

Quasi-uniform Space

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. In this article, using mostly Pervin [9], Kunzi [6], [8], [7], Williams [11] and Bourbaki [3] works, we formalize in Mizar [2] the notions of quasi-uniform space, semi-uniform space and locally uniform space.

We define the topology induced by a quasi-uniform space. Finally we formalize from the sets of the form $((X \setminus \Omega) \times X) \cup (X \times \Omega)$, the Csaszar-Pervin quasi-uniform space induced by a topological space.

MSC: 54E15 03B35

Keywords: quasi-uniform space; quasi-uniformity; Pervin space; Csaszar-Pervin quasi-uniformity

MML identifier: UNIFORM2, version: 8.1.05 5.37.1275

1. PRELIMINARIES

From now on X denotes a set, A denotes a subset of X , and R, S denote binary relations on X .

Now we state the propositions:

- (1) $(X \setminus A) \times X \cup X \times A \subseteq X \times X$.
- (2) $(X \setminus A) \times X \cup X \times A = A \times A \cup (X \setminus A) \times X$.

PROOF: $(X \setminus A) \times X \cup X \times A \subseteq A \times A \cup (X \setminus A) \times X$ by (1), [4, (87)]. \square

- (3) $R \cdot S = \{\langle x, y \rangle, \text{ where } x, y \text{ are elements of } X : \text{ there exists an element } z \text{ of } X \text{ such that } \langle x, z \rangle \in R \text{ and } \langle z, y \rangle \in S\}$.

PROOF: $R \cdot S \subseteq \{\langle x, y \rangle, \text{ where } x, y \text{ are elements of } X : \text{ there exists an element } z \text{ of } X \text{ such that } \langle x, z \rangle \in R \text{ and } \langle z, y \rangle \in S\}$ by [4, (87)]. $\{\langle x, y \rangle, \text{ where } x, y \text{ are elements of } X : \text{ there exists an element } z \text{ of } X \text{ such that } \langle x, z \rangle \in R \text{ and } \langle z, y \rangle \in S\} \subseteq R \cdot S$. \square

Let X be a set and \mathcal{B} be a family of subsets of X . One can check that $[\mathcal{B}]$ is non empty.

Let \mathcal{B} be a family of subsets of $X \times X$. Note that every element of \mathcal{B} is relation-like.

Let B be an element of \mathcal{B} . We introduce the notation $B[\sim]$ as a synonym of B^\sim .

Let us observe that the functor $B[\sim]$ yields a subset of $X \times X$. Let B_1, B_2 be elements of \mathcal{B} . We introduce the notation $B_1 \otimes B_2$ as a synonym of $B_1 \cdot B_2$.

One can verify that the functor $B_1 \otimes B_2$ yields a subset of $X \times X$. Now we state the propositions:

- (4) Let us consider a set X , and a family G of subsets of X . If G is upper, then $\text{FinMeetCl}(G)$ is upper.
- (5) If R is symmetric in X , then R^\sim is symmetric in X .

2. UNIFORM SPACE STRUCTURE

We consider uniform space structures which extend 1-sorted structures and are systems

$$\langle \text{a carrier, entourages} \rangle$$

where the carrier is a set, the entourages constitute a family of subsets of (the carrier) \times (the carrier).

Let U be a uniform space structure. We say that U is void if and only if

(Def. 1) the entourages of U is empty.

Let X be a set. The functor $\text{UniformSpace}(X)$ yielding a strict uniform space structure is defined by the term

(Def. 2) $\langle X, \emptyset_{2^{X \times X}} \rangle$.

The functors: the trivial uniform space and the non empty trivial uniform space yielding strict uniform space structures are defined by terms

(Def. 3) $\langle \emptyset, 2_*^{\emptyset \times \emptyset} \rangle$,

(Def. 4) there exists a family S_1 of subsets of $\{\emptyset\} \times \{\emptyset\}$ such that $S_1 = \{\{\emptyset\} \times \{\emptyset\}\}$ and the non empty trivial uniform space = $\langle \{\emptyset\}, S_1 \rangle$,

respectively. Let X be an empty set. One can verify that $\text{UniformSpace}(X)$ is empty.

Let X be a non empty set. One can check that $\text{UniformSpace}(X)$ is non empty.

Let X be a set. Note that $\text{UniformSpace}(X)$ is void and the trivial uniform space is empty and non void and the non empty trivial uniform space is non empty and non void and there exists a uniform space structure which is empty,

strict, and void and there exists a uniform space structure which is empty, strict, and non void and there exists a uniform space structure which is non empty, strict, and void and there exists a uniform space structure which is non empty, strict, and non void.

Let X be a set and S_1 be a family of subsets of $X \times X$. The functor $S_1[\sim]$ yielding a family of subsets of $X \times X$ is defined by the term

(Def. 5) the set of all $S[\sim]$ where S is an element of S_1 .

Let U be a uniform space structure. The functor $U[\sim]$ yielding a uniform space structure is defined by the term

(Def. 6) \langle the carrier of U , (the entourages of U) $[\sim]$ \rangle .

Let U be a non empty uniform space structure. One can verify that $U[\sim]$ is non empty.

3. AXIOMS

Let U be a uniform space structure. We say that U is upper if and only if

(Def. 7) the entourages of U is upper.

We say that U is \cap -closed if and only if

(Def. 8) the entourages of U is \cap -closed.

We say that U satisfies axiom U1 if and only if

(Def. 9) for every element S of the entourages of U , $\text{id}_\alpha \subseteq S$, where α is the carrier of U .

We say that U satisfies axiom U2 if and only if

(Def. 10) for every element S of the entourages of U , $S[\sim] \in$ the entourages of U .

We say that U satisfies axiom U3 if and only if

(Def. 11) for every element S of the entourages of U , there exists an element W of the entourages of U such that $W \otimes W \subseteq S$.

Let us consider a non void uniform space structure U . Now we state the propositions:

(6) U satisfies axiom U1 if and only if for every element S of the entourages of U , there exists a binary relation R on the carrier of U such that $R = S$ and R is reflexive in the carrier of U .

(7) U satisfies axiom U1 if and only if for every element S of the entourages of U , there exists a total, reflexive binary relation R on the carrier of U such that $R = S$. The theorem is a consequence of (6).

Note that every uniform space structure which is void does not satisfy also axiom U2.

Now we state the proposition:

- (8) Let us consider a uniform space structure U . Suppose U satisfies axiom U2. Let us consider an element S of the entourages of U , and elements x, y of U . Suppose $\langle x, y \rangle \in S$. Then $\langle y, x \rangle \in \bigcup(\text{the entourages of } U)$.

Let us consider a non void uniform space structure U . Now we state the propositions:

- (9) Suppose for every element S of the entourages of U , there exists a binary relation R on the carrier of U such that $S = R$ and R is symmetric in the carrier of U . Then U satisfies axiom U2. The theorem is a consequence of (5).
- (10) Suppose for every element S of the entourages of U , there exists a binary relation R on the carrier of U such that $S = R$ and R is symmetric. Then U satisfies axiom U2. The theorem is a consequence of (9).
- (11) If for every element S of the entourages of U , there exists a tolerance R of the carrier of U such that $S = R$, then U satisfies axiom U1 and axiom U2. The theorem is a consequence of (7) and (10).

Let X be an empty set. Observe that $\text{UniformSpace}(X)$ is upper and \cap -closed and satisfies axiom U1 and axiom U3 and does not satisfy axiom U2 and $\text{UniformSpace}(\{\emptyset\})$ is upper and \cap -closed and does not satisfy axiom U2 and the trivial uniform space is upper and \cap -closed and satisfies axiom U1, axiom U2, and axiom U3 and the non empty trivial uniform space is upper and \cap -closed and satisfies axiom U1, axiom U2, and axiom U3.

There exists a uniform space structure which is strict, empty, non void, upper, and \cap -closed and satisfies axiom U1, axiom U2, and axiom U3 and every strict uniform space structure which is empty satisfies also axiom U1 and there exists a uniform space structure which is strict, non empty, non void, upper, and \cap -closed and satisfies axiom U1, axiom U2, and axiom U3.

Let S_4 be a non empty uniform space structure satisfying axiom U1, x be an element of S_4 , and V be an element of the entourages of S_4 . The functor $\text{Nbh}(V, x)$ yielding a non empty subset of S_4 is defined by the term

(Def. 12) $\{y, \text{ where } y \text{ is an element of } S_4 : \langle x, y \rangle \in V\}$.

Now we state the proposition:

- (12) Let us consider a non empty uniform space structure U satisfying axiom U1, an element x of the carrier of U , and an element V of the entourages of U . Then $x \in \text{Nbh}(V, x)$.

Let U be a \cap -closed uniform space structure and V_1, V_2 be elements of the entourages of U . One can check that the functor $V_1 \cap V_2$ yields an element of the entourages of U . Now we state the proposition:

- (13) Let us consider a non empty, \cap -closed uniform space structure U satisfying axiom U1, an element x of U , and elements V, W of the entourages

of U . Then $\text{Nbh}(V, x) \cap \text{Nbh}(W, x) = \text{Nbh}(V \cap W, x)$.

Let U be a non empty uniform space structure satisfying axiom U1. Let us observe that the entourages of U has non empty elements and the entourages of U is non empty.

Let x be a point of U . The functor Neighborhood x yielding a family of subsets of U is defined by the term

(Def. 13) the set of all $\text{Nbh}(V, x)$ where V is an element of the entourages of U .

Let us note that Neighborhood x is non empty.

Now we state the proposition:

(14) Let us consider a non empty uniform space structure S_4 satisfying axiom U1, an element x of the carrier of S_4 , and an element V of the entourages of S_4 . Then

- (i) $\text{Nbh}(V, x) = V^\circ\{x\}$, and
- (ii) $\text{Nbh}(V, x) = \text{rng}(V \upharpoonright \{x\})$, and
- (iii) $\text{Nbh}(V, x) = V^\circ x$, and
- (iv) $\text{Nbh}(V, x) = [x]_V$, and
- (v) $\text{Nbh}(V, x) = \text{neighbourhood}(x, V)$.

PROOF: $\text{Nbh}(V, x) = V^\circ\{x\}$ by [4, (87)]. \square

Let U be a non empty uniform space structure satisfying axiom U1. The functor Neighborhood U yielding a function from the carrier of U into $2^{2^{\text{(the carrier of } U\text{)}}}$ is defined by

(Def. 14) for every element x of U , $it(x) = \text{Neighborhood } x$.

We say that U is topological if and only if

(Def. 15) $\langle \text{the carrier of } U, \text{Neighborhood } U \rangle$ is a topology from neighbourhoods.

4. QUASI-UNIFORM SPACE

A quasi-uniform space is an upper, \cap -closed uniform space structure satisfying axiom U1 and axiom U3. In the sequel Q denotes a quasi-uniform space.

Now we state the propositions:

- (15) If the entourages of Q is empty, then the entourages of $Q[\sim] = \{\emptyset\}$.
- (16) Suppose the entourages of $Q[\sim] = \{\emptyset\}$ and the entourages of $Q[\sim]$ is upper. Then the carrier of Q is empty.

Let Q be a non void quasi-uniform space. One can check that $Q[\sim]$ is upper and \cap -closed and satisfies axiom U1 and axiom U3.

Let X be a set and \mathcal{B} be a family of subsets of $X \times X$. We say that \mathcal{B} satisfies axiom UP1 if and only if

(Def. 16) for every element B of \mathcal{B} , $\text{id}_X \subseteq B$.

We say that \mathcal{B} satisfies axiom UP3 if and only if

(Def. 17) for every element B_1 of \mathcal{B} , there exists an element B_2 of \mathcal{B} such that $B_2 \otimes B_2 \subseteq B_1$.

Now we state the propositions:

(17) Let us consider a non empty set X , and an empty family \mathcal{B} of subsets of $X \times X$. Then \mathcal{B} does not satisfy axiom UP1.

(18) Let us consider a set X , and a family \mathcal{B} of subsets of $X \times X$. Suppose \mathcal{B} is quasi-basis and satisfies axiom UP1 and axiom UP3. Then $\langle X, [\mathcal{B}] \rangle$ is a quasi-uniform space.

5. SEMI-UNIFORM SPACE

A semi-uniform space is an upper, \cap -closed uniform space structure satisfying axiom U1 and axiom U2. From now on S_4 denotes a semi-uniform space.

Let us observe that every semi-uniform space is non void.

Now we state the proposition:

(19) If S_4 is empty, then $\emptyset \in$ the entourages of S_4 .

Let S_4 be an empty semi-uniform space. One can verify that the entourages of S_4 has the empty element.

6. LOCALLY UNIFORM SPACE

Let S_4 be a non empty semi-uniform space. We say that S_4 satisfies axiom UL if and only if

(Def. 18) for every element S of the entourages of S_4 and for every element x of S_4 , there exists an element W of the entourages of S_4 such that $\{y, \text{ where } y \text{ is an element of } S_4 : \langle x, y \rangle \in W \otimes W\} \subseteq \text{Nbh}(S, x)$.

One can verify that every non empty semi-uniform space which satisfies axiom U3 satisfies also axiom UL and there exists a non empty semi-uniform space which satisfies axiom UL.

A locally uniform space is a non empty semi-uniform space satisfying axiom UL. Now we state the propositions:

(20) Let us consider a non empty, upper uniform space structure U satisfying axiom U1, and an element x of the carrier of U . Then Neighborhood x is upper.

- (21) Let us consider a non empty uniform space structure U satisfying axiom U1, an element x of U , and an element V of the entourages of U . Then $x \in \text{Nbh}(V, x)$.
- (22) Let us consider a non empty, \cap -closed uniform space structure U satisfying axiom U1, and an element x of U . Then Neighborhood x is \cap -closed. The theorem is a consequence of (13).
- (23) Let us consider a non empty, upper, \cap -closed uniform space structure U satisfying axiom U1, and an element x of U . Then Neighborhood x is a filter of the carrier of U . The theorem is a consequence of (22) and (20).

Let us observe that every locally uniform space is topological.

7. TOPOLOGICAL SPACE INDUCED BY A UNIFORM SPACE STRUCTURE

Let U be a topological, non empty uniform space structure satisfying axiom U1. The FMT induced by U yielding a non empty, strict topology from neighbourhoods is defined by the term

(Def. 19) $\langle \text{the carrier of } U, \text{Neighborhood } U \rangle$.

The topological space induced by U yielding a topological space is defined by the term

(Def. 20) $\text{FMT2TopSpace}(\text{the FMT induced by } U)$.

8. THE QUASI-UNIFORM PERVIN SPACE INDUCED BY A TOPOLOGICAL SPACE

Let X be a set and A be a subset of X . The functor $\text{qBlock}(A)$ yielding a subset of $X \times X$ is defined by the term

(Def. 21) $(X \setminus A) \times X \cup X \times A$.

Now we state the proposition:

- (24) (i) $\text{id}_X \subseteq \text{qBlock}(A)$, and
- (ii) $\text{qBlock}(A) \cdot \text{qBlock}(A) \subseteq \text{qBlock}(A)$.

PROOF: $\text{id}_X \subseteq \text{qBlock}(A)$ by [4, (96)]. \square

Let T be a topological space. The functor $\text{qBlocks}(T)$ yielding a family of subsets of $(\text{the carrier of } T) \times (\text{the carrier of } T)$ is defined by the term

(Def. 22) the set of all $\text{qBlock}(O)$ where O is an element of the topology of T .

Let T be a non empty topological space. One can check that $\text{qBlocks}(T)$ is non empty.

Let T be a topological space. The functor $\text{FMCqBlocks}(T)$ yielding a family of subsets of $(\text{the carrier of } T) \times (\text{the carrier of } T)$ is defined by the term

(Def. 23) $\text{FinMeetCl}(\text{qBlocks}(T))$.

Let X be a set. One can check that every non empty family of subsets of $X \times X$ which is \cap -closed is also quasi-basis.

In the sequel T denotes a topological space.

Let us consider T . One can check that $\text{FMCqBlocks}(T)$ is \cap -closed and $\text{FMCqBlocks}(T)$ is quasi-basis and $\text{FMCqBlocks}(T)$ satisfies axiom UP1 and $\text{FMCqBlocks}(T)$ satisfies axiom UP3.

Let T be a topological space. The Pervin quasi-uniformity of T yielding a strict quasi-uniform space is defined by the term

(Def. 24) $\langle \text{the carrier of } T, [\text{FMCqBlocks}(T)] \rangle$.

Now we state the propositions:

(25) Let us consider a non empty topological space T , and an element V of the entourages of the Pervin quasi-uniformity of T . Then there exists an element b of $\text{FinMeetCl}(\text{qBlocks}(T))$ such that $b \subseteq V$.

(26) Let us consider a non empty topological space T , and a subset V of $(\text{the carrier of } T) \times (\text{the carrier of } T)$. Suppose there exists an element b of $\text{FinMeetCl}(\text{qBlocks}(T))$ such that $b \subseteq V$. Then V is an element of the entourages of the Pervin quasi-uniformity of T .

(27) $\text{qBlocks}(T) \subseteq$ the entourages of the Pervin quasi-uniformity of T .

Let us consider a non void quasi-uniform space Q . Now we state the propositions:

(28) $(\text{The carrier of } Q) \times (\text{the carrier of } Q) \in$ the entourages of Q .

(29) Suppose the carrier of $T =$ the carrier of Q and $\text{qBlocks}(T) \subseteq$ the entourages of Q . Then the entourages of the Pervin quasi-uniformity of $T \subseteq$ the entourages of Q .

PROOF: The entourages of the Pervin quasi-uniformity of $T \subseteq$ the entourages of Q by (28), [1, (1)]. \square

Let T be a non empty topological space. One can check that the Pervin quasi-uniformity of T is non empty and the Pervin quasi-uniformity of T is topological.

Now we state the propositions:

(30) Let us consider a non empty topological space T , an element x of $\text{qBlocks}(T)$, and an element y of the Pervin quasi-uniformity of T . Then $\{z, \text{ where } z \text{ is an element of the Pervin quasi-uniformity of } T : \langle y, z \rangle \in x\} \in$ the topology of T .

(31) Let us consider a non empty topological space T , an element x of the carrier of the Pervin quasi-uniformity of T , and an element b of $\text{FinMeetCl}(\text{qBlocks}(T))$. Then $\{y, \text{ where } y \text{ is an element of } T : \langle x, y \rangle \in b\} \in$ the to-

pology of T . The theorem is a consequence of (30).

- (32) Let us consider a non empty, strict quasi-uniform space U , a non empty, strict formal topological space F , and an element x of F . Suppose $F = \langle \text{the carrier of } U, \text{Neighborhood } U \rangle$. Then there exists an element y of U such that
- (i) $x = y$, and
 - (ii) $U_F(x) = \text{Neighborhood } y$.
- (33) Let us consider a non empty topological space T . Then the open set family of the FMT induced by the Pervin quasi-uniformity of $T =$ the topology of T .
- PROOF: The open set family of the FMT induced by the Pervin quasi-uniformity of $T \subseteq$ the topology of T by (32), [5, (18)], (31), [12, (25)]. The topology of $T \subseteq$ the open set family of the FMT induced by the Pervin quasi-uniformity of T by (32), [10, (4)], [5, (18)], [4, (87)]. \square
- (34) Let us consider a non empty, strict topological space T . Then the topological space induced by the Pervin quasi-uniformity of $T = T$. The theorem is a consequence of (33).

ACKNOWLEDGEMENT: The author wants to express his gratitude to the anonymous referee for his/her work for the introduction of new notations and to make the presentation more readable.

REFERENCES

- [1] William W. Armstrong, Yatsuka Nakamura, and Piotr Rudnicki. Armstrong's axioms. *Formalized Mathematics*, 11(1):39–51, 2003.
- [2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.
- [3] Nicolas Bourbaki. *General Topology: Chapters 1–4*. Springer Science and Business Media, 2013.
- [4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [5] Roland Coghetto. Convergent filter bases. *Formalized Mathematics*, 23(3):189–203, 2015. doi:10.1515/forma-2015-0016.
- [6] Hans-Peter A. Künzi. Quasi-uniform spaces - eleven years later. In *Topology Proceedings*, volume 18, pages 143–171, 1993.
- [7] Hans-Peter A. Künzi. An introduction to quasi-uniform spaces. *Beyond Topology*, 486: 239–304, 2009.
- [8] Hans-Peter A. Künzi and Carolina Ryser. The Bourbaki quasi-uniformity. In *Topology Proceedings*, volume 20, pages 161–183, 1995.
- [9] William J. Pervin. Quasi-uniformization of topological spaces. *Mathematische Annalen*, 147(4):316–317, 1962.

- [10] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(2):233–236, 1996.
- [11] James Williams. Locally uniform spaces. *Transactions of the American Mathematical Society*, 168:435–469, 1972.
- [12] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(1):231–237, 1990.

Received June 30, 2016
