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Compactness in Metric Spaces

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Summary. In this article, we mainly formalize in Mizar [2] the equivalence among a few compactness definitions of metric spaces, norm spaces, and the real line. In the first section, we formalized general topological properties of metric spaces. We discussed openness and closedness of subsets in metric spaces in terms of convergence of element sequences. In the second section, we firstly formalize the definition of sequentially compact, and then discuss the equivalence of compactness, countable compactness, sequential compactness, and totally boundedness with completeness in metric spaces.

In the third section, we discuss compactness in norm spaces. We formalize the equivalence of compactness and sequential compactness in norm space. In the fourth section, we formalize topological properties of the real line in terms of convergence of real number sequences. In the last section, we formalize the equivalence of compactness and sequential compactness in the real line. These formalizations are based on [20], [5], [17], [14], and [4].

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1. Topological Properties of Metric Spaces

Now we state the propositions:

(1) Let us consider a non empty set M, and a sequence x of M. Suppose $\operatorname{rng} x$ is finite. Then there exists an element z of M such that

(i) $x^{-1}(\{z\}) \subseteq \mathbb{N}$, and

(ii) $x^{-1}(\{z\})$ is infinite.

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PROOF: Define $\mathcal{X}(\text{object}) = x^{-1}(\{\$_1\})$. Set $K = \{\mathcal{X}(w)$, where w is an element of $M : w \in \operatorname{rng} x\}$. K is finite from [18, Sch. 21]. For every set Y such that $Y \in K$ holds Y is finite. $\operatorname{dom} x \subseteq \bigcup K$ by [6, (3)]. $\square$

(2) Let us consider a subset X of $\mathbb{N}$. Suppose X is infinite. Then there exists an increasing sequence N of $\mathbb{N}$ such that $\operatorname{rng} N \subseteq X$.

PROOF: Reconsider $B = 2^X$ as a non empty set. Reconsider $N_0 = \min^* X$ as an element of $\mathbb{N}$. Reconsider $Y_0 = X$ as an element of B. Define $\mathcal{P}[\text{object}, \text{object}, \text{set}, \text{object}, \text{set}] \equiv \$_5 = \$_3 \setminus \{\$_2\}$ and $\$_4 = \min^* \$_5$. For every natural number n and for every element x of $\mathbb{N}$ and for every element y of B, there exists an element x_1 of $\mathbb{N}$ and there exists an element y_1 of B such that $\mathcal{P}[n, x, y, x_1, y_1]$. Consider N being a sequence of $\mathbb{N}$, Y being a sequence of B such that $N(0) = N_0$ and $Y(0) = Y_0$ and for every natural number n, $\mathcal{P}[n, N(n), Y(n), N(n+1), Y(n+1)]$ from [13, Sch. 3]. Define $\mathcal{Q}[\text{natural number}] \equiv N(\$_1) = \min^*(Y(\$_1))$ and $N(\$_1) \in Y(\$_1)$ and $Y(\$_1)$ is infinite and $Y(\$_1) \subseteq \mathbb{N}$. For every natural number i such that $\mathcal{Q}[i]$ holds $\mathcal{Q}[i+1]$ by [8, (31)]. For every natural number i, $\mathcal{Q}[i]$ from [1, Sch. 2]. $\operatorname{rng} N \subseteq X$ by [7, (11)]. For every natural number i, $N(i) < N(i+1)$. $\square$

(3) Let us consider a non empty metric space M, and a subset V of M_{top}. Suppose V is open. Then there exists a family F of subsets of M such that

(i) $F = \{\operatorname{Ball}(x, r)$, where x is an element of M, r is a real number : $r > 0$ and $\operatorname{Ball}(x, r) \subseteq V\}$, and

(ii) $V = \bigcup F$.

PROOF: Set $F = \{\operatorname{Ball}(x, r)$, where x is an element of M, r is a real number: $r > 0$ and $\operatorname{Ball}(x, r) \subseteq V\}$. For every object z such that $z \in F$ holds $z \in$ the open set family of M by [3, (29)]. Reconsider $Q = \bigcup F$ as a subset of M_{top}. For every object z, $z \in V$ iff $z \in Q$ by [9, (15)], [12, (1), (11)]. $\square$

(4) Let us consider a non empty metric space M, a subset X of M_{top}, and an element p of M. Then $p \in \overline{X}$ if and only if for every real number r such that $0 < r$ holds X meets $\operatorname{Ball}(p, r)$.

(5) Let us consider a non empty metric space M, a subset X of M_{top}, and an object x. Then $x \in \overline{X}$ if and only if there exists a sequence S of M such that for every natural number n, $S(n) \in X$ and S is convergent and $\lim S = x$.

(6) Let us consider a non empty metric space M, and a subset X of M_{top}. Then X is closed if and only if for every sequence S of M such that for every natural number n, $S(n) \in X$ and S is convergent holds $\lim S \in X$. The theorem is a consequence of (5).

(7) Let us consider non empty metric spaces X, Y, and a function f from $X_{\rm top}$ into $Y_{\rm top}$. Then f is continuous if and only if for every sequence S of X and for every sequence T of Y such that S is convergent and $T = f \cdot S$ holds T is convergent and $\lim T = f(\lim S)$.

PROOF: For every subset B of $Y_{\rm top}$ such that B is closed holds $f^{-1}(B)$ is closed by [7, (15)], (6). □

2. COMPACTNESS IN METRIC SPACES

Let M be a non empty metric space and X be a subset of M. We say that X is sequentially compact if and only if

(Def. 1) for every sequence S_1 of M such that $\operatorname{rng} S_1 \subseteq X$ there exists a sequence S_2 of M such that there exists an increasing sequence N of $\mathbb{N}$ such that $S_2 = S_1 \cdot N$ and S_2 is convergent and $\lim S_2 \in X$.

Let us observe that every subset of M which is empty is also sequentially compact.

We say that M is sequentially compact if and only if

(Def. 2) Ω_M is sequentially compact.

Now we state the proposition:

(8) Let us consider a non empty metric space M, a subset X of M, a subset Y of $M_{\rm top}$, an element x of M, and an element y of $M_{\rm top}$. Suppose $X = Y$ and $x = y$. Then y is an accumulation point of Y if and only if for every real number r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ meets $X \setminus \{x\}$.

Let us consider a non empty metric space M. Now we state the propositions:

(9) If $M_{\rm top}$ is countably-compact, then M is sequentially compact.

PROOF: For every subset X of M such that X is infinite there exists an element x of M such that for every real number r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ meets $X \setminus \{x\}$ by [16, (28)], [11, (16)], (8). For every sequence x of M such that $\operatorname{rng} x \subseteq \Omega_M$ there exists a sequence x_1 of M such that there exists an increasing sequence N of $\mathbb{N}$ such that $x_1 = x \cdot N$ and x_1 is convergent and $\lim x_1 \in \Omega_M$ by (1), (2), [7, (4), (38), (15)]. □

(10) If M is sequentially compact, then $M_{\rm top}$ is countably-compact.

PROOF: For every subset X of M such that X is infinite there exists an element x of M such that for every real number r such that $0 < r$ holds $\mathrm{Ball}(x, r)$ meets $X \setminus \{x\}$ by [15, (3)], [7, (2)], [19, (26)], [7, (112)]. For every subset A of $M_{\rm top}$ such that A is infinite holds $\operatorname{Der} A$ is not empty by (8), [11, (16)]. □

(11) $M_{\rm top}$ is compact if and only if M is sequentially compact. The theorem is a consequence of (9).

(12) M is totally bounded and complete if and only if M is sequentially compact. The theorem is a consequence of (11).

Let us consider a non empty metric space M and a non empty subset S of M. Now we state the propositions:

(13) S is sequentially compact if and only if $M \restriction S$ is sequentially compact.
PROOF: For every sequence S_1 of M such that $\operatorname{rng} S_1 \subseteq S$ there exists a sequence S_2 of M such that there exists an increasing sequence N of $\mathbb{N}$ such that $S_2 = S_1 \cdot N$ and S_2 is convergent and $\lim S_2 \in S$ by [7, (6)]. □

(14) S is sequentially compact if and only if $M \restriction S$ is compact. The theorem is a consequence of (11) and (13).

(15) Let us consider a non empty metric space M, a subset S of M, and a subset T of $M_{\rm top}$. If $T = S$, then T is compact iff S is sequentially compact. The theorem is a consequence of (11) and (13).

(16) Let us consider a non empty metric space M, a non empty subset S of M, and a non empty subset T of $M_{\rm top}$. Suppose $T = S$. Then $M_{\rm top} \restriction T$ is countably-compact if and only if S is sequentially compact. The theorem is a consequence of (11) and (13).

(17) Let us consider a non empty metric space M, and a non empty subset S of M. Then $M \restriction S$ is totally bounded and complete if and only if S is sequentially compact. The theorem is a consequence of (12) and (13).

3. Compactness in Norm Spaces

Now we state the propositions:

(18) Let us consider a real normed space N, a subset S of N, and a subset T of MetricSpaceNorm N. If $S = T$, then S is compact iff T is sequentially compact.

(19) Let us consider a real normed space N, a subset S of N, and a subset T of TopSpaceNorm N. If $S = T$, then S is compact iff T is compact. The theorem is a consequence of (15) and (18).

4. Topological Properties of the Real Line

Let us consider a sequence S_1 of the metric space of real numbers, a sequence s of real numbers, a real number g, and an element g_1 of the metric space of real numbers. Now we state the propositions:

(20) Suppose $S_1 = s$ and $g = g_1$. Then for every real number p such that $0 < p$ there exists a natural number n such that for every natural number

m such that $n \leqslant m$ holds $|s(m) - g| < p$ if and only if for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\rho(S_1(m), g_1) < p$.

Proof: For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s(m) - g| < p$ by [9, (11)]. $\square$

(21) If $S_1 = s$ and $g = g_1$, then s is convergent and $\lim s = g$ iff S_1 is convergent and $\lim S_1 = g_1$. The theorem is a consequence of (20).

(22) Let us consider a sequence S_1 of the metric space of real numbers, and a sequence s of real numbers. Suppose $S_1 = s$ and s is convergent. Then

(i) S_1 is convergent, and

(ii) $\lim S_1 = \lim s$.

The theorem is a consequence of (20).

5. Compactness in the Real Line

Now we state the propositions:

(23) Let us consider a subset N of $\mathbb{R}$, and a subset M of $\mathbb{R}^{\mathbf{1}}$. Suppose $N = M$. Then for every family F of subsets of $\mathbb{R}$ such that F is a cover of N and for every subset P of $\mathbb{R}$ such that $P \in F$ holds P is open there exists a family G of subsets of $\mathbb{R}$ such that $G \subseteq F$ and G is cover of N and finite if and only if for every family F_1 of subsets of $\mathbb{R}^{\mathbf{1}}$ such that F_1 is cover of M and open there exists a family G_1 of subsets of $\mathbb{R}^{\mathbf{1}}$ such that $G_1 \subseteq F_1$ and G_1 is cover of M and finite.

Proof: Reconsider $F_1 = F$ as a family of subsets of $\mathbb{R}^{\mathbf{1}}$. For every subset P_1 of $\mathbb{R}^{\mathbf{1}}$ such that $P_1 \in F_1$ holds P_1 is open by [10, (39)]. Consider G_1 being a family of subsets of $\mathbb{R}^{\mathbf{1}}$ such that $G_1 \subseteq F_1$ and G_1 is cover of M and finite. $\square$

(24) Let us consider a subset N of $\mathbb{R}$. Then N is compact if and only if for every family F of subsets of $\mathbb{R}$ such that F is a cover of N and for every subset P of $\mathbb{R}$ such that $P \in F$ holds P is open there exists a family G of subsets of $\mathbb{R}$ such that $G \subseteq F$ and G is cover of N and finite. The theorem is a consequence of (23).

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