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Conservation Rules of Direct Sum Decomposition of Groups

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Summary. In this article, conservation rules of the direct sum decomposition of groups are mainly discussed. In the first section, we prepare miscellaneous definitions and theorems for further formalization in Mizar [5]. In the next three sections, we formalized the fact that the property of direct sum decomposition is preserved against the substitutions of the subscript set, flattening of direct sum, and layering of direct sum, respectively. We referred to [14], [13] [6] and [11] in the formalization.

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1. Preliminaries

Let I, J be non empty sets, a be a function from I into J, and F be a multiplicative magma family of J. Observe that the functor $F \cdot a$ yields a multiplicative magma family of I. Let F be a group family of J. Let us observe that the functor $F \cdot a$ yields a group family of I. Let G be a group and F be a subgroup family of J and G. The functor $F \cdot a$ yielding a subgroup family of I and G is defined by the term

(Def. 1) $F \cdot a$.

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The scheme *Sch1* deals with a set $\mathcal{A}$ and a 1-sorted structure $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding a set and states that

(Sch. 1) There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every element x of $\mathcal{B}$ such that $x \in \mathcal{A}$ holds $f(x) = \mathcal{F}(x)$.

Let I be a set. Let us note that there exists a many sorted set indexed by I which is non-empty and disjoint valued.

Now we state the propositions:

(1) Let us consider a non-empty, disjoint valued function f. If $\bigcup f$ is finite, then $\operatorname{dom} f$ is finite.
PROOF: For every objects x, y such that x, $y \in \operatorname{dom} f$ and $f(x) = f(y)$ holds $x = y$ by [7, (3)]. $\square$

(2) Let us consider non empty sets X, Y, sets X_0, Y_0, and a function f from X into Y. Suppose f is bijective and $\operatorname{rng}(f{\restriction}X_0) = Y_0$. Then $(f{\restriction}X_0)^{-1} = f^{-1}{\restriction}Y_0$.
PROOF: For every object x such that $x \in \operatorname{dom}(f^{-1}{\restriction}Y_0)$ holds $(f^{-1}{\restriction}Y_0)(x) = (f{\restriction}X_0)^{-1}(x)$ by [18, (62)], [7, (49), (33)], [18, (59)]. $\square$

2. Conservation Rule of Direct Sum Decomposition for Substitution of Subscript Set

Now we state the proposition:

(3) Let us consider non empty sets I, J, a function a from I into J, a multiplicative magma family F of J, and an element x of $\prod F$. Then $x \cdot a \in \prod (F \cdot a)$.
PROOF: Reconsider $y = x \cdot a$ as a many sorted set indexed by I. Reconsider $z =$ the support of $F \cdot a$ as a many sorted set indexed by I. For every object i such that $i \in I$ holds $y(i) \in z(i)$ by [7, (13)]. $\square$

Let I, J be non empty sets, a be a function from I into J, and F be a multiplicative magma family of J. The functor $\operatorname{Trans}\prod(F, a)$ yielding a function from $\prod F$ into $\prod (F \cdot a)$ is defined by

(Def. 2) for every element x of $\prod F$, $it(x) = x \cdot a$.

Now we state the proposition:

(4) Let us consider non empty sets I, J, a function a from I into J, and a multiplicative magma family F of J. Then $\operatorname{Trans}\prod(F, a)$ is multiplicative.
PROOF: Reconsider $f = \operatorname{Trans}\prod(F, a)$ as a function from $\prod F$ into $\prod (F \cdot a)$. For every elements x, y of $\prod F$, $f(x \cdot y) = f(x) \cdot f(y)$ by (3), [7, (13)], [10, (1)], [18, (27)]. $\square$

Let I, J be non empty sets, a be a function from I into J, and F be a group family of J. Let us observe that the functor $\mathrm{Trans}\prod(F,a)$ yields a homomorphism from $\prod F$ to $\prod(F \cdot a)$. Now we state the propositions:

(5) Let us consider non empty sets I, J, a function a from I into J, a multiplicative magma family F of J, and an element y of $\prod(F \cdot a)$. If a is bijective, then $y \cdot a^{-1} \in \prod F$.
PROOF: Set $x = y \cdot a^{-1}$. For every object j such that $j \in J$ holds $x(j) \in$ (the support of F)(j) by [7, (32), (13)]. □

(6) Let us consider non empty sets I, J, a function a from I into J, and functions x, y. Suppose $\mathrm{dom}\, x = I$ and $\mathrm{dom}\, y = J$ and a is bijective. Then $x = y \cdot a$ if and only if $y = x \cdot a^{-1}$.

(7) Let us consider non empty sets I, J, a multiplicative magma family F of J, and a function a from I into J. Suppose a is bijective. Then

(i) $\mathrm{dom}\,\mathrm{Trans}\prod(F,a) = \Omega_{\prod F}$, and

(ii) $\mathrm{rng}\,\mathrm{Trans}\prod(F,a) = \Omega_{\prod(F \cdot a)}$.

The theorem is a consequence of (5) and (6).

(8) Let us consider non empty sets I, J, a function a from I into J, and a multiplicative magma family F of J. If a is bijective, then $\mathrm{Trans}\prod(F,a)$ is bijective.
PROOF: Reconsider $f = \mathrm{Trans}\prod(F,a)$ as a function from $\prod F$ into $\prod(F \cdot a)$. $\mathrm{dom}\, f = \Omega_{\prod F}$ and $\mathrm{rng}\, f = \Omega_{\prod(F \cdot a)}$. For every objects x, y such that x, $y \in \mathrm{dom}\, f$ and $f(x) = f(y)$ holds $x = y$ by [7, (86)]. □

Let us consider non empty sets I, J, a function a from I into J, a group family F of J, and a function x. Now we state the propositions:

(9) If a is one-to-one, then $a^{\circ}(\mathrm{support}(x \cdot a, F \cdot a)) \subseteq \mathrm{support}(x, F)$.
PROOF: For every object j such that $j \in a^{\circ}(\mathrm{support}(x \cdot a, F \cdot a))$ holds $j \in \mathrm{support}(x, F)$ by [7, (13)]. □

(10) If a is onto, then $\mathrm{support}(x, F) \subseteq a^{\circ}(\mathrm{support}(x \cdot a, F \cdot a))$.
PROOF: For every object j such that $j \in \mathrm{support}(x, F)$ holds $j \in a^{\circ}(\mathrm{support}(x \cdot a, F \cdot a))$ by [8, (11)], [7, (13)]. □

(11) If a is one-to-one, then if $x \in \mathrm{sum}\, F$, then $x \cdot a \in \mathrm{sum}(F \cdot a)$. The theorem is a consequence of (3) and (9).

(12) If a is bijective, then $x \in \mathrm{sum}\, F$ iff $x \cdot a \in \mathrm{sum}(F \cdot a)$ and $\mathrm{dom}\, x = J$. The theorem is a consequence of (11).

Let I, J be non empty sets, a be a function from I into J, and F be a group family of J. Assume a is bijective. The functor $\mathrm{Trans}\sum(F,a)$ yielding a function from $\mathrm{sum}\, F$ into $\mathrm{sum}(F \cdot a)$ is defined by the term

(Def. 3) $\mathrm{Trans}\prod(F, a) \restriction \mathrm{sum}\, F$.

Now we state the proposition:

(13) Let us consider groups G, H, a subgroup H_0 of H, and a homomorphism f from G to H. Suppose $\mathrm{rng}\, f \subseteq \Omega_{H_0}$. Then f is a homomorphism from G to H_0.

PROOF: Reconsider $g = f$ as a function from G into H_0. For every elements a, b of G, $g(a \cdot b) = g(a) \cdot g(b)$ by [16, (43)]. $\square$

Let I, J be non empty sets, a be a function from I into J, and F be a group family of J. Assume a is bijective. Let us observe that the functor $\mathrm{Trans}\sum(F, a)$ yields a homomorphism from $\mathrm{sum}\, F$ to $\mathrm{sum}(F \cdot a)$. Now we state the propositions:

(14) Let us consider non empty sets I, J, a function a from I into J, and a group family F of J. If a is bijective, then $\mathrm{Trans}\sum(F, a)$ is bijective.

PROOF: Reconsider $f = \mathrm{Trans}\prod(F, a)$ as a homomorphism from $\prod F$ to $\prod(F \cdot a)$. Reconsider $g = \mathrm{Trans}\sum(F, a)$ as a homomorphism from $\mathrm{sum}\, F$ to $\mathrm{sum}(F \cdot a)$. f is bijective. For every object y such that $y \in \Omega_{\mathrm{sum}(F \cdot a)}$ holds $y \in \mathrm{rng}\, g$ by [16, (42)], (5), (6), (12). $\square$

(15) Let us consider a group G, non empty sets I, J, a direct sum components F of G and J, and a function a from I into J. If a is bijective, then $F \cdot a$ is a direct sum components of G and I. The theorem is a consequence of (14).

(16) Let us consider a non empty set I, and a group G. Then every internal direct sum components of G and I is a subgroup family of I and G.

(17) Let us consider non empty sets I, J, a group G, a function x from I into G, a function y from J into G, and a function a from I into J. Suppose a is onto and $x = y \cdot a$. Then $\mathrm{support}\, y = a^{\circ}(\mathrm{support}\, x)$.

(18) Let us consider non empty sets I, J, a commutative group G, a finite-support function x from I into G, a finite-support function y from J into G, and a function a from I into J. If a is bijective and $x = y \cdot a$, then $\prod x = \prod y$.

PROOF: Reconsider $S_1 = \mathrm{support}\, x$ as a finite set. Reconsider $S_2 = \mathrm{support}\, y$ as a finite set. Reconsider $s_1 = \mathrm{CFS}(S_1)$ as a finite sequence of elements of S_1. Reconsider $s_2 = \mathrm{CFS}(S_2)$ as a finite sequence of elements of S_2. Reconsider $x_1 = x \restriction S_1$ as a function from S_1 into G. Consider x_2 being a finite sequence of elements of G such that $\prod x_1 = \prod x_2$ and $x_2 = x_1 \cdot s_1$. Reconsider $y_1 = y \restriction S_2$ as a function from S_2 into G. Consider y_2 being a finite sequence of elements of G such that $\prod y_1 = \prod y_2$ and $y_2 = y_1 \cdot s_2$. $S_2 = a^{\circ} S_1$. $\overline{\overline{S_1}} = \overline{\overline{S_2}}$ by [1, (66)], [8, (25)], [17, (63)], [8, (17), (29)]. Reconsider $n = \overline{\overline{S_1}}$ as a natural number. Reconsider $a_1 = a \restriction S_1$ as a function from S_1 into J. Reconsider $a_2 = s_2{}^{-1}$ as a function from S_2 into $\mathrm{Seg}\, n$.

Reconsider $p = a_2 \cdot a_1 \cdot s_1$ as a function. If S_2 is not empty, then $x_2 = y_2 \cdot p$ by [18, (27)], [7, (3), (12), (47)]. □

(19) Let us consider non empty sets I, J, a group G, a finite-support function x from I into G, a finite-support function y from J into G, and a function a from I into J. Suppose a is bijective and $x = y \cdot a$ and for every elements i, j of I, $x(i) \cdot x(j) = x(j) \cdot x(i)$. Then $\prod x = \prod y$. The theorem is a consequence of (18).

(20) Let us consider a group G, non empty sets I, J, an internal direct sum components F of G and J, and a function a from I into J. Suppose a is bijective. Then $F \cdot a$ is an internal direct sum components of G and I.
Proof: Reconsider $E = F \cdot a$ as a direct sum components of G and I. For every element i of I, $E(i)$ is a subgroup of G by [7, (13)]. There exists a homomorphism h from $\operatorname{sum} E$ to G such that h is bijective and for every finite-support function x from I into G such that $x \in \operatorname{sum} E$ holds $h(x) = \prod x$ by (14), [17, (62), (63)], [12, (25)]. □

3. Conservation Rule of Direct Sum Decomposition for Flattening

Let I be a non empty set and J be a many sorted set indexed by I.

A J-indexed family of multiplicative magma families is a many sorted set indexed by I and is defined by

(Def. 4) for every element i of I, $it(i)$ is a multiplicative magma family of $J(i)$.

A J-indexed family of group families is a J-indexed family of multiplicative magma families and is defined by

(Def. 5) for every element i of I, $it(i)$ is a group family of $J(i)$.

Let N be a J-indexed family of multiplicative magma families and i be an element of I. One can verify that the functor $N(i)$ yields a multiplicative magma family of $J(i)$. Let N be a J-indexed family of group families. Observe that the functor $N(i)$ yields a group family of $J(i)$. Let J be a disjoint valued many sorted set indexed by I and F be a J-indexed family of group families. One can verify that the functor $\bigcup F$ yields a group family of $\bigcup J$. Now we state the proposition:

(21) Let us consider a non empty set I, a disjoint valued many sorted set J indexed by I, a J-indexed family of group families F, an element j of I, and an object i. If $i \in J(j)$, then $(\bigcup F)(i) = F(j)(i)$.

Let I be a non empty set, J be a many sorted set indexed by I, and F be a J-indexed family of multiplicative magma families. The functor $\operatorname{ProdBundle}(F)$ yielding a multiplicative magma family of I is defined by

(Def. 6) for every element i of I, $it(i) = \prod(F(i))$.

Let F be a J-indexed family of group families.

Note that the functor $\mathrm{ProdBundle}(F)$ yields a group family of I. The functor $\mathrm{SumBundle}(F)$ yielding a group family of I is defined by

(Def. 7) for every element i of I, $it(i) = \mathrm{sum}(F(i))$.

Let F be a J-indexed family of multiplicative magma families. The functor $\mathrm{d}\prod F$ yielding a multiplicative magma is defined by the term

(Def. 8) $\prod \mathrm{ProdBundle}(F)$.

Let J be a non-empty many sorted set indexed by I. One can check that $\mathrm{d}\prod F$ is non empty and constituted functions.

Let F be a J-indexed family of group families. Observe that $\mathrm{d}\prod F$ is group-like and associative.

The functor $\mathrm{d}\sum F$ yielding a group is defined by the term

(Def. 9) $\mathrm{sum}\,\mathrm{SumBundle}(F)$.

Note that $\mathrm{d}\sum F$ is non empty and constituted functions.

Let us consider a non empty set I and group families F_1, F_2 of I.

Let us assume that for every element i of I, $F_1(i)$ is a subgroup of $F_2(i)$. Now we state the propositions:

(22) $\prod F_1$ is a subgroup of $\prod F_2$.

PROOF: For every object x such that $x \in \Omega_{\prod F_1}$ holds $x \in \Omega_{\prod F_2}$. Reconsider $f_2 =$ (the multiplication of $\prod F_2$) $\restriction \Omega_{\prod F_1}$ as a function from $\Omega_{\prod F_1} \times \Omega_{\prod F_1}$ into $\Omega_{\prod F_2}$. Reconsider $f_1 =$ the multiplication of $\prod F_1$ as a function from $\Omega_{\prod F_1} \times \Omega_{\prod F_1}$ into $\Omega_{\prod F_2}$. For every sets x, y such that $x, y \in \Omega_{\prod F_1}$ holds $f_1(x, y) = f_2(x, y)$ by [10, (1)], [16, (43)], [7, (49)], [9, (87)]. $\square$

(23) $\mathrm{sum}\,F_1$ is a subgroup of $\mathrm{sum}\,F_2$.

PROOF: For every object x such that $x \in \Omega_{\mathrm{sum}\,F_1}$ holds $x \in \Omega_{\mathrm{sum}\,F_2}$ by [16, (40)], (22), [16, (42), (44)]. $\prod F_1$ is a subgroup of $\prod F_2$. $\square$

(24) Let us consider a non empty set I, a non-empty many sorted set J indexed by I, and a J-indexed family of group families F. Then $\mathrm{d}\sum F$ is a subgroup of $\mathrm{d}\prod F$. The theorem is a consequence of (22).

Let I be a non empty set, J be a non-empty, disjoint valued many sorted set indexed by I, and F be a J-indexed family of group families. One can verify that the functor $\mathrm{d}\sum F$ yields a subgroup of $\mathrm{d}\prod F$. The functor $\mathrm{dProd2Prod}(F)$ yielding a homomorphism from $\mathrm{d}\prod F$ to $\prod \bigcup F$ is defined by

(Def. 10) for every element x of $\mathrm{d}\prod F$ and for every element i of I, $x(i) = it(x) \restriction J(i)$.

Now we state the proposition:

(25) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a J-indexed family of group families F, an element y of $\prod \bigcup F$, and an element i of I. Then $y \restriction J(i) \in \prod(F(i))$.
PROOF: Set $x = y \restriction J(i)$. Set $z =$ the support of $F(i)$. For every object j such that $j \in J(i)$ holds $x(j) \in z(j)$ by [7, (49), (1)]. $\square$

Let I be a non empty set, J be a non-empty, disjoint valued many sorted set indexed by I, and F be a J-indexed family of group families. Note that $\mathrm{dProd2Prod}(F)$ is bijective.

The functor $\mathrm{Prod2dProd}(F)$ yielding a homomorphism from $\prod \bigcup F$ to $\mathrm{d}\prod F$ is defined by the term

(Def. 11) $(\mathrm{dProd2Prod}(F))^{-1}$.

Now we state the proposition:

(26) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a J-indexed family of group families F, an element x of $\prod \bigcup F$, and an element i of I. Then $x \restriction J(i) = (\mathrm{Prod2dProd}(F))(x)(i)$.

Let I be a non empty set, J be a non-empty, disjoint valued many sorted set indexed by I, and F be a J-indexed family of group families. Note that $\mathrm{Prod2dProd}(F)$ is bijective.

(27) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, and a J-indexed family of group families F. Then $\mathrm{Prod2dProd}(F) = (\mathrm{dProd2Prod}(F))^{-1}$.

Let I be a non empty set, J be a non-empty, disjoint valued many sorted set indexed by I, F be a J-indexed family of group families, and x be a function. The functor $\mathrm{rsupport}(x, F)$ yielding a disjoint valued many sorted set indexed by I is defined by

(Def. 12) for every element i of I, $it(i) = \mathrm{support}(x \restriction J(i), F(i))$.

Now we state the propositions:

(28) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a J-indexed family of group families F, and a function x. Then $\mathrm{support}(x, \bigcup F) = \bigcup \mathrm{rsupport}(x, F)$.
PROOF: Set $y = \mathrm{rsupport}(x, F)$. For every object j, $j \in \mathrm{support}(x, \bigcup F)$ iff $j \in \bigcup y$ by (21), [7, (49), (3)], [9, (74)]. $\square$

(29) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a J-indexed family of group families F, and functions x, y, z. Suppose $z \in \mathrm{d}\prod F$ and $x = (\mathrm{dProd2Prod}(F))(z)$. Then

(i) $\mathrm{rsupport}(x, F) \restriction \mathrm{support}(z, \mathrm{SumBundle}(F))$ is a non-empty, disjoint valued many sorted set indexed by $\mathrm{support}(z, \mathrm{SumBundle}(F))$, and

(ii) $\mathrm{support}(x, \bigcup F) = \bigcup(\mathrm{rsupport}(x, F) \restriction \mathrm{support}(z, \mathrm{SumBundle}(F)))$.

PROOF: Set $s_1 = \mathrm{rsupport}(x, F)$. Set $s_2 = \mathrm{support}(z, \mathrm{SumBundle}(F))$. Set $f = s_1 \restriction s_2$. For every objects s, t such that $s \neq t$ holds $f(s)$ misses $f(t)$ by [7, (47)]. $\emptyset \notin \mathrm{rng}\, f$ by [7, (47)], [10, (5)], [16, (44)]. $\mathrm{support}(x, \bigcup F) = \bigcup s_1$. For every object k such that $k \in \mathrm{support}(x, \bigcup F)$ holds $k \in \bigcup(s_1 \restriction s_2)$ by [10, (6)], [16, (44)], [18, (57)], [7, (47), (3)]. □

(30) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a J-indexed family of group families F, and a function y. Suppose $y \in \mathrm{sum} \bigcup F$. Then there exists a function x such that

(i) $y = (\mathrm{dProd2Prod}(F))(x)$, and

(ii) $x \in \mathrm{d}\sum F$.

PROOF: Consider x being an element of $\Omega_{\mathrm{d}\prod F}$ such that $y = (\mathrm{dProd2Prod}(F))(x)$. Set $s_1 = \mathrm{rsupport}(y, F)$. $\mathrm{support}(y, \bigcup F) = \bigcup s_1$. For every element i of I, $x(i) \in (\mathrm{SumBundle}(F))(i)$ by [7, (3)], [9, (74)], [12, (8)]. Set $S = \mathrm{SumBundle}(F)$. Reconsider $W =$ the support of S as a many sorted set indexed by I. For every object i such that $i \in I$ holds $x(i) \in W(i)$. Reconsider $s_2 = s_1 \restriction \mathrm{support}(x, \mathrm{SumBundle}(F))$ as a non-empty, disjoint valued many sorted set indexed by $\mathrm{support}(x, \mathrm{SumBundle}(F))$. $\bigcup s_2$ is finite. $\mathrm{dom}\, s_2$ is finite. □

(31) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a J-indexed family of group families F, and functions x, y. Suppose x, $x \in \mathrm{d}\sum F$. Then $(\mathrm{dProd2Prod}(F))(x) \in \mathrm{sum} \bigcup F$.
PROOF: Reconsider $y = (\mathrm{dProd2Prod}(F))(x)$ as an element of $\prod \bigcup F$. Set $s_1 = \mathrm{rsupport}(y, F)$. Reconsider $s_2 = s_1 \restriction \mathrm{support}(x, \mathrm{SumBundle}(F))$ as a non-empty, disjoint valued many sorted set indexed by $\mathrm{support}(x, \mathrm{SumBundle}(F))$. For every object i such that $i \in \mathrm{dom}\, s_2$ holds $s_2(i)$ is finite by [16, (40)], [7, (49)]. $\mathrm{support}(y, \bigcup F)$ is finite. □

(32) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, and a J-indexed family of group families F. Then $\mathrm{rng}(\mathrm{dProd2Prod}(F) \restriction \mathrm{d}\sum F) = \Omega_{\mathrm{sum} \bigcup F}$.
PROOF: For every object y, $y \in \mathrm{rng}(\mathrm{dProd2Prod}(F) \restriction \Omega_{\mathrm{d}\sum F})$ iff $y \in \Omega_{\mathrm{sum} \bigcup F}$ by [18, (61)], (31), [7, (47)], (30). □

Let I be a non empty set, J be a non-empty, disjoint valued many sorted set indexed by I, and F be a J-indexed family of group families. The functor $\mathrm{dSum2Sum}(F)$ yielding a homomorphism from $\mathrm{d}\sum F$ to $\mathrm{sum} \bigcup F$ is defined by the term

(Def. 13) $\mathrm{dProd2Prod}(F) \restriction \mathrm{d}\sum F$.

One can verify that $\mathrm{dSum2Sum}(F)$ is bijective.

The functor Sum2dSum(F) yielding a homomorphism from sum $\bigcup F$ to d$\sum F$ is defined by the term

(Def. 14) $(\text{dSum2Sum}(F))^{-1}$.

Now we state the proposition:

(33) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, and a J-indexed family of group families F. Then $\text{Sum2dSum}(F) = \text{Prod2dProd}(F){\restriction}\,\text{sum} \bigcup F$. The theorem is a consequence of (2).

Let I be a non empty set, J be a non-empty, disjoint valued many sorted set indexed by I, and F be a J-indexed family of group families. One can check that Sum2dSum(F) is bijective.

Now we state the proposition:

(34) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, and a J-indexed family of group families F. Then $\text{dSum2Sum}(F) = (\text{Sum2dSum}(F))^{-1}$.

Let I be a non empty set, G be a group, and F be an internal direct sum components of G and I. The functor InterHom(F) yielding a homomorphism from sum F to G is defined by

(Def. 15) *it* is bijective and for every finite-support function x from I into G such that $x \in \text{sum}\, F$ holds $it(x) = \prod x$.

Let J be a non-empty, disjoint valued many sorted set indexed by I, M be a direct sum components of G and I, N be a J-indexed family of group families, and h be a many sorted set indexed by I. Assume for every element i of I, there exists a homomorphism h_0 from $(\text{SumBundle}(N))(i)$ to $M(i)$ such that $h_0 = h(i)$ and h_0 is bijective. The functor ProdHom(G, M, N, h) yielding a homomorphism from d$\sum N$ to sum M is defined by

(Def. 16) $it = \text{SumMap}(\text{SumBundle}(N), M, h)$ and *it* is bijective.

Now we state the propositions:

(35) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a group G, a direct sum components M of G and I, and a J-indexed family of group families N. Suppose for every element i of I, $N(i)$ is a direct sum components of $M(i)$ and $J(i)$. Then $\bigcup N$ is a direct sum components of G and $\bigcup J$.

PROOF: Consider f_2 being a homomorphism from sum M to G such that f_2 is bijective. Define $\mathcal{P}(\text{object}) = \Omega_{\text{sum}(N(\$_1(\in I)))}$. Consider D_2 being a function such that $\text{dom}\, D_2 = I$ and for every object i such that $i \in I$ holds $D_2(i) = \mathcal{P}(i)$ from [7, Sch. 3]. Define $\mathcal{Q}(\text{object}) = \Omega_{M(\$_1(\in I))}$. Consider R_1 being a function such that $\text{dom}\, R_1 = I$ and for every object i such that $i \in I$ holds $R_1(i) = \mathcal{Q}(i)$ from [7, Sch. 3]. Define $\mathcal{R}[\text{object}, \text{object}] \equiv$ there exists a homomorphism f_3 from $\text{sum}(N(\$_1(\in I)))$ to $M(\$_1(\in I))$ such that $f_3 = \$_2$ and f_3 is bijective. For every element i of I, there exists an element y of $\bigcup D_2 \dot{\to} \bigcup R_1$ such that $\mathcal{R}[i, y]$ by [7, (3)], [9, (74)]. Consider f_1 being a function from I into $\bigcup D_2 \dot{\to} \bigcup R_1$ such that for every element i of I, $\mathcal{R}[i, f_1(i)]$ from [8, Sch. 3]. For every element i of I, there exists a homomorphism h_0 from $(\text{SumBundle}(N))(i)$ to $M(i)$ such that $h_0 = f_1(i)$ and h_0 is bijective. $\square$

(36) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a group G, an internal direct sum components M of G and I, and a J-indexed family of group families N. Suppose for every element i of I, $N(i)$ is an internal direct sum components of $M(i)$ and $J(i)$. Then $\bigcup N$ is an internal direct sum components of G and $\bigcup J$.
PROOF: Consider f_3 being a homomorphism from $\text{sum}\, M$ to G such that f_3 is bijective and for every finite-support function x from I into G such that $x \in \text{sum}\, M$ holds $f_3(x) = \prod x$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an internal direct sum components N_1 of $M(\$_1(\in I))$ and $J(\$_1(\in I))$ such that $N_1 = N(\$_1)$ and $\$_2 = \text{InterHom}(N_1)$. For every object x such that $x \in I$ there exists an object y such that $\mathcal{Q}[x, y]$. Consider f_1 being a function such that $\text{dom}\, f_1 = I$ and for every object i such that $i \in I$ holds $\mathcal{Q}[i, f_1(i)]$ from [7, Sch. 2]. Set $f_2 = \text{ProdHom}(G, M, N, f_1)$. For every element i of I, there exists a homomorphism h_0 from $(\text{SumBundle}(N))(i)$ to $M(i)$ such that $h_0 = f_1(i)$ and h_0 is bijective and for every finite-support function x from $J(i)$ into $M(i)$ such that $x \in (\text{SumBundle}(N))(i)$ holds $h_0(x) = \prod x$. For every element i of I, there exists a homomorphism h_0 from $(\text{SumBundle}(N))(i)$ to $M(i)$ such that $h_0 = f_1(i)$ and h_0 is bijective. Reconsider $h = f_3 \cdot f_2 \cdot \text{Sum2dSum}(N)$ as a homomorphism from $\text{sum} \bigcup N$ to G. Reconsider $U_2 = \bigcup J$ as a non empty set. Reconsider $U_3 = \bigcup N$ as a direct sum components of G and U_2. For every object j such that $j \in U_2$ holds $U_3(j)$ is a subgroup of G by (21), [16, (56)]. For every finite-support function x from U_2 into G such that $x \in \text{sum}\, U_3$ holds $h(x) = \prod x$ by [16, (42), (40)], [7, (13)], [8, (5), (15)]. $\square$

4. Conservation Rule of Direct Sum Decomposition for Layering

Now we state the propositions:

(37) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a group G, a group family M of I, and a J-indexed family of group families N. Suppose $\bigcup N$ is a direct sum components of G and $\bigcup J$ and for every element i of I, $N(i)$ is a direct sum

components of $M(i)$ and $J(i)$. Then M is a direct sum components of G and I.

Proof: Set $U_3 = \bigcup N$. Consider f_4 being a homomorphism from sum U_3 to G such that f_4 is bijective. Define $\mathcal{P}(\text{object}) =$ the carrier of sum$(N(\$_1(\in I)))$. Consider D_2 being a function such that $\operatorname{dom} D_2 = I$ and for every object i such that $i \in I$ holds $D_2(i) = \mathcal{P}(i)$ from [7, Sch. 3]. Define $\mathcal{Q}(\text{object}) =$ the carrier of $M(\$_1(\in I))$. Consider R_1 being a function such that $\operatorname{dom} R_1 = I$ and for every object i such that $i \in I$ holds $R_1(i) = \mathcal{Q}(i)$ from [7, Sch. 3]. Define $\mathcal{R}[\text{object}, \text{object}] \equiv$ there exists a homomorphism f_3 from $M(\$_1(\in I))$ to sum$(N(\$_1(\in I)))$ such that $f_3 = \$_2$ and f_3 is bijective. For every element i of I, there exists an element y of $\bigcup R_1 \dot{\rightarrow} \bigcup D_2$ such that $\mathcal{R}[i, y]$ by [17, (62), (63)], [7, (3)], [9, (74)]. Consider f_1 being a function from I into $\bigcup R_1 \dot{\rightarrow} \bigcup D_2$ such that for every element i of I, $\mathcal{R}[i, f_1(i)]$ from [8, Sch. 3]. For every element i of I, there exists a homomorphism h_0 from $M(i)$ to $(\text{SumBundle}(N))(i)$ such that $h_0 = f_1(i)$ and h_0 is bijective. $\square$

(38) Let us consider a non empty set I, a non-empty, disjoint valued many sorted set J indexed by I, a group G, a subgroup family M of I and G, and a J-indexed family of group families N. Suppose $\bigcup N$ is an internal direct sum components of G and $\bigcup J$ and for every element i of I, $N(i)$ is an internal direct sum components of $M(i)$ and $J(i)$. Then M is an internal direct sum components of G and I.

Proof: Reconsider $U_2 = \bigcup J$ as a non empty set. Consider f_4 being a homomorphism from sum $\bigcup N$ to G such that f_4 is bijective and for every finite-support function x from U_2 into G such that $x \in \text{sum} \bigcup N$ holds $f_4(x) = \prod x$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists an internal direct sum components N_1 of $M(\$_1(\in I))$ and $J(\$_1(\in I))$ such that $N_1 = N(\$_1)$ and $\$_2 = (\text{InterHom}(N_1))^{-1}$. For every object x such that $x \in I$ there exists an object y such that $\mathcal{Q}[x, y]$.

Consider f_1 being a function such that $\operatorname{dom} f_1 = I$ and for every object i such that $i \in I$ holds $\mathcal{Q}[i, f_1(i)]$ from [7, Sch. 2]. Reconsider $f_3 = \text{SumMap}(M, (\text{SumBundle}(N)), f_1)$ as a homomorphism from sum M to $\mathrm{d}\sum N$. For every element i of I, there exists a homomorphism h_0 from $M(i)$ to $(\text{SumBundle}(N))(i)$ such that $h_0 = f_1(i)$ and h_0 is bijective by [17, (62), (63)]. Reconsider $h = f_4 \cdot \text{dSum2Sum}(N) \cdot f_3$ as a homomorphism from sum M to G. For every element i of I, there exists a homomorphism h_0 from $(\text{SumBundle}(N))(i)$ to $M(i)$ such that $h_0{}^{-1} = f_1(i)$ and h_0 is bijective and for every finite-support function x from $J(i)$ into $M(i)$ such that $x \in (\text{SumBundle}(N))(i)$ holds $h_0(x) = \prod x$. For every element i of I, there exists a homomorphism h_0 from $(\text{SumBundle}(N))(i)$ to $M(i)$ such that ${h_0}^{-1} = f_1(i)$ and h_0 is bijective. For every finite-support function x from I into G such that $x \in \operatorname{sum} M$ holds $h(x) = \prod x$ by [16, (40)], [7, (13)], [8, (5), (15)]. □

(39) Let us consider a non empty set I_2, and a group family F_2 of I_2. Suppose for every element i of I_2, $\overline{\overline{F_2(i)}} = 1$. Then $\overline{\overline{\alpha}} = 1$, where α is the carrier of $\operatorname{sum} F_2$.

PROOF: For every object x such that $x \in \Omega_{\operatorname{sum} F_2}$ holds $x = \mathbf{1}_{\operatorname{sum} F_2}$ by [16, (42)], [1, (30)], [2, (102)], [10, (5)]. □

(40) Let us consider a non empty set I, a group G, and a finite-support function x from I into G. Suppose for every object i such that $i \in I$ holds $x(i) = \mathbf{1}_G$. Then $\prod x = \mathbf{1}_G$.

(41) Let us consider a non empty set I, a group G, a finite-support function x from I into G, and an element a of G. If $I = \{1, 2\}$ and $x = \langle a, \mathbf{1}_G \rangle$, then $\prod x = a$.

PROOF: Reconsider $i_1 = 1$ as an element of I. Set $y = (I \longmapsto \mathbf{1}_G) +\cdot (i_1, a)$. For every object i such that $i \in \operatorname{dom} x$ holds $x(i) = y(i)$ by [3, (44)], [4, (31), (32)], [15, (7)]. □

(42) Let us consider a group G, non empty sets I_1, I_2, a direct sum components F_1 of G and I_1, and a group family F_2 of I_2. Suppose I_1 misses I_2 and for every element i of I_2, $\overline{\overline{F_2(i)}} = 1$. Then $F_1 +\cdot F_2$ is a direct sum components of G and $I_1 \cup I_2$.

PROOF: Reconsider $I = \{1, 2\}$ as a non empty set. Set $J = \{\langle 1, I_1 \rangle, \langle 2, I_2 \rangle\}$. For every objects x, y_1, y_2 such that $\langle x, y_1 \rangle$, $\langle x, y_2 \rangle \in J$ holds $y_1 = y_2$. $\emptyset \notin \operatorname{rng} J$. For every objects i, j such that $i \neq j$ holds $J(i)$ misses $J(j)$. Reconsider $M = \langle \operatorname{sum} F_1, \operatorname{sum} F_2 \rangle$ as a group family of I. $\overline{\overline{\Omega_{\operatorname{sum} F_2}}} = 1$. Consider w being an object such that $\{w\} = \Omega_{\operatorname{sum} F_2}$. For every functions x, y such that x, $y \in \Omega_{\prod M}$ and $x(1) = y(1)$ holds $x = y$ by [12, (5)], [3, (44)].

Consider h_1 being a homomorphism from $\operatorname{sum} F_1$ to G such that h_1 is bijective. Set $C_1 =$ the carrier of $\prod M$. Set $C_2 =$ the carrier of G. Define $\mathcal{P}[\text{element of } C_1, \text{element of } C_2] \equiv \$_2 = h_1(\$_1(1))$. For every element x of C_1, there exists an element y of C_2 such that $\mathcal{P}[x, y]$ by [12, (5)], [3, (44)], [8, (5)]. Consider h being a function from C_1 into C_2 such that for every element x of C_1, $\mathcal{P}[x, h(x)]$ from [8, Sch. 3]. For every objects x_1, x_2 such that x_1, $x_2 \in C_1$ and $h(x_1) = h(x_2)$ holds $x_1 = x_2$ by [12, (5)], [3, (44)], [8, (19)]. For every object y such that $y \in C_2$ there exists an object x such that $x \in C_1$ and $y = h(x)$ by [8, (11)], [3, (44)]. For every elements a, b of C_1, $h(a \cdot b) = h(a) \cdot h(b)$ by [3, (44)], [12, (5)], [10, (1)]. Reconsider $M = \langle \operatorname{sum} F_1, \operatorname{sum} F_2 \rangle$ as a direct sum components of G and I. Set $N = \langle F_1, F_2 \rangle$. For every element i of I, $N(i)$ is a group family of $J(i)$ by [3, (44)]. For every element i of I, $N(i)$ is a direct sum components of $M(i)$ and $J(i)$ by [3, (44)]. For every object x such that $x \in \operatorname{dom} F_1 \cup \operatorname{dom} F_2$ holds if $x \in \operatorname{dom} F_2$, then $(\bigcup N)(x) = F_2(x)$ and if $x \notin \operatorname{dom} F_2$, then $(\bigcup N)(x) = F_1(x)$ by (21), [3, (44)]. $\square$

(43) Let us consider a group G, non empty sets I_1, I_2, an internal direct sum components F_1 of G and I_1, and a subgroup family F_2 of I_2 and G. Suppose I_1 misses I_2 and $F_2 = I_2 \longmapsto \{\mathbf{1}\}_G$. Then $F_1 {+\cdot} F_2$ is an internal direct sum components of G and $I_1 \cup I_2$.

Proof: Reconsider $I = \{1, 2\}$ as a non empty set. Set $J = \{\langle 1, I_1 \rangle, \langle 2, I_2 \rangle\}$. For every objects x, y_1, y_2 such that $\langle x, y_1 \rangle$, $\langle x, y_2 \rangle \in J$ holds $y_1 = y_2$. $\emptyset \notin \operatorname{rng} J$. For every objects i, j such that $i \neq j$ holds $J(i)$ misses $J(j)$. Reconsider $M = \langle G, \{\mathbf{1}\}_G \rangle$ as a group family of I. For every functions x, y such that x, $y \in \Omega_{\prod M}$ and $x(1) = y(1)$ holds $x = y$ by [12, (5)], [3, (44)]. Set $h_1 = \mathrm{id}_{(\text{the carrier of } G)}$. Set $C_1 =$ the carrier of $\prod M$. Set $C_2 =$ the carrier of G. Define $\mathcal{P}[\text{element of } C_1, \text{element of } C_2] \equiv \$_2 = h_1(\$_1(1))$. For every element x of C_1, there exists an element y of C_2 such that $\mathcal{P}[x, y]$ by [12, (5)], [3, (44)], [8, (5)]. Consider h being a function from C_1 into C_2 such that for every element x of C_1, $\mathcal{P}[x, h(x)]$ from [8, Sch. 3]. For every objects x_1, x_2 such that x_1, $x_2 \in C_1$ and $h(x_1) = h(x_2)$ holds $x_1 = x_2$ by [12, (5)], [3, (44)], [8, (19)]. For every object y such that $y \in C_2$ there exists an object x such that $x \in C_1$ and $y = h(x)$ by [8, (11)], [3, (44)]. For every elements a, b of C_1, $h(a \cdot b) = h(a) \cdot h(b)$ by [3, (44)], [12, (5)], [10, (1)].

Reconsider $M = \langle G, \{\mathbf{1}\}_G \rangle$ as a direct sum components of G and I. For every element i of I, $M(i)$ is a subgroup of G by [3, (44)], [16, (54)]. For every finite-support function x from I into G such that $x \in \operatorname{sum} M$ holds $h(x) = \prod x$ by [10, (9)], [3, (44)], (41). Set $N = \langle F_1, F_2 \rangle$. For every element i of I, $N(i)$ is a group family of $J(i)$ by [3, (44)]. For every element i of I, $N(i)$ is an internal direct sum components of $M(i)$ and $J(i)$ by [3, (44)], [15, (7)], [1, (30)], (39). For every object x such that $x \in \operatorname{dom} F_1 \cup \operatorname{dom} F_2$ holds if $x \in \operatorname{dom} F_2$, then $(\bigcup N)(x) = F_2(x)$ and if $x \notin \operatorname{dom} F_2$, then $(\bigcup N)(x) = F_1(x)$ by (21), [3, (44)]. $\square$

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