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Riemann-Stieltjes Integral[1]

Keiko Narita
Hirosaki-city
Aomori, Japan

Kazuhisa Nakasho
Akita Prefectural University
Akita, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, the definitions and basic properties of Riemann-Stieltjes integral are formalized in Mizar [1]. In the first section, we showed the preliminary definition. We proved also some properties of finite sequences of real numbers. In Sec. 2, we defined variation. Using the definition, we also defined bounded variation and total variation, and proved theorems about related properties.

In Sec. 3, we defined Riemann-Stieltjes integral. Referring to the way of the article [7], we described the definitions. In the last section, we proved theorems about linearity of Riemann-Stieltjes integral. Because there are two types of linearity in Riemann-Stieltjes integral, we proved linearity in two ways. We showed the proof of theorems based on the description of the article [7]. These formalizations are based on [8], [5], [3], and [4].

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1. Properties of Real Finite Sequences

Let A be a subset of $\mathbb{R}$ and ϱ be a real-valued function. The functor $\mathrm{vol}(A, \varrho)$ yielding a real number is defined by the term

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(Def. 1) $\begin{cases} 0, & \textbf{if } A \text{ is empty}, \\ \varrho(\sup A) - \varrho(\inf A), & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(1) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a partition D of A, a function ϱ from A into $\mathbb{R}$, a non empty, closed interval subset B of $\mathbb{R}$, and a finite sequence v of elements of $\mathbb{R}$. Suppose $B \subseteq A$ and $\operatorname{len} D = \operatorname{len} v$ and for every natural number i such that $i \in \operatorname{dom} v$ holds $v(i) = \operatorname{vol}(B \cap \operatorname{divset}(D, i), \varrho)$. Then $\sum v = \operatorname{vol}(B, \varrho)$.

(2) Let us consider natural numbers n, m, a function a from $\operatorname{Seg} n \times \operatorname{Seg} m$ into $\mathbb{R}$, and finite sequences p, q of elements of $\mathbb{R}$. Suppose $\operatorname{dom} p = \operatorname{Seg} n$ and for every natural number i such that $i \in \operatorname{dom} p$ there exists a finite sequence r of elements of $\mathbb{R}$ such that $\operatorname{dom} r = \operatorname{Seg} m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \operatorname{dom} r$ holds $r(j) = a(i, j)$ and $\operatorname{dom} q = \operatorname{Seg} m$ and for every natural number j such that $j \in \operatorname{dom} q$ there exists a finite sequence s of elements of $\mathbb{R}$ such that $\operatorname{dom} s = \operatorname{Seg} n$ and $q(j) = \sum s$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) = a(i, j)$. Then $\sum p = \sum q$.

2. The Definitions of Bounded Variation

Let A be a non empty, closed interval subset of $\mathbb{R}$, ϱ be a real-valued function, and t be a partition of A. A var-volume of ϱ and t is a finite sequence of elements of $\mathbb{R}$ and is defined by

(Def. 2) $\operatorname{len} it = \operatorname{len} t$ and for every natural number k such that $k \in \operatorname{dom} t$ holds $it(k) = |\operatorname{vol}(\operatorname{divset}(t, k), \varrho)|$.

Now we state the propositions:

(3) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, a partition t of A, a var-volume F of ϱ and t, and a natural number k. If $k \in \operatorname{dom} F$, then $0 \leqslant F(k)$.

(4) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, a partition t of A, and a var-volume F of ϱ and t. Then $0 \leqslant \sum F$. The theorem is a consequence of (3).

Let A be a non empty, closed interval subset of $\mathbb{R}$ and ϱ be a function from A into $\mathbb{R}$. We say that ϱ is bounded-variation if and only if

(Def. 3) there exists a real number d such that $0 < d$ and for every partition t of A and for every var-volume F of ϱ and t, $\sum F \leqslant d$.

Assume ϱ is bounded-variation. The functor $\operatorname{TotalVD}(\varrho)$ yielding a real number is defined by

(Def. 4) there exists a non empty subset V of $\mathbb{R}$ such that V is upper bounded and $V = \{r$, where r is a real number : there exists a partition t of A and there exists a var-volume F of ϱ and t such that $r = \sum F\}$ and $it = \sup V$.

Now we state the propositions:

(5) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, and a partition T of A. Suppose ϱ is bounded-variation. Let us consider a var-volume F of ϱ and T. Then $\sum F \leqslant \mathrm{TotalVD}(\varrho)$.

(6) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, and a function ϱ from A into $\mathbb{R}$. If ϱ is bounded-variation, then $0 \leqslant \mathrm{TotalVD}(\varrho)$. The theorem is a consequence of (4).

3. The Definitions of Riemann-Stieltjes Integral

Let A be a non empty, closed interval subset of $\mathbb{R}$, ϱ be a function from A into $\mathbb{R}$, and u be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Assume ϱ is bounded-variation and $\operatorname{dom} u = A$. Let t be a partition of A.

A middle volume of ϱ, u and t is a finite sequence of elements of $\mathbb{R}$ and is defined by

(Def. 5) $\operatorname{len} it = \operatorname{len} t$ and for every natural number k such that $k \in \operatorname{dom} t$ there exists a real number r such that $r \in \operatorname{rng}(u \restriction \operatorname{divset}(t, k))$ and $it(k) = r \cdot \operatorname{vol}(\operatorname{divset}(t, k), \varrho)$.

Let T be a division sequence of A. A middle volume sequence of ϱ, u and T is a sequence of $\mathbb{R}^*$ and is defined by

(Def. 6) for every element k of $\mathbb{N}$, $it(k)$ is a middle volume of ϱ, u and $T(k)$.

Let S be a middle volume sequence of ϱ, u and T and k be a natural number. One can check that the functor $S(k)$ yields a middle volume of ϱ, u and $T(k)$. From now on A denotes a non empty, closed interval subset of $\mathbb{R}$, ϱ denotes a function from A into $\mathbb{R}$, u denotes a partial function from $\mathbb{R}$ to $\mathbb{R}$, T denotes a division sequence of A, S denotes a middle volume sequence of ϱ, u and T, and k denotes a natural number.

Let A be a non empty, closed interval subset of $\mathbb{R}$, ϱ be a function from A into $\mathbb{R}$, u be a partial function from $\mathbb{R}$ to $\mathbb{R}$, T be a division sequence of A, and S be a middle volume sequence of ϱ, u and T. The functor middle-sum(S) yielding a sequence of real numbers is defined by

(Def. 7) for every natural number i, $it(i) = \sum(S(i))$.

We say that u is Riemann-Stieltjes integrable with ϱ if and only if

(Def. 8) there exists a real number I such that for every division sequence T of A for every middle volume sequence S of ϱ, u and T such that δ_T is convergent and $\lim \delta_T = 0$ holds middle-sum(S) is convergent and $\lim$ middle-sum(S) $= I$.

Assume ϱ is bounded-variation and $\operatorname{dom} u = A$ and u is Riemann-Stieltjes integrable with ϱ. The functor $\int\limits_{\varrho} u(x)dx$ yielding a real number is defined by

(Def. 9) for every division sequence T of A and for every middle volume sequence S of ϱ, u and T such that δ_T is convergent and $\lim \delta_T = 0$ holds middle-sum(S) is convergent and $\lim$ middle-sum(S) $=$ *it*.

4. Linearity of Riemann-Stieltjes Integral

Now we state the propositions:

(7) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a real number r, a function ϱ from A into $\mathbb{R}$, and partial functions u, w from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and $\operatorname{dom} u = A$ and $\operatorname{dom} w = A$ and $w = r \cdot u$ and u is Riemann-Stieltjes integrable with ϱ. Then

(i) w is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} w(x)dx = r \cdot \int\limits_{\varrho} u(x)dx$.

(8) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, and partial functions u, w from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and $\operatorname{dom} u = A$ and $\operatorname{dom} w = A$ and $w = -u$ and u is Riemann-Stieltjes integrable with ϱ. Then

(i) w is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} w(x)dx = -\int\limits_{\varrho} u(x)dx$.

The theorem is a consequence of (7).

Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a function ϱ from A into $\mathbb{R}$, and partial functions u, v, w from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(9) Suppose ϱ is bounded-variation and $\operatorname{dom} u = A$ and $\operatorname{dom} v = A$ and $\operatorname{dom} w = A$ and $w = u + v$ and u is Riemann-Stieltjes integrable with ϱ and v is Riemann-Stieltjes integrable with ϱ. Then

(i) w is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} w(x)dx = \int\limits_{\varrho} u(x)dx + \int\limits_{\varrho} v(x)dx.$

(10) Suppose ϱ is bounded-variation and $\operatorname{dom} u = A$ and $\operatorname{dom} v = A$ and $\operatorname{dom} w = A$ and $w = u - v$ and u is Riemann-Stieltjes integrable with ϱ and v is Riemann-Stieltjes integrable with ϱ. Then

(i) w is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} w(x)dx = \int\limits_{\varrho} u(x)dx - \int\limits_{\varrho} v(x)dx.$

The theorem is a consequence of (8) and (9).

(11) Let us consider non empty, closed interval subsets A, B of $\mathbb{R}$, a real number r, and functions ϱ, ϱ_1 from A into $\mathbb{R}$. Suppose $B \subseteq A$ and $\varrho = r \cdot \varrho_1$. Then $\operatorname{vol}(B, \varrho) = r \cdot \operatorname{vol}(B, \varrho_1)$.

Proof: Set $x_1 = \sup B$. Set $x_2 = \inf B$. $|x_2 - x_1| = x_1 - x_2$ by [6, (11)], [2, (44)]. □

(12) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, a real number r, functions ϱ, ϱ_1 from A into $\mathbb{R}$, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and ϱ_1 is bounded-variation and $\operatorname{dom} u = A$ and $\varrho = r \cdot \varrho_1$ and u is Riemann-Stieltjes integrable with ϱ_1. Then

(i) u is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} u(x)dx = r \cdot \int\limits_{\varrho_1} u(x)dx.$

The theorem is a consequence of (11).

(13) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, functions ϱ, ϱ_1 from A into $\mathbb{R}$, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and ϱ_1 is bounded-variation and $\operatorname{dom} u = A$ and $\varrho = -\varrho_1$ and u is Riemann-Stieltjes integrable with ϱ_1. Then

(i) u is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} u(x)dx = -\int\limits_{\varrho_1} u(x)dx.$

The theorem is a consequence of (12).

(14) Let us consider non empty, closed interval subsets A, B of $\mathbb{R}$, and functions ϱ, ϱ_1, ϱ_2 from A into $\mathbb{R}$. Suppose $B \subseteq A$ and $\varrho = \varrho_1 + \varrho_2$. Then $\operatorname{vol}(B, \varrho) = \operatorname{vol}(B, \varrho_1) + \operatorname{vol}(B, \varrho_2)$.

Proof: Set $x_1 = \sup B$. Set $x_2 = \inf B$. $|x_2 - x_1| = x_1 - x_2$ by [6, (11)], [2, (44)]. □

(15) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, functions ϱ, ϱ_1, ϱ_2 from A into $\mathbb{R}$, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and ϱ_1 is bounded-variation and ϱ_2 is bounded-variation and $\operatorname{dom} u = A$ and $\varrho = \varrho_1 + \varrho_2$ and u is Riemann-Stieltjes integrable with ϱ_1 and u is Riemann-Stieltjes integrable with ϱ_2. Then

(i) u is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} u(x)dx = \int\limits_{\varrho_1} u(x)dx + \int\limits_{\varrho_2} u(x)dx$.

The theorem is a consequence of (14).

(16) Let us consider non empty, closed interval subsets A, B of $\mathbb{R}$, and functions ϱ, ϱ_1, ϱ_2 from A into $\mathbb{R}$. Suppose $B \subseteq A$ and $\varrho = \varrho_1 - \varrho_2$. Then $\operatorname{vol}(B, \varrho) = \operatorname{vol}(B, \varrho_1) - \operatorname{vol}(B, \varrho_2)$. The theorem is a consequence of (14).

(17) Let us consider a non empty, closed interval subset A of $\mathbb{R}$, functions ϱ, ϱ_1, ϱ_2 from A into $\mathbb{R}$, and a partial function u from $\mathbb{R}$ to $\mathbb{R}$. Suppose ϱ is bounded-variation and ϱ_1 is bounded-variation and ϱ_2 is bounded-variation and $\operatorname{dom} u = A$ and $\varrho = \varrho_1 - \varrho_2$ and u is Riemann-Stieltjes integrable with ϱ_1 and u is Riemann-Stieltjes integrable with ϱ_2. Then

(i) u is Riemann-Stieltjes integrable with ϱ, and

(ii) $\int\limits_{\varrho} u(x)dx = \int\limits_{\varrho_1} u(x)dx - \int\limits_{\varrho_2} u(x)dx$.

The theorem is a consequence of (16).

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