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Double Sequences and Iterated Limits in Regular Space

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. First, we define in Mizar [5], the Cartesian product of two filters bases and the Cartesian product of two filters. After comparing the product of two Fréchet filters on $\mathbb{N}$ ($\mathcal{F}_1$) with the Fréchet filter on $\mathbb{N} \times \mathbb{N}$ ($\mathcal{F}_2$), we compare $lim_{\mathcal{F}_1}$ and $lim_{\mathcal{F}_2}$ for all double sequences in a non empty topological space.

Endou, Okazaki and Shidama formalized in [14] the "convergence in Pringsheim's sense" for double sequence of real numbers. We show some basic correspondences between the p-convergence and the filter convergence in a topological space. Then we formalize that the double sequence $(x_{m,n} = \frac{1}{m+1})_{(m,n)} \in \mathbb{N} \times \mathbb{N}$ converges in "Pringsheim's sense" but not in Frechet filter on $\mathbb{N} \times \mathbb{N}$ sense.

In the next section, we generalize some definitions: "is convergent in the first coordinate", "is convergent in the second coordinate", "the lim in the first coordinate of", "the lim in the second coordinate of" according to [14], in Hausdorff space.

Finally, we generalize two theorems: (3) and (4) from [14] in the case of double sequences and we formalize the "iterated limit" theorem ("Double limit" [7], p. 81, par. 8.5 *"Double limite"* [6] (TG I,57)), all in regular space. We were inspired by the exercises (2.11.4), (2.17.5) [17] and the corrections B.10 [18].

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1. Preliminaries

From now on x denotes an object, X, Y, Z denote sets, i, j, k, l, m, n denote natural numbers, r, s denote real numbers, n_1 denotes an element of the ordered $\mathbb{N}$, and A denotes a subset of $\mathbb{N} \times \mathbb{N}$.

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Now we state the propositions:

(1) Let us consider a finite subset W of X. If $X \setminus W \subseteq Z$, then $X \setminus Z$ is finite.

(2) If $Z \subseteq X$ and $X \setminus Z$ is finite, then there exists a finite subset W of X such that $X \setminus W = Z$.

(3) Let us consider sets X_1, X_2, a family S_1 of subsets of X_1, and a family S_2 of subsets of X_2. Then $\{s$, where s is a subset of $X_1 \times X_2$: there exist sets s_1, s_2 such that $s_1 \in S_1$ and $s_2 \in S_2$ and $s = s_1 \times s_2\}$ is a family of subsets of $X_1 \times X_2$.

(4) If $x \in X \times Y$, then x is pair.

(5) If $0 < r$, then there exists m such that m is not zero and $\frac{1}{m} < r$.

(6) Let us consider points x, y of the metric space of real numbers. Then there exist real numbers x_1, y_1 such that

  (i) $x = x_1$, and

  (ii) $y = y_1$, and

  (iii) $\rho(x, y) = \rho_{\mathbb{R}}(x, y)$, and

  (iv) $\rho(x, y) = \rho^1(\langle x \rangle, \langle y \rangle)$, and

  (v) $\rho(x, y) = |x_1 - y_1|$.

(7) Let us consider points x, y of $(\mathcal{E}^1)_{\mathrm{top}}$. Then there exist points x_2, y_2 of the metric space of real numbers and there exist real numbers x_1, y_1 such that $x_2 = x_1$ and $y_2 = y_1$ and $x = \langle x_1 \rangle$ and $y = \langle y_1 \rangle$ and $\rho(x_2, y_2) = \rho_{\mathbb{R}}(x_1, y_1)$ and $\rho(x_2, y_2) = \rho^1(\langle x_1 \rangle, \langle y_1 \rangle)$ and $\rho(x_2, y_2) = |x_1 - y_1|$.

(8) Let us consider points x, y of $\mathcal{E}^1$, and real numbers r, s. If $x = \langle r \rangle$ and $y = \langle s \rangle$, then $\rho(x, y) = |r - s|$. The theorem is a consequence of (7).

One can check that $\mathbb{N} \times \mathbb{N}$ is countable and $\mathbb{N} \times \mathbb{N}$ is denumerable.

Now we state the propositions:

(9) the set of all $\langle 0, n \rangle$ where n is a natural number is infinite.
PROOF: Define $\mathcal{F}(\text{object}) = \langle 0, \$_1 \rangle$. Consider f being a function such that $\operatorname{dom} f = \mathbb{N}$ and for every object x such that $x \in \mathbb{N}$ holds $f(x) = \mathcal{F}(x)$ from [9, Sch. 3]. f is one-to-one. $\operatorname{rng} f =$ the set of all $\langle 0, n \rangle$ where n is a natural number by [9, (3)]. □

(10) If $i \leqslant k$ and $j \leqslant l$, then $\mathbb{Z}_i \times \mathbb{Z}_j \subseteq \mathbb{Z}_k \times \mathbb{Z}_l$.

(11) $(\mathbb{N} \setminus \mathbb{Z}_m) \times (\mathbb{N} \setminus \mathbb{Z}_n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_m \times \mathbb{Z}_n$.

(12) If $n = n_1$ and $n \leqslant m$, then $m \in {\uparrow}n_1$.

(13) If $n = n_1$ and $m \in {\uparrow}n_1$, then $n \leqslant m$.

(14) If $n = n_1$, then ${\uparrow}n_1 = \mathbb{N} \setminus \mathbb{Z}_n$.

PROOF: $\uparrow n_1 \subseteq \mathbb{N} \setminus \mathbb{Z}_n$ by [12, (50)], (13), [1, (44)]. $\mathbb{N} \setminus \mathbb{Z}_n \subseteq \uparrow n_1$ by [1, (44)], [12, (50)]. □

(15) $\pi_1(A) = \{x$, where x is an element of $\mathbb{N}$: there exists an element y of $\mathbb{N}$ such that $\langle x, y\rangle \in A\}$.

(16) $\pi_2(A) = \{y$, where y is an element of $\mathbb{N}$: there exists an element x of $\mathbb{N}$ such that $\langle x, y\rangle \in A\}$.

(17) Let us consider a finite subset A of $\mathbb{N} \times \mathbb{N}$. Then there exists m and there exists n such that $A \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$. The theorem is a consequence of (15) and (16).

(18) Let us consider a non empty set X. Then every filter of X is a proper filter of $2^X_\subseteq$.

(19) Let us consider a non empty set X, and a filter $\mathcal{F}$ of X. Then there exists a filter base $\mathcal{B}$ of X such that

(i) $\mathcal{B} = \mathcal{F}$, and

(ii) $[\mathcal{B}) = \mathcal{F}$.

(20) Let us consider a non empty topological space T, and a filter $\mathcal{F}$ of the carrier of T. If $x \in \mathrm{LimFilter}(\mathcal{F})$, then x is a cluster point of $\mathcal{F}$,T.

(21) Let us consider an element B of the base of Frechet filter. Then there exists n such that $B = \mathbb{N} \setminus \mathbb{Z}_n$. The theorem is a consequence of (14).

(22) Let us consider a subset B of $\mathbb{N}$. Suppose $B = \mathbb{N} \setminus \mathbb{Z}_n$. Then B is an element of the base of Frechet filter. The theorem is a consequence of (14).

2. Cartesian Product of Two Filters

From now on X, Y, X_1, X_2 denote non empty sets, $\mathcal{A}_1$, $\mathcal{B}_1$ denote filter bases of X_1, $\mathcal{A}_2$, $\mathcal{B}_2$ denote filter bases of X_2, $\mathcal{F}_1$ denotes a filter of X_1, $\mathcal{F}_2$ denotes a filter of X_2, $\mathcal{B}_3$ denotes a generalized basis of $\mathcal{F}_1$.

Let X_1, X_2 be non empty sets, $\mathcal{B}_1$ be a filter base of X_1, and $\mathcal{B}_2$ be a filter base of X_2. The functor $\mathcal{B}_1 \times \mathcal{B}_2$ yielding a filter base of $X_1 \times X_2$ is defined by the term

(Def. 1) the set of all $B_1 \times B_2$ where B_1 is an element of $\mathcal{B}_1$, B_2 is an element of $\mathcal{B}_2$.

Now we state the propositions:

(23) Suppose $\mathcal{F}_1 = [\mathcal{B}_1)$ and $\mathcal{F}_1 = [\mathcal{A}_1)$ and $\mathcal{F}_2 = [\mathcal{B}_2)$ and $\mathcal{F}_2 = [\mathcal{A}_2)$. Then $[\mathcal{B}_1 \times \mathcal{B}_2) = [\mathcal{A}_1 \times \mathcal{A}_2)$.

(24) If $\mathcal{B}_3 = \mathcal{B}_1$, then $[\mathcal{B}_1] = \mathcal{F}_1$.

(25) There exists $\mathcal{B}_1$ such that $[\mathcal{B}_1) = \mathcal{F}_1$. The theorem is a consequence of (24).

Let X_1, X_2 be non empty sets, $\mathcal{F}_1$ be a filter of X_1, and $\mathcal{F}_2$ be a filter of X_2. The functor $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ yielding a filter of $X_1 \times X_2$ is defined by

(Def. 2) there exists a filter base $\mathcal{B}_1$ of X_1 and there exists a filter base $\mathcal{B}_2$ of X_2 such that $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$ and $it = [\mathcal{B}_1 \times \mathcal{B}_2)$.

Let $\mathcal{B}_1$ be a generalized basis of $\mathcal{F}_1$ and $\mathcal{B}_2$ be a generalized basis of $\mathcal{F}_2$. The functor $\mathcal{B}_1 \times \mathcal{B}_2$ yielding a generalized basis of $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ is defined by

(Def. 3) there exists a filter base $\mathcal{B}_3$ of X_1 and there exists a filter base $\mathcal{B}_4$ of X_2 such that $\mathcal{B}_1 = \mathcal{B}_3$ and $\mathcal{B}_2 = \mathcal{B}_4$ and $it = \mathcal{B}_3 \times \mathcal{B}_4$.

Let n be a natural number. The functor $\uparrow^2(n)$ yielding a subset of $\mathbb{N} \times \mathbb{N}$ is defined by

(Def. 4) for every element x of $\mathbb{N} \times \mathbb{N}$, $x \in it$ iff there exist natural numbers n_1, n_2 such that $n_1 = (x)_{\mathbf{1}}$ and $n_2 = (x)_{\mathbf{2}}$ and $n \leqslant n_1$ and $n \leqslant n_2$.

Now we state the proposition:

(26) $\langle n, n \rangle \in \uparrow^2(n)$.

Let us consider n. One can check that $\uparrow^2(n)$ is non empty.

Now we state the propositions:

(27) If $\langle i, j \rangle \in \uparrow^2(n)$, then $\langle i+k, j \rangle$, $\langle i, j+l \rangle \in \uparrow^2(n)$.

(28) $\uparrow^2(n)$ is an infinite subset of $\mathbb{N} \times \mathbb{N}$. The theorem is a consequence of (17).

(29) If $n_1 = n$, then $\uparrow^2(n) = \uparrow n_1 \times \uparrow n_1$. The theorem is a consequence of (12) and (13).

(30) If $m = n - 1$, then $\uparrow^2(n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \operatorname{Seg} m \times \operatorname{Seg} m$.

PROOF: Reconsider $y = x$ as an element of $\mathbb{N} \times \mathbb{N}$. Consider n_1, n_2 being natural numbers such that $n_1 = (y)_{\mathbf{1}}$ and $n_2 = (y)_{\mathbf{2}}$ and $n \leqslant n_1$ and $n \leqslant n_2$. $x \notin \operatorname{Seg} m \times \operatorname{Seg} m$ by [3, (1)]. $\square$

(31) $\uparrow^2(n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_n \times \mathbb{Z}_n$.

PROOF: Reconsider $y = x$ as an element of $\mathbb{N} \times \mathbb{N}$. Consider n_1, n_2 being natural numbers such that $n_1 = (y)_{\mathbf{1}}$ and $n_2 = (y)_{\mathbf{2}}$ and $n \leqslant n_1$ and $n \leqslant n_2$. $x \notin \mathbb{Z}_n \times \mathbb{Z}_n$ by [16, (10)]. $\square$

(32) $\uparrow^2(n) = (\mathbb{N} \setminus \mathbb{Z}_n) \times (\mathbb{N} \setminus \mathbb{Z}_n)$. The theorem is a consequence of (14) and (29).

(33) There exists n such that $\uparrow^2(n) \subseteq (\mathbb{N} \setminus \mathbb{Z}_i) \times (\mathbb{N} \setminus \mathbb{Z}_j)$. The theorem is a consequence of (4).

(34) If $n = \max(i, j)$, then $\uparrow^2(n) \subseteq (\uparrow^2(i)) \cap (\uparrow^2(j))$.

Let n be a natural number. The functor $\downarrow^2(n)$ yielding a subset of $\mathbb{N} \times \mathbb{N}$ is defined by

(Def. 5) for every element x of $\mathbb{N} \times \mathbb{N}$, $x \in it$ iff there exist natural numbers n_1, n_2 such that $n_1 = (x)_{\mathbf{1}}$ and $n_2 = (x)_{\mathbf{2}}$ and $n_1 < n$ and $n_2 < n$.

Now we state the propositions:

(35) $\downarrow^2(n) = \mathbb{Z}_n \times \mathbb{Z}_n$.
PROOF: $\downarrow^2(n) \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ by [1, (44)]. Consider y_2, y_1 being objects such that $y_2 \in \mathbb{Z}_n$ and $y_1 \in \mathbb{Z}_n$ and $x = \langle y_2, y_1 \rangle$. □

(36) Let us consider a finite subset A of $\mathbb{N} \times \mathbb{N}$. Then there exists n such that $A \subseteq \downarrow^2(n)$.
PROOF: Consider m, n such that $A \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$. Reconsider $m_1 = \max(m, n)$ as a natural number. $A \subseteq \downarrow^2(m_1)$ by [1, (39)], [11, (96)], (35). □

(37) $\downarrow^2(n)$ is a finite subset of $\mathbb{N} \times \mathbb{N}$. The theorem is a consequence of (35).

3. Comparison between Cartesian Product of Frechet Filter on $\mathbb{N}$ and the Frechet Filter of $\mathbb{N} \times \mathbb{N}$

Let us consider an element x of (the base of Frechet filter) $\times$ (the base of Frechet filter). Now we state the propositions:

(38) There exists i and there exists j such that $x = (\mathbb{N} \setminus \mathbb{Z}_i) \times (\mathbb{N} \setminus \mathbb{Z}_j)$. The theorem is a consequence of (21).

(39) There exists n such that $\uparrow^2(n) \subseteq x$. The theorem is a consequence of (38) and (33).

(40) (The base of Frechet filter) $\times$ (the base of Frechet filter) is a filter base of $\mathbb{N} \times \mathbb{N}$.

(41) There exists a generalized basis $\mathcal{B}$ of FrechetFilter($\mathbb{N}$) such that

  (i) $\mathcal{B} =$ the base of Frechet filter, and

  (ii) $\mathcal{B} \times \mathcal{B}$ is a generalized basis of $\langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$.

The functor $\uparrow^2_{\mathbb{N}}$ yielding a filter base of $\mathbb{N} \times \mathbb{N}$ is defined by the term

(Def. 6) the set of all $\uparrow^2(n)$ where n is a natural number.

Now we state the propositions:

(42) $\uparrow^2_{\mathbb{N}}$ and (the base of Frechet filter) $\times$ (the base of Frechet filter) are equivalent generators. The theorem is a consequence of (22), (32), and (39).

(43) $[$(the base of Frechet filter) $\times$ (the base of Frechet filter)$) = \langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$. The theorem is a consequence of (41).

(44) $[\uparrow^2_{\mathbb{N}}) = \langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$.

(45) $\langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$ is finer than FrechetFilter($\mathbb{N} \times \mathbb{N}$). The theorem is a consequence of (17), (11), (22), and (43).

(46) (i) $\mathbb{N} \times \mathbb{N} \setminus$ the set of all $\langle 0, n \rangle$ where n is a natural number $\in$ $\langle$Frechet Filter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$, and

(ii) $\mathbb{N} \times \mathbb{N} \setminus$ the set of all $\langle 0, n \rangle$ where n is a natural number $\notin$ Frechet Filter($\mathbb{N} \times \mathbb{N}$).

Proof: Set $X = \mathbb{N} \times \mathbb{N} \setminus$ the set of all $\langle 0, n \rangle$ where n is a natural number. $\uparrow^2(1) \subseteq X$ by (32), [1, (44)]. $X \notin$ FrechetFilter($\mathbb{N} \times \mathbb{N}$) by [12, (51)], [15, (5)], (9). $\square$

(47) FrechetFilter($\mathbb{N} \times \mathbb{N}$) $\neq$ $\langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$.

4. Topological Space and Double Sequence

In the sequel T denotes a non empty topological space, s denotes a function from $\mathbb{N} \times \mathbb{N}$ into the carrier of T, M denotes a subset of the carrier of T, and $\mathcal{F}_1$, $\mathcal{F}_2$ denote filters of the carrier of T. Now we state the propositions:

(48) If $\mathcal{F}_2$ is finer than $\mathcal{F}_1$, then LimFilter($\mathcal{F}_1$) $\subseteq$ LimFilter($\mathcal{F}_2$).

(49) Let us consider a function f from X into Y, and filters $\mathcal{F}_1$, $\mathcal{F}_2$ of X. Suppose $\mathcal{F}_2$ is finer than $\mathcal{F}_1$. Then the image of filter $\mathcal{F}_2$ under f is finer than the image of filter $\mathcal{F}_1$ under f.

(50) $s^{-1}(M) \in$ FrechetFilter($\mathbb{N} \times \mathbb{N}$) if and only if there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(M) = \mathbb{N} \times \mathbb{N} \setminus A$.

(51) $s^{-1}(M) \in$ $\langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$ if and only if there exists n such that $\uparrow^2(n) \subseteq s^{-1}(M)$. The theorem is a consequence of (43), (39), and (42).

(52) The image of filter FrechetFilter($\mathbb{N} \times \mathbb{N}$) under $s = \{M$, where M is a subset of the carrier of T : there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(M) = \mathbb{N} \times \mathbb{N} \setminus A\}$. The theorem is a consequence of (50).

(53) The image of filter $\langle$FrechetFilter($\mathbb{N}$), FrechetFilter($\mathbb{N}$)$\rangle$ under $s = \{M$, where M is a subset of the carrier of T : there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(M)\}$. The theorem is a consequence of (51).

Let us consider a point x of T. Now we state the propositions:

(54) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N}\times\mathbb{N})} s$ if and only if for every neighbourhood A of x, there exists a finite subset B of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(A) = \mathbb{N} \times \mathbb{N} \setminus B$. The theorem is a consequence of (52).

(55) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N}\times\mathbb{N})} s$ if and only if for every neighbourhood A of x, $\mathbb{N} \times \mathbb{N} \setminus s^{-1}(A)$ is finite. The theorem is a consequence of (54), (1), and (2).

(56) $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every neighbourhood A of x, there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(A)$. The theorem is a consequence of (53).

Let us consider a point x of T and a generalized basis $\mathcal{B}$ of BooleanFilter ToFilter(the neighborhood system of x). Now we state the propositions:

(57) $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of $\mathcal{B}$, there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(B)$. The theorem is a consequence of (56).

(58) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of $\mathcal{B}$, there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(B) = \mathbb{N} \times \mathbb{N} \setminus A$. The theorem is a consequence of (54), (1), and (55).

(59) $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of $\mathcal{B}$, there exists a natural number n such that $s^{\circ}(\uparrow^2(n)) \subseteq B$. The theorem is a consequence of (57).

(60) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of $\mathcal{B}$, there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{\circ}(\mathbb{N} \times \mathbb{N} \setminus A) \subseteq B$.

PROOF: For every neighbourhood A of x, $\mathbb{N} \times \mathbb{N} \setminus s^{-1}(A)$ is finite by [4, (2)], [19, (143)], [9, (76)]. □

(61) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of $\mathcal{B}$, there exists n and there exists m such that $s^{\circ}(\mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_n \times \mathbb{Z}_m) \subseteq B$. The theorem is a consequence of (60) and (17).

(62) $x \in s^{\circ}(\uparrow^2(n))$ if and only if there exists i and there exists j such that $n \leqslant i$ and $n \leqslant j$ and $x = s(i, j)$.

(63) $x \in s^{\circ}(\mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_i \times \mathbb{Z}_j)$ if and only if there exist natural numbers n, m such that ($i \leqslant n$ or $j \leqslant m$) and $x = s(n, m)$.

PROOF: Consider n, m being natural numbers such that $i \leqslant n$ or $j \leqslant m$ and $x = s(n, m)$. $\langle n, m \rangle \notin \mathbb{Z}_i \times \mathbb{Z}_j$ by [1, (44)]. □

Let us consider a point x of T and a generalized basis $\mathcal{B}$ of BooleanFilter ToFilter(the neighborhood system of x). Now we state the propositions:

(64) $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of $\mathcal{B}$, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $s(n_1, n_2) \in B$. The theorem is a consequence of (62) and (59).

(65) $x \in \lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of $\mathcal{B}$, there exists i and there exists j such that for every m and n such that $i \leqslant m$ or $j \leqslant n$ holds $s(m, n) \in B$. The theorem is a consequence of (61).

(66) $\lim_{\mathrm{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s \subseteq \lim_{[\uparrow^2_{\mathbb{N}})} s$. The theorem is a consequence of (42), (43), (45), (48), and (49).

5. Metric Space and Double Sequence

Now we state the propositions:

(67) Let us consider a non empty metric space M, a point p of M, a point x of M_{top}, and a function s from $\mathbb{N} \times \mathbb{N}$ into M_{top}. Suppose $x = p$. Then $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $s(n_1, n_2) \in \{q$, where q is a point of $M : \rho(p, q) < \frac{1}{m}\}$.

Proof: $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ iff for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $s(n_1, n_2) \in \{q$, where q is a point of $M : \rho(p, q) < \frac{1}{m}\}$ by [13, (6)], (64). □

(68) Let us consider a non empty metric space M, a point p of M, a point x of M_{top}, a function s from $\mathbb{N} \times \mathbb{N}$ into M_{top}, and a function s_2 from $\mathbb{N} \times \mathbb{N}$ into M. Suppose $x = p$ and $s = s_2$. Then $x \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $s_2(n_1, n_2) \in \{q$, where q is a point of $M : \rho(p, q) < \frac{1}{m}\}$.

6. One-dimensional Euclidean Metric Space and Double Sequence

In the sequel R denotes a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$.

Now we state the proposition:

(69) Let us consider a point x of $(\mathcal{E}^1)_{\mathrm{top}}$, a point y of $\mathcal{E}^1$, a generalized basis $\mathcal{B}$ of BooleanFilterToFilter(the neighborhood system of x), and an element b of $\mathcal{B}$. Suppose $x = y$ and $\mathcal{B} = \mathrm{Balls}\, x$. Then there exists a natural number n such that $b = \{q$, where q is an element of $\mathcal{E}^1 : \rho(y, q) < \frac{1}{n}\}$.

Let s be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. The functor $\# s$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by the term

(Def. 7) s.

Now we state the propositions:

(70) Let us consider a function s from $\mathbb{N} \times \mathbb{N}$ into $(\mathcal{E}^1)_{\mathrm{top}}$, and a point y of $\mathcal{E}^1$. Then $s^{\circ}(\uparrow^2(n)) \subseteq \{q$, where q is an element of $\mathcal{E}^1 : \rho(y, q) < \frac{1}{m}\}$ if and only if for every object x such that $x \in s^{\circ}(\uparrow^2(n))$ there exist real numbers r_1, r_2 such that $x = \langle r_1 \rangle$ and $y = \langle r_2 \rangle$ and $|r_2 - r_1| < \frac{1}{m}$. The theorem is a consequence of (8).

(71) $r \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R$ if and only if for every non zero natural number m, there exists a natural number n such that for every

natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $|R(n_1, n_2) - r| < \frac{1}{m}$.

Proof: Reconsider $p = r$ as a point of the metric space of real numbers. for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $R(n_1, n_2) \in \{q$, where q is a point of the metric space of real numbers : $\rho(p, q) < \frac{1}{m}\}$ iff for every non zero natural number m, there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $|R(n_1, n_2) - r| < \frac{1}{m}$ by (6), [8, (60)]. $\square$

7. Basic Relations Convergence in Pringsheim's Sense and Filter Convergence

Now we state the propositions:

(72) Suppose $\lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R \neq \emptyset$. Then there exists a real number x such that $\lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R = \{x\}$.

(73) If R is P-convergent, then P-lim $R \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R$. The theorem is a consequence of (71).

(74) R is P-convergent if and only if $\lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R \neq \emptyset$. The theorem is a consequence of (71) and (5).

(75) Suppose R is P-convergent. Then $\{\text{P-lim}\, R\} = \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R$. The theorem is a consequence of (73) and (72).

(76) Suppose $\lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R$ is not empty. Then

(i) R is P-convergent, and

(ii) $\{\text{P-lim}\, R\} = \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# R$.

8. Example: Double Sequence Converges in Pringsheim's Sense but not in Frechet Filter of $\mathbb{N} \times \mathbb{N}$ Sense

The functor DblSeq-ex1 yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 8) for every natural numbers m, n, $it(m, n) = \frac{1}{m+1}$.

Now we state the propositions:

(77) Let us consider a non zero natural number m. Then there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leqslant n_1$ and $n \leqslant n_2$ holds $|(\text{DblSeq-ex1})(n_1, n_2) - 0| < \frac{1}{m}$.

(78) $0 \in \lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}), \mathrm{FrechetFilter}(\mathbb{N}) \rangle} \# \text{DblSeq-ex1}$.

(79) $\lim_{\mathrm{FrechetFilter}(\mathbb{N}\times\mathbb{N})} \#\,\mathrm{DblSeq\text{-}ex1} = \emptyset$. The theorem is a consequence of (66), (42), (43), (72), (78), and (65).

(80) $\lim_{\langle \mathrm{FrechetFilter}(\mathbb{N}),\mathrm{FrechetFilter}(\mathbb{N})\rangle} \#\,\mathrm{DblSeq\text{-}ex1} \neq \lim_{\mathrm{FrechetFilter}(\mathbb{N}\times\mathbb{N})} \#\,\mathrm{DblSeq\text{-}ex1}$.

9. Correspondence with some Definitions from [14]

Let X_1, X_2 be non empty sets, $\mathcal{F}_1$ be a filter of X_1, Y be a Hausdorff, non empty topological space, and f be a function from $X_1 \times X_2$ into Y. Assume for every element x of X_2, $\lim_{\mathcal{F}_1} \mathrm{curry}'(f, x) \neq \emptyset$. The functor $\lim_1(f, \mathcal{F}_1)$ yielding a function from X_2 into Y is defined by

(Def. 9) for every element x of X_2, $\{it(x)\} = \lim_{\mathcal{F}_1} \mathrm{curry}'(f, x)$.

Let $\mathcal{F}_2$ be a filter of X_2. Assume for every element x of X_1, $\lim_{\mathcal{F}_2} \mathrm{curry}(f, x) \neq \emptyset$. The functor $\lim_2(f, \mathcal{F}_2)$ yielding a function from X_1 into Y is defined by

(Def. 10) for every element x of X_1, $\{it(x)\} = \lim_{\mathcal{F}_2} \mathrm{curry}(f, x)$.

Now we state the propositions:

(81) Every function from X into $\mathbb{R}$ is a function from X into $\mathbb{R}^{\mathbf{1}}$.

(82) Every sequence of $\mathbb{R}$ is a function from $\mathbb{N}$ into $\mathbb{R}^{\mathbf{1}}$.

From now on f denotes a function from $\Omega_{\text{the ordered } \mathbb{N}}$ into $\mathbb{R}^{\mathbf{1}}$ and s_1 denotes a function from $\mathbb{N}$ into $\mathbb{R}$.

Now we state the propositions:

(83) Suppose $f = s_1$ and $\mathrm{LimF}(f) \neq \emptyset$. Then

(i) s_1 is convergent, and

(ii) there exists a real number z such that $z \in \mathrm{LimF}(f)$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s_1(m) - z| < p$.

Proof: Consider x being an object such that $x \in \mathrm{LimF}(f)$. Reconsider $y = x$ as a point of (the metric space of real numbers)$_{\mathrm{top}}$. Reconsider $z = y$ as a real number. Consider y_1 being a point of the metric space of real numbers such that $y_1 = y$ and $\mathrm{Balls}\, y = \{\mathrm{Ball}(y_1, \frac{1}{n})$, where n is a natural number : $n \neq 0\}$. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s_1(m) - z| < p$ by (5), [12, (84), (50)], [2, (18)]. □

(84) If $f = s_1$ and $\mathrm{LimF}(f) \neq \emptyset$, then $\mathrm{LimF}(f) = \{\lim s_1\}$.

Proof: Consider x being an object such that $x \in \mathrm{LimF}(f)$. Consider u being an object such that $\mathrm{LimF}(f) = \{u\}$. $\mathrm{LimF}(f) = \{\lim s_1\}$ by (83), [11, (3)]. □

(85) Let us consider a function f from Ω_α into T, and a sequence s of T. If $f = s$, then $\mathrm{LimF}(f) = \mathrm{LimF}(s)$, where α is the ordered $\mathbb{N}$.

(86) Let us consider a function f from Ω_α into T, and a function g from $\mathbb{N}$ into T. If $f = g$, then $\mathrm{LimF}(f) = \mathrm{LimF}(g)$, where α is the ordered $\mathbb{N}$.

(87) Let us consider a function f from $\mathbb{N}$ into $\mathbb{R}^{\mathbf{1}}$. Suppose $f = s_1$ and $\mathrm{LimF}(f) \neq \emptyset$. Then $\mathrm{LimF}(f) = \{\lim s_1\}$. The theorem is a consequence of (84).

(88) for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}'(\# R, x) \neq \emptyset$ if and only if R is convergent in the first coordinate. The theorem is a consequence of (5).

(89) for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}(\# R, x) \neq \emptyset$ if and only if R is convergent in the second coordinate. The theorem is a consequence of (5).

Let us consider an element t of $\mathbb{N}$, a function f from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}^{\mathbf{1}}$, and a function s_1 from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Now we state the propositions:

(90) Suppose $f = s_1$ and for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}(f, x) \neq \emptyset$. Then $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}(f, t) = \{\lim \mathrm{curry}(s_1, t)\}$. The theorem is a consequence of (87).

(91) Suppose $f = s_1$ and for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}'(f, x) \neq \emptyset$. Then $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}'(f, t) = \{\lim \mathrm{curry}'(s_1, t)\}$. The theorem is a consequence of (87).

(92) Let us consider a Hausdorff, non empty topological space Y, and a function f from $\mathbb{N}\times\mathbb{N}$ into Y. Suppose for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}'(f, x) \neq \emptyset$ and $f = R$ and $Y = \mathbb{R}^{\mathbf{1}}$. Then $\mathrm{lim}_1(f, \mathrm{FrechetFilter}(\mathbb{N})) =$ the lim in the first coordinate of R. The theorem is a consequence of (91).

(93) Let us consider a non empty, Hausdorff topological space Y, and a function f from $\mathbb{N}\times\mathbb{N}$ into Y. Suppose for every element x of $\mathbb{N}$, $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} \mathrm{curry}(f, x) \neq \emptyset$ and $f = R$ and $Y = \mathbb{R}^{\mathbf{1}}$. Then $\mathrm{lim}_2(f, \mathrm{FrechetFilter}(\mathbb{N})) =$ the lim in the second coordinate of R. The theorem is a consequence of (90).

10. Regular Space, Double Limit and Iterated Limit

From now on Y denotes a non empty topological space, x denotes a point of Y, and f denotes a function from $X_1 \times X_2$ into Y.

Now we state the proposition:

(94) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1\rangle = \mathcal{F}_1$ and $[\mathcal{B}_2\rangle = \mathcal{F}_2$. Let us consider a subset V of Y. Suppose V is open and $x \in V$. Then there exists an ele-

ment B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that $f^\circ(B_1 \times B_2) \subseteq V$.

Let us consider a neighbourhood U of x. Now we state the propositions:

(95) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that $f^\circ(B_1 \times B_2) \subseteq \operatorname{Int} U$.

(96) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that for every element y of B_1, $f^\circ(\{y\} \times B_2) \subseteq \operatorname{Int} U$. The theorem is a consequence of (95).

(97) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that for every element z of X_1 for every element y of Y such that $z \in B_1$ and $y \in \lim_{\mathcal{F}_2} \operatorname{curry}(f, z)$ holds $y \in \overline{\operatorname{Int} U}$.

Proof: Consider B_1 being an element of $\mathcal{B}_1$, B_2 being an element of $\mathcal{B}_2$ such that $f^\circ(B_1 \times B_2) \subseteq \operatorname{Int} U$. For every element y of B_1, $f^\circ(\{y\} \times B_2) \subseteq \operatorname{Int} U$ by [11, (95)], [19, (125)]. For every element z of B_1 and for every element y of Y such that $y \in \lim_{\mathcal{F}_2} \operatorname{curry}(f, z)$ holds the image of filter $\mathcal{F}_2$ under $\operatorname{curry}(f, z)$ is a proper filter of $2^{\Omega_Y}_{\subseteq}$ and $\operatorname{Int} U \in$ the image of filter $\mathcal{F}_2$ under $\operatorname{curry}(f, z)$ and y is a cluster point of the image of filter $\mathcal{F}_2$ under $\operatorname{curry}(f, z)$,$Y$ by (18), [19, (132)], [10, (95)], (20). For every element z of B_1 and for every element y of Y such that $y \in \lim_{\mathcal{F}_2} \operatorname{curry}(f, z)$ holds $y \in \overline{\operatorname{Int} U}$ by [4, (25)]. □

(98) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1) = \mathcal{F}_1$ and $[\mathcal{B}_2) = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of $\mathcal{B}_1$ and there exists an element B_2 of $\mathcal{B}_2$ such that for every element z of X_2 for every element y of Y such that $z \in B_2$ and $y \in \lim_{\mathcal{F}_1} \operatorname{curry}'(f, z)$ holds $y \in \overline{\operatorname{Int} U}$.

Proof: Consider B_1 being an element of $\mathcal{B}_1$, B_2 being an element of $\mathcal{B}_2$ such that $f^\circ(B_1 \times B_2) \subseteq \operatorname{Int} U$. For every element y of B_2, $f^\circ(B_1 \times \{y\}) \subseteq \operatorname{Int} U$ by [11, (95)], [19, (125)]. For every element z of B_2 and for every element y of Y such that $y \in \lim_{\mathcal{F}_1} \operatorname{curry}'(f, z)$ holds the image of filter $\mathcal{F}_1$ under $\operatorname{curry}'(f, z)$ is a proper filter of $2^{\Omega_Y}_{\subseteq}$ and $\operatorname{Int} U \in$ the image of filter $\mathcal{F}_1$ under $\operatorname{curry}'(f, z)$ and y is a cluster point of the image of filter $\mathcal{F}_1$ under $\operatorname{curry}'(f, z)$,$Y$ by (18), [19, (132)], [10, (95)], (20). For every element z of B_2 and for every element y of Y such that $y \in \lim_{\mathcal{F}_1} \operatorname{curry}'(f, z)$ holds $y \in \overline{\operatorname{Int} U}$ by [4, (25)]. □

Let us consider a Hausdorff, regular, non empty topological space Y and a function f from $X_1 \times X_2$ into Y. Now we state the propositions:

(99) Suppose for every element x of X_2, $\lim_{\mathcal{F}_1} \operatorname{curry}'(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle}$

$f \subseteq \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (19) and (98).

(100) Suppose for every element x of X_1, $\lim_{\mathcal{F}_2} \mathrm{curry}(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \subseteq \lim_{\mathcal{F}_1} \lim_2(f, \mathcal{F}_2)$. The theorem is a consequence of (19) and (97).

Let us consider non empty sets X_1, X_2, a filter $\mathcal{F}_1$ of X_1, a filter $\mathcal{F}_2$ of X_2, a Hausdorff, regular, non empty topological space Y, and a function f from $X_1 \times X_2$ into Y. Now we state the propositions:

(101) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_1, $\lim_{\mathcal{F}_2} \mathrm{curry}(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f = \lim_{\mathcal{F}_1} \lim_2(f, \mathcal{F}_2)$. The theorem is a consequence of (100).

(102) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_2, $\lim_{\mathcal{F}_1} \mathrm{curry}'(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f = \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (99).

(103) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_1, $\lim_{\mathcal{F}_2} \mathrm{curry}(f, x) \neq \emptyset$ and for every element x of X_2, $\lim_{\mathcal{F}_1} \mathrm{curry}'(f, x) \neq \emptyset$. Then $\lim_{\mathcal{F}_1} \lim_2 (f, \mathcal{F}_2) = \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (102) and (101).

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