,3} = (r)_{\mathbf{3}}$.

The theorem is a consequence of (1).

(22) Let us consider real numbers p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3. Suppose $M = \langle\langle p_1, q_1, r_1\rangle, \langle p_2, q_2, r_2\rangle, \langle p_3, q_3, r_3\rangle\rangle$. Then $M^{\mathrm{T}} = \langle\langle p_1, p_2, p_3\rangle, \langle q_1, q_2, q_3\rangle, \langle r_1, r_2, r_3\rangle\rangle$. The theorem is a consequence of (1) and (20).

(23) Suppose $M = \langle p, q, r \rangle$. Then $M^{\mathrm{T}} = \langle \langle (p)_{\mathbf{1}}, (q)_{\mathbf{1}}, (r)_{\mathbf{1}} \rangle, \langle (p)_{\mathbf{2}}, (q)_{\mathbf{2}}, (r)_{\mathbf{2}} \rangle, \langle (p)_{\mathbf{3}}, (q)_{\mathbf{3}}, (r)_{\mathbf{3}} \rangle \rangle$. The theorem is a consequence of (1) and (21).

(24) $\mathrm{lines}(M) = \{\mathrm{Line}(M, 1), \mathrm{Line}(M, 2), \mathrm{Line}(M, 3)\}$.
PROOF: $\mathrm{lines}(M) \subseteq \{\mathrm{Line}(M, 1), \mathrm{Line}(M, 2), \mathrm{Line}(M, 3)\}$ by [14, (103)], [19, (1)]. $\{\mathrm{Line}(M, 1), \mathrm{Line}(M, 2), \mathrm{Line}(M, 3)\} \subseteq \mathrm{lines}(M)$ by [3, (1)], [14, (103)]. □

(25) Suppose $M = \langle \langle (p)_{\mathbf{1}}, (p)_{\mathbf{2}}, (p)_{\mathbf{3}} \rangle, \langle (q)_{\mathbf{1}}, (q)_{\mathbf{2}}, (q)_{\mathbf{3}} \rangle, \langle (r)_{\mathbf{1}}, (r)_{\mathbf{2}}, (r)_{\mathbf{3}} \rangle \rangle$. Then

- (i) $\mathrm{Line}(M, 1) = p$, and
- (ii) $\mathrm{Line}(M, 2) = q$, and
- (iii) $\mathrm{Line}(M, 3) = r$.

(26) Let us consider an object x. Then $x \in \mathrm{lines}(M^{\mathrm{T}})$ if and only if there exists a natural number i such that $i \in \mathrm{Seg}\,3$ and $x = M_{\square, i}$.

2. Grassmann-Plücker Relation

Now we state the propositions:

(27) $\langle |p, q, r| \rangle = (p)_{\mathbf{1}} \cdot (q)_{\mathbf{2}} \cdot (r)_{\mathbf{3}} - (p)_{\mathbf{3}} \cdot (q)_{\mathbf{2}} \cdot (r)_{\mathbf{1}} - (p)_{\mathbf{1}} \cdot (q)_{\mathbf{3}} \cdot (r)_{\mathbf{2}} + (p)_{\mathbf{2}} \cdot (q)_{\mathbf{3}} \cdot (r)_{\mathbf{1}} - (p)_{\mathbf{2}} \cdot (q)_{\mathbf{1}} \cdot (r)_{\mathbf{3}} + (p)_{\mathbf{3}} \cdot (q)_{\mathbf{1}} \cdot (r)_{\mathbf{2}}$.

(28) GRASSMANNN-PLÜCKER-RELATION IN RANK 3:
$\langle |p, q, r| \rangle \cdot \langle |p, s, t| \rangle - \langle |p, q, s| \rangle \cdot \langle |p, r, t| \rangle + \langle |p, q, t| \rangle \cdot \langle |p, r, s| \rangle = 0$. The theorem is a consequence of (27).

(29) $\langle |p, q, r| \rangle = -\langle |p, r, q| \rangle$. The theorem is a consequence of (27).

(30) $\langle |p, q, r| \rangle = -\langle |q, p, r| \rangle$. The theorem is a consequence of (27).

(31) $\langle |a \cdot p, q, r| \rangle = a \cdot \langle |p, q, r| \rangle$. The theorem is a consequence of (27).

(32) $\langle |p, a \cdot q, r| \rangle = a \cdot \langle |p, q, r| \rangle$. The theorem is a consequence of (30) and (31).

(33) $\langle |p, q, a \cdot r| \rangle = a \cdot \langle |p, q, r| \rangle$. The theorem is a consequence of (29) and (32).

(34) Suppose $M = \langle \langle (p)_{\mathbf{1}}, (q)_{\mathbf{1}}, (r)_{\mathbf{1}} \rangle, \langle (p)_{\mathbf{2}}, (q)_{\mathbf{2}}, (r)_{\mathbf{2}} \rangle, \langle (p)_{\mathbf{3}}, (q)_{\mathbf{3}}, (r)_{\mathbf{3}} \rangle \rangle$. Then $\langle |p, q, r| \rangle = \mathrm{Det}\, M$. The theorem is a consequence of (22).

(35) Suppose $M = \langle \langle (p)_{\mathbf{1}}, (p)_{\mathbf{2}}, (p)_{\mathbf{3}} \rangle, \langle (q)_{\mathbf{1}}, (q)_{\mathbf{2}}, (q)_{\mathbf{3}} \rangle, \langle (r)_{\mathbf{1}}, (r)_{\mathbf{2}}, (r)_{\mathbf{3}} \rangle \rangle$. Then $\langle |p, q, r| \rangle = \mathrm{Det}\, M$.

Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension k. Now we state the propositions:

(36) $\mathrm{Det}\, M = 0_{\mathbb{R}_{\mathrm{F}}}$ if and only if $\mathrm{rk}(M) < k$.

(37) $\mathrm{rk}(M) < k$ if and only if lines(M) is linearly dependent or M is not without repeated line.

(38) Let us consider a matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension $k \times m$. Then Mx2Tran (M) is a function from RLSp2RVSp$(\mathcal{E}^k_{\mathrm{T}})$ into RLSp2RVSp$(\mathcal{E}^m_{\mathrm{T}})$.

(39) Let us consider a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension k. Then Mx2Tran(M) is a linear transformation from RLSp2RVSp$(\mathcal{E}^k_{\mathrm{T}})$ to RLSp2RVSp$(\mathcal{E}^k_{\mathrm{T}})$.
Proof: Reconsider $M_1 = \mathrm{Mx2Tran}(M)$ as a function from RLSp2RVSp$(\mathcal{E}^k_{\mathrm{T}})$ into RLSp2RVSp$(\mathcal{E}^k_{\mathrm{T}})$. For every elements x, y of RLSp2RVSp$(\mathcal{E}^k_{\mathrm{T}})$, $M_1(x+y) = M_1(x) + M_1(y)$ by [15, (22)]. For every scalar a of $\mathbb{R}_{\mathrm{F}}$ and for every vector x of RLSp2RVSp$(\mathcal{E}^k_{\mathrm{T}})$, $M_1(a \cdot x) = a \cdot M_1(x)$ by [15, (23)]. $\square$

(40) Suppose $M = \langle\langle(p)_{\mathbf{1}}, (p)_{\mathbf{2}}, (p)_{\mathbf{3}}\rangle, \langle(q)_{\mathbf{1}}, (q)_{\mathbf{2}}, (q)_{\mathbf{3}}\rangle, \langle(r)_{\mathbf{1}}, (r)_{\mathbf{2}}, (r)_{\mathbf{3}}\rangle\rangle$ and $\mathrm{rk}(M) < 3$. Then there exists a and there exists b and there exists c such that $a \cdot p + b \cdot q + c \cdot r = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$). The theorem is a consequence of (37), (25), (24), (39), and (7).

(41) If $a \cdot p + b \cdot q + c \cdot r = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$), then $\langle|p, q, r|\rangle = 0$. The theorem is a consequence of (14) and (30).

(42) Suppose $\langle|p, q, r|\rangle = 0$. Then there exists a and there exists b and there exists c such that $a \cdot p + b \cdot q + c \cdot r = 0_{\mathcal{E}^3_{\mathrm{T}}}$ and ($a \neq 0$ or $b \neq 0$ or $c \neq 0$). The theorem is a consequence of (19), (35), (36), and (40).

(43) p, q and r are lineary dependent if and only if $\langle|p, q, r|\rangle = 0$. The theorem is a consequence of (41) and (42).

3. Some Properties about the Cross Product

Now we state the propositions:

(44) $|(p, p \times q)| = 0$.

(45) $|(p, q \times p)| = 0$.

(46) (i) $\langle|o, p, (o \times p) \times (q \times r)|\rangle = 0$, and

(ii) $\langle|q, r, (o \times p) \times (q \times r)|\rangle = 0$.

The theorem is a consequence of (44) and (45).

(47) (i) o, p and $(o \times p) \times (q \times r)$ are lineary dependent, and

(ii) q, r and $(o \times p) \times (q \times r)$ are lineary dependent.

The theorem is a consequence of (46) and (43).

(48) (i) $0_{\mathcal{E}^3_{\mathrm{T}}} \times p = 0_{\mathcal{E}^3_{\mathrm{T}}}$, and

(ii) $p \times 0_{\mathcal{E}^3_{\mathrm{T}}} = 0_{\mathcal{E}^3_{\mathrm{T}}}$.

(49) $\langle|p, q, 0_{\mathcal{E}^3_\mathrm{T}}|\rangle = 0$. The theorem is a consequence of (48).

(50) If $p \times q = 0_{\mathcal{E}^3_\mathrm{T}}$ and $r = [1, 1, 1]$, then p, q and r are lineary dependent.

PROOF: Reconsider $r = [1, 1, 1]$ as an element of $\mathcal{E}^3_\mathrm{T}$. $\langle|p, q, r|\rangle = 0$ by [8, (2)], (27). $\square$

(51) If p is not zero and q is not zero and $p \times q = 0_{\mathcal{E}^3_\mathrm{T}}$, then p and q are proportional.

(52) Let us consider non zero points p, q, r, s of $\mathcal{E}^3_\mathrm{T}$. Suppose $(p \times q) \times (r \times s)$ is zero. Then

(i) p and q are proportional, or

(ii) r and s are proportional, or

(iii) $p \times q$ and $r \times s$ are proportional.

The theorem is a consequence of (51).

(53) $\langle|p, q, p \times q|\rangle = |(q, q)| \cdot |(p, p)| - |(q, p)| \cdot |(p, q)|$.

(54) $|(p \times q, p \times q)| = |(q, q)| \cdot |(p, p)| - |(q, p)| \cdot |(p, q)|$.

(55) If p is not zero and $|(p, q)| = 0$ and $|(p, r)| = 0$ and $|(p, s)| = 0$, then $\langle|q, r, s|\rangle = 0$. The theorem is a consequence of (13) and (27).

(56) $\langle|p, q, p \times q|\rangle = |p \times q|^{\mathbf{2}}$. The theorem is a consequence of (53) and (54).

(57) The projective space over $\mathcal{E}^3_\mathrm{T}$ is a projective plane defined in terms of collinearity.

PROOF: Set $P =$ the projective space over $\mathcal{E}^3_\mathrm{T}$. There exist elements u, v, w_1 of $\mathcal{E}^3_\mathrm{T}$ such that for every real numbers a, b, c such that $a \cdot u + b \cdot v + c \cdot w_1 = 0_{\mathcal{E}^3_\mathrm{T}}$ holds $a = 0$ and $b = 0$ and $c = 0$ by [6, (22)], [8, (4)], [11, (39)], [8, (2)]. For every elements p, p_1, q, q_1 of P, there exists an element r of P such that p, p_1 and r are collinear and q, q_1 and r are collinear by [9, (26)], (52), [9, (22)], [18, (2)]. $\square$

4. Real Projective Plane and Homography

Let us consider elements u, v, w, x of $\mathcal{E}^3_\mathrm{T}$. Now we state the propositions:

(58) Suppose u is not zero and x is not zero and the direction of $u =$ the direction of x. Then $\langle|u, v, w|\rangle = 0$ if and only if $\langle|x, v, w|\rangle = 0$. The theorem is a consequence of (31).

(59) Suppose v is not zero and x is not zero and the direction of $v =$ the direction of x. Then $\langle|u, v, w|\rangle = 0$ if and only if $\langle|u, x, w|\rangle = 0$. The theorem is a consequence of (32).

(60) Suppose w is not zero and x is not zero and the direction of w = the direction of x. Then $\langle|u, v, w|\rangle = 0$ if and only if $\langle|u, v, x|\rangle = 0$. The theorem is a consequence of (33).

(61) (i) $(1_{\mathbb{R}}\,\mathrm{matrix}(3))(1) = e_1$, and

(ii) $(1_{\mathbb{R}}\,\mathrm{matrix}(3))(2) = e_2$, and

(iii) $(1_{\mathbb{R}}\,\mathrm{matrix}(3))(3) = e_3$.

(62) (i) the base finite sequence of 3 and $1 = e_1$, and

(ii) the base finite sequence of 3 and $2 = e_2$, and

(iii) the base finite sequence of 3 and $3 = e_3$.

(63) Let us consider a finite sequence p_2 of elements of D. Suppose $\mathrm{len}\, p_2 = 3$. Then

(i) $\langle p_2\rangle_{\square,1} = \langle p_2(1)\rangle$, and

(ii) $\langle p_2\rangle_{\square,2} = \langle p_2(2)\rangle$, and

(iii) $\langle p_2\rangle_{\square,3} = \langle p_2(3)\rangle$.

The theorem is a consequence of (3).

(64) (i) $\langle e_1\rangle_{\square,1} = \langle 1\rangle$, and

(ii) $\langle e_1\rangle_{\square,2} = \langle 0\rangle$, and

(iii) $\langle e_1\rangle_{\square,3} = \langle 0\rangle$.

The theorem is a consequence of (63).

(65) (i) $\langle e_2\rangle_{\square,1} = \langle 0\rangle$, and

(ii) $\langle e_2\rangle_{\square,2} = \langle 1\rangle$, and

(iii) $\langle e_2\rangle_{\square,3} = \langle 0\rangle$.

The theorem is a consequence of (63).

(66) (i) $\langle e_3\rangle_{\square,1} = \langle 0\rangle$, and

(ii) $\langle e_3\rangle_{\square,2} = \langle 0\rangle$, and

(iii) $\langle e_3\rangle_{\square,3} = \langle 1\rangle$.

The theorem is a consequence of (63).

(67) (i) $(I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}})_{\square,1} = \langle 1, 0, 0\rangle$, and

(ii) $(I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}})_{\square,2} = \langle 0, 1, 0\rangle$, and

(iii) $(I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}})_{\square,3} = \langle 0, 0, 1\rangle$.

The theorem is a consequence of (1) and (15).

(68) (i) $\mathrm{Line}(I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}}, 1) = \langle 1, 0, 0\rangle$, and

(ii) $\mathrm{Line}(I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}}, 2) = \langle 0, 1, 0\rangle$, and

(iii) $\mathrm{Line}(I^{3\times 3}_{\mathbb{R}_\mathrm{F}}, 3) = \langle 0, 0, 1\rangle$.

The theorem is a consequence of (1).

(69) (i) $\langle e_1\rangle^\mathrm{T} = \langle\langle 1\rangle, \langle 0\rangle, \langle 0\rangle\rangle$, and

(ii) $\langle e_2\rangle^\mathrm{T} = \langle\langle 0\rangle, \langle 1\rangle, \langle 0\rangle\rangle$, and

(iii) $\langle e_3\rangle^\mathrm{T} = \langle\langle 0\rangle, \langle 0\rangle, \langle 1\rangle\rangle$.

The theorem is a consequence of (64), (65), and (66).

From now on p_1 denotes a finite sequence of elements of D.

Now we state the propositions:

(70) Let us consider a finite sequence p_1 of elements of D. If $k \in \mathrm{dom}\, p_1$, then $\langle p_1\rangle_{1,k} = p_1(k)$.

(71) If $k \in \mathrm{dom}\, p_1$, then $\langle p_1\rangle_{\square,k} = \langle p_1(k)\rangle$. The theorem is a consequence of (70).

(72) Let us consider an element p_2 of $\mathcal{R}^3$. Suppose $p_1 = p_2$. Then $(\mathbb{R} \to \mathbb{R}_\mathrm{F})\,\mathrm{ColVec2Mx}(p_2) = \langle p_1\rangle^\mathrm{T}$. The theorem is a consequence of (71).

In the sequel P denotes a square matrix over $\mathbb{R}_\mathrm{F}$ of dimension 3.

Now we state the propositions:

(73) Suppose $P = \langle\langle(p)_\mathbf{1}, (p)_\mathbf{2}, (p)_\mathbf{3}\rangle, \langle(q)_\mathbf{1}, (q)_\mathbf{2}, (q)_\mathbf{3}\rangle, \langle(r)_\mathbf{1}, (r)_\mathbf{2}, (r)_\mathbf{3}\rangle\rangle$. Then

(i) $\mathrm{Line}(P, 1) = p$, and

(ii) $\mathrm{Line}(P, 2) = q$, and

(iii) $\mathrm{Line}(P, 3) = r$.

(74) Suppose $P = \langle\langle(p)_\mathbf{1}, (p)_\mathbf{2}, (p)_\mathbf{3}\rangle, \langle(q)_\mathbf{1}, (q)_\mathbf{2}, (q)_\mathbf{3}\rangle, \langle(r)_\mathbf{1}, (r)_\mathbf{2}, (r)_\mathbf{3}\rangle\rangle$. Then

(i) $P_{\square,1} = \langle(p)_\mathbf{1}, (q)_\mathbf{1}, (r)_\mathbf{1}\rangle$, and

(ii) $P_{\square,2} = \langle(p)_\mathbf{2}, (q)_\mathbf{2}, (r)_\mathbf{2}\rangle$, and

(iii) $P_{\square,3} = \langle(p)_\mathbf{3}, (q)_\mathbf{3}, (r)_\mathbf{3}\rangle$.

(75) $\mathrm{width}\langle p_1\rangle = \mathrm{len}\, p_1$.

(76) Suppose $\mathrm{len}\, p_1 = 3$. Then

(i) $\mathrm{Line}(\langle p_1\rangle^\mathrm{T}, 1) = \langle p_1(1)\rangle$, and

(ii) $\mathrm{Line}(\langle p_1\rangle^\mathrm{T}, 2) = \langle p_1(2)\rangle$, and

(iii) $\mathrm{Line}(\langle p_1\rangle^\mathrm{T}, 3) = \langle p_1(3)\rangle$.

The theorem is a consequence of (75) and (63).

(77) If $\mathrm{len}\, p_1 = 3$, then $\langle p_1\rangle^\mathrm{T} = \langle\langle p_1(1)\rangle, \langle p_1(2)\rangle, \langle p_1(3)\rangle\rangle$. The theorem is a consequence of (76).

Let us consider D. Let p be a finite sequence of elements of D. Assume $\mathrm{len}\, p = 3$. The functor $\mathrm{F2M}(p)$ yielding a finite sequence of elements of D^1 is defined by the term

(Def. 1) $\langle\langle p(1)\rangle, \langle p(2)\rangle, \langle p(3)\rangle\rangle$.

Let us consider a finite sequence p of elements of $\mathbb{R}$. Now we state the propositions:

(78) If $\operatorname{len} p = 3$, then $\operatorname{len} \mathrm{F2M}(p) = 3$.

(79) If $\operatorname{len} p = 3$, then p is a 3-element finite sequence of elements of $\mathbb{R}$.

(80) If $p = [0, 0, 0]$, then $\mathrm{F2M}(p) = \langle\langle 0\rangle, \langle 0\rangle, \langle 0\rangle\rangle$.

(81) Suppose $\operatorname{len} p_1 = 3$. Then $\langle\langle p_1\rangle_{\square,1}, \langle p_1\rangle_{\square,2}, \langle p_1\rangle_{\square,3}\rangle = \mathrm{F2M}(p_1)$. The theorem is a consequence of (63).

Let us consider D. Let p be a finite sequence of elements of D^1. Assume $\operatorname{len} p = 3$. The functor $\mathrm{M2F}(p)$ yielding a finite sequence of elements of D is defined by the term

(Def. 2) $\langle p(1)(1), p(2)(1), p(3)(1)\rangle$.

Now we state the proposition:

(82) Let us consider a finite sequence p of elements of $\mathbb{R}^1$. Suppose $\operatorname{len} p = 3$. Then $\mathrm{M2F}(p)$ is a point of $\mathcal{E}^3_{\mathrm{T}}$.

Let p be a finite sequence of elements of $\mathbb{R}^1$ and a be a real number. Assume $\operatorname{len} p = 3$. The functor $a \cdot p$ yielding a finite sequence of elements of $\mathbb{R}^1$ is defined by

(Def. 3) there exist real numbers p_1, p_2, p_3 such that $p_1 = p(1)(1)$ and $p_2 = p(2)(1)$ and $p_3 = p(3)(1)$ and $it = \langle\langle a \cdot p_1\rangle, \langle a \cdot p_2\rangle, \langle a \cdot p_3\rangle\rangle$.

Let us consider a finite sequence p of elements of $\mathbb{R}^1$. Now we state the propositions:

(83) If $\operatorname{len} p = 3$, then $\mathrm{M2F}(a \cdot p) = a \cdot \mathrm{M2F}(p)$.

(84) If $\operatorname{len} p = 3$, then $\langle\langle p(1)(1)\rangle, \langle p(2)(1)\rangle, \langle p(3)(1)\rangle\rangle = p$.

(85) If $\operatorname{len} p = 3$, then $\mathrm{F2M}(\mathrm{M2F}(p)) = p$. The theorem is a consequence of (84).

(86) Let us consider a finite sequence p of elements of $\mathbb{R}$. If $\operatorname{len} p = 3$, then $\mathrm{M2F}(\mathrm{F2M}(p)) = p$.

(87) (i) $\langle e_1\rangle^{\mathrm{T}} = \mathrm{F2M}(e_1)$, and

(ii) $\langle e_2\rangle^{\mathrm{T}} = \mathrm{F2M}(e_2)$, and

(iii) $\langle e_3\rangle^{\mathrm{T}} = \mathrm{F2M}(e_3)$.

The theorem is a consequence of (69).

(88) Let us consider a finite sequence p of elements of D. If $\operatorname{len} p = 3$, then $\langle p\rangle^{\mathrm{T}} = \mathrm{F2M}(p)$. The theorem is a consequence of (77).

(89) $\mathrm{Line}(\langle p_1\rangle, 1) = p_1$.

(90) Let us consider a matrix M over D of dimension 3×1. Then

(i) $\mathrm{Line}(M, 1) = \langle M_{1,1} \rangle$, and

(ii) $\mathrm{Line}(M, 2) = \langle M_{2,1} \rangle$, and

(iii) $\mathrm{Line}(M, 3) = \langle M_{3,1} \rangle$.

From now on R denotes a ring.

Now we state the propositions:

(91) Let us consider a square matrix N over R of dimension 3, and a finite sequence p of elements of R. If $\mathrm{len}\, p = 3$, then $N \cdot \langle p \rangle^{\mathrm{T}}$ is 3,1-size.

(92) Let us consider a finite sequence p_1 of elements of R, and a square matrix N over R of dimension 3. Suppose $\mathrm{len}\, p_1 = 3$. Then

(i) $\mathrm{Line}(N \cdot \langle p_1 \rangle^{\mathrm{T}}, 1) = \langle (N \cdot \langle p_1 \rangle^{\mathrm{T}})_{1,1} \rangle$, and

(ii) $\mathrm{Line}(N \cdot \langle p_1 \rangle^{\mathrm{T}}, 2) = \langle (N \cdot \langle p_1 \rangle^{\mathrm{T}})_{2,1} \rangle$, and

(iii) $\mathrm{Line}(N \cdot \langle p_1 \rangle^{\mathrm{T}}, 3) = \langle (N \cdot \langle p_1 \rangle^{\mathrm{T}})_{3,1} \rangle$.

The theorem is a consequence of (91) and (90).

(93) $(\langle p_1 \rangle^{\mathrm{T}})_{\square,1} = p_1$. The theorem is a consequence of (89).

(94) Let us consider finite sequences p_1, q_1, r_1 of elements of $\mathbb{R}_{\mathrm{F}}$. Suppose $p = p_1$ and $q = q_1$ and $r = r_1$ and $\langle |p, q, r| \rangle \neq 0$. Then there exists a square matrix M over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) M is invertible, and

(ii) $M \cdot p_1 = \mathrm{F2M}(e_1)$, and

(iii) $M \cdot q_1 = \mathrm{F2M}(e_2)$, and

(iv) $M \cdot r_1 = \mathrm{F2M}(e_3)$.

Proof: Reconsider $P = \langle \langle (p)_{\mathbf{1}}, (p)_{\mathbf{2}}, (p)_{\mathbf{3}} \rangle, \langle (q)_{\mathbf{1}}, (q)_{\mathbf{2}}, (q)_{\mathbf{3}} \rangle, \langle (r)_{\mathbf{1}}, (r)_{\mathbf{2}}, (r)_{\mathbf{3}} \rangle \rangle$ as a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. $\langle |p, q, r| \rangle = \mathrm{Det}\, P$. Consider N being a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that N is inverse of P^{T}. $N \cdot \langle p_1 \rangle^{\mathrm{T}}$ is a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3×1 and $N \cdot \langle q_1 \rangle^{\mathrm{T}}$ is a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3×1 and $N \cdot \langle r_1 \rangle^{\mathrm{T}}$ is a matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3×1. $N \cdot \langle p_1 \rangle^{\mathrm{T}} = \mathrm{F2M}(e_1)$ by (78), [3, (91), (45), (1)]. $N \cdot \langle q_1 \rangle^{\mathrm{T}} = \mathrm{F2M}(e_2)$ by (78), [3, (91), (45), (1)]. $N \cdot \langle r_1 \rangle^{\mathrm{T}} = \mathrm{F2M}(e_3)$ by (78), [3, (91), (45), (1)]. $\square$

(95) Let us consider finite sequences p_1, q_1, r_1 of elements of $\mathbb{R}_{\mathrm{F}}$, and finite sequences p_2, q_2, r_2 of elements of $\mathbb{R}^1$. Suppose $P = \langle \langle (p)_{\mathbf{1}}, (q)_{\mathbf{1}}, (r)_{\mathbf{1}} \rangle, \langle (p)_{\mathbf{2}}, (q)_{\mathbf{2}}, (r)_{\mathbf{2}} \rangle, \langle (p)_{\mathbf{3}}, (q)_{\mathbf{3}}, (r)_{\mathbf{3}} \rangle \rangle$ and $p = p_1$ and $q = q_1$ and $r = r_1$ and $p_2 = M \cdot p_1$ and $q_2 = M \cdot q_1$ and $r_2 = M \cdot r_1$. Then $(M \cdot P)^{\mathrm{T}} = \langle \mathrm{M2F}(p_2), \mathrm{M2F}(q_2), \mathrm{M2F}(r_2) \rangle$.

Proof: $P^{\mathrm{T}} = \langle \langle (p)_{\mathbf{1}}, (p)_{\mathbf{2}}, (p)_{\mathbf{3}} \rangle, \langle (q)_{\mathbf{1}}, (q)_{\mathbf{2}}, (q)_{\mathbf{3}} \rangle, \langle (r)_{\mathbf{1}}, (r)_{\mathbf{2}}, (r)_{\mathbf{3}} \rangle \rangle$. $\mathrm{width}\, M = \mathrm{len} \langle p_1 \rangle^{\mathrm{T}}$ and $\mathrm{width}\, M = \mathrm{len} \langle q_1 \rangle^{\mathrm{T}}$ and $\mathrm{width}\, M = \mathrm{len} \langle r_1 \rangle^{\mathrm{T}}$ by (75), [11, (50)]. $\mathrm{len}\, p_2 = 3$ and $\mathrm{len}\, q_2 = 3$ and $\mathrm{len}\, r_2 = 3$. $\square$

(96) Let us consider finite sequences p_2, q_2, r_2 of elements of $\mathbb{R}^1$. Suppose $M = \langle \mathrm{M2F}(p_2), \mathrm{M2F}(q_2), \mathrm{M2F}(r_2) \rangle$ and $\mathrm{Det}\, M = 0$ and $\mathrm{M2F}(p_2) = p$ and $\mathrm{M2F}(q_2) = q$ and $\mathrm{M2F}(r_2) = r$. Then $\langle |p, q, r| \rangle = 0$. The theorem is a consequence of (35).

(97) Let us consider points p_3, q_3, r_3 of $\mathcal{E}^3_{\mathrm{T}}$, finite sequences p_2, q_2, r_2 of elements of $\mathbb{R}^1$, and finite sequences p_1, q_1, r_1 of elements of $\mathbb{R}_{\mathrm{F}}$. Suppose M is invertible and $p = p_1$ and $q = q_1$ and $r = r_1$ and $p_2 = M \cdot p_1$ and $q_2 = M \cdot q_1$ and $r_2 = M \cdot r_1$ and $\mathrm{M2F}(p_2) = p_3$ and $\mathrm{M2F}(q_2) = q_3$ and $\mathrm{M2F}(r_2) = r_3$. Then $\langle |p, q, r| \rangle = 0$ if and only if $\langle |p_3, q_3, r_3| \rangle = 0$. The theorem is a consequence of (19), (23), (95), and (35).

(98) If $0 < m$, then every matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $m \times 1$ is a finite sequence of elements of $\mathbb{R}^1$.

Proof: Consider s being a finite sequence such that $s \in \mathrm{rng}\, M$ and $\mathrm{len}\, s = 1$. Consider n being a natural number such that for every object x such that $x \in \mathrm{rng}\, M$ there exists a finite sequence s such that $s = x$ and $\mathrm{len}\, s = n$. Consider s_1 being a finite sequence such that $s_1 = s$ and $\mathrm{len}\, s_1 = n$. $\mathrm{rng}\, M \subseteq \mathbb{R}^1$ by [5, (132)]. $\square$

(99) Let us consider a finite sequence u_1 of elements of $\mathbb{R}_{\mathrm{F}}$. Suppose $\mathrm{len}\, u_1 = 3$. Then $\langle u_1 \rangle^{\mathrm{T}} = I^{3 \times 3}_{\mathbb{R}_{\mathrm{F}}} \cdot \langle u_1 \rangle^{\mathrm{T}}$. The theorem is a consequence of (77), (91), (2), (68), (7), and (93).

(100) Let us consider an element u of $\mathcal{E}^3_{\mathrm{T}}$, and a finite sequence u_1 of elements of $\mathbb{R}_{\mathrm{F}}$. Suppose $u = u_1$ and $\langle u_1 \rangle^{\mathrm{T}} = \langle \langle 0 \rangle, \langle 0 \rangle, \langle 0 \rangle \rangle$. Then $u = 0_{\mathcal{E}^3_{\mathrm{T}}}$. The theorem is a consequence of (77).

(101) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements u, μ of $\mathcal{E}^3_{\mathrm{T}}$, a finite sequence u_1 of elements of $\mathbb{R}_{\mathrm{F}}$, and a finite sequence u_2 of elements of $\mathbb{R}^1$. Suppose u is not zero and $u = u_1$ and $u_2 = N \cdot u_1$ and $\mu = \mathrm{M2F}(u_2)$. Then μ is not zero. The theorem is a consequence of (75), (85), (80), (8), (99), and (100).

Let N be an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. The homography of N yielding a function from the projective space over $\mathcal{E}^3_{\mathrm{T}}$ into the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by

(Def. 4) for every point x of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, there exist elements u, v of $\mathcal{E}^3_{\mathrm{T}}$ and there exists a finite sequence u_1 of elements of $\mathbb{R}_{\mathrm{F}}$ and there exists a finite sequence p of elements of $\mathbb{R}^1$ such that $x =$ the direction of u and u is not zero and $u = u_1$ and $p = N \cdot u_1$ and $v = \mathrm{M2F}(p)$ and v is not zero and $it(x) =$ the direction of v.

Now we state the proposition:

(102) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and points p, q, r of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Then p, q and r are

collinear if and only if (the homography of N)(p), (the homography of N)(q) and (the homography of N)(r) are collinear.

Proof: If p, q and r are collinear, then (the homography of N)(p), (the homography of N)(q) and (the homography of N)(r) are collinear by [10, (23)], (43), [9, (22), (1)]. If (the homography of N)(p), (the homography of N)(q) and (the homography of N)(r) are collinear, then p, q and r are collinear. $\square$

References

[1] Susanne Apel. *The geometry of brackets and the area principle.* Phd thesis, Technische Universität München, Fakultät für Mathematik, 2014.

[2] Susanne Apel and Jürgen Richter-Gebert. Cancellation patterns in automatic geometric theorem proving. In *Automated Deduction in Geometry*, pages 1–33. Springer, 2010.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[7] Laurent Fuchs and Laurent Thery. A formalization of Grassmann-Cayley algebra in Coq and its application to theorem proving in projective geometry. In *Automated Deduction in Geometry*, pages 51–67. Springer, 2010.

[8] Kanchun, Hiroshi Yamazaki, and Yatsuka Nakamura. Cross products and tripple vector products in 3-dimensional Euclidean space. *Formalized Mathematics*, 11(**4**):381–383, 2003.

[9] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[10] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces – part I. *Formalized Mathematics*, 1(**4**):767–776, 1990.

[11] Xiquan Liang, Piqing Zhao, and Ou Bai. Vector functions and their differentiation formulas in 3-dimensional Euclidean spaces. *Formalized Mathematics*, 18(**1**):1–10, 2010. doi:10.2478/v10037-010-0001-2.

[12] Nicolas Magaud, Julien Narboux, and Pascal Schreck. Formalizing projective plane geometry in Coq. In *Automated Deduction in Geometry*, pages 141–162. Springer, 2008.

[13] Timothy James McKenzie Makarios. A mechanical verification of the independence of Tarski's Euclidean Axiom. Victoria University of Wellington, New Zealand, 2012. Master's thesis.

[14] Karol Pąk. Basic properties of the rank of matrices over a field. *Formalized Mathematics*, 15(**4**):199–211, 2007. doi:10.2478/v10037-007-0024-5.

[15] Karol Pąk. Linear transformations of Euclidean topological spaces. *Formalized Mathematics*, 19(**2**):103–108, 2011. doi:10.2478/v10037-011-0016-3.

[16] Jürgen Richter-Gebert. Mechanical theorem proving in projective geometry. *Annals of Mathematics and Artificial Intelligence*, 13(1-2):139–172, 1995.

[17] Jürgen Richter-Gebert. *Perspectives on projective geometry: a guided tour through real and complex geometry.* Springer Science & Business Media, 2011.

[18] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.
[19] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

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