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Circumcenter, Circumcircle and Centroid of a Triangle

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Summary. We introduce, using the Mizar system [1], some basic concepts of Euclidean geometry: the half length and the midpoint of a segment, the perpendicular bisector of a segment, the medians (the cevians that join the vertices of a triangle to the midpoints of the opposite sides) of a triangle.

We prove the existence and uniqueness of the circumcenter of a triangle (the intersection of the three perpendicular bisectors of the sides of the triangle). The extended law of sines and the formula of the radius of the Morley's trisector triangle are formalized [3].

Using the generalized Ceva's Theorem, we prove the existence and uniqueness of the centroid (the common point of the medians [4]) of a triangle.

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1. Preliminaries

From now on n denotes a natural number, λ, λ_2, μ, μ_2 denote real numbers, x_1, x_2 denote elements of $\mathcal{R}^n$, A_1, B_1, C_1 denote points of $\mathcal{E}^n_{\mathrm{T}}$, and a denotes a real number.

Now we state the propositions:

(1) If $A_1 = (1-\lambda)\cdot x_1 + \lambda \cdot x_2$ and $B_1 = (1-\mu)\cdot x_1 + \mu\cdot x_2$, then $B_1 - A_1 = (\mu-\lambda)\cdot(x_2 - x_1)$.

(2) If $|a| = |1-a|$, then $a = \frac{1}{2}$.

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In the sequel P, A, B denote elements of $\mathcal{R}^n$ and L denotes an element of $\mathrm{Lines}(\mathcal{R}^n)$.

Now we state the propositions:

(3) $\mathrm{Line}(P, P) = \{P\}$.

(4) If $A_1 = A$ and $B_1 = B$, then $\mathrm{Line}(A_1, B_1) = \mathrm{Line}(A, B)$.

(5) If $A_1 \neq C_1$ and $C_1 \in \mathcal{L}(A_1, B_1)$ and A_1, $C_1 \in L$ and L is a line, then $B_1 \in L$. The theorem is a consequence of (4).

Let n be a natural number and S be a subset of $\mathcal{R}^n$. We say that S is a point if and only if

(Def. 1) there exists an element P of $\mathcal{R}^n$ such that $S = \{P\}$.

Now we state the propositions:

(6) (i) L is a line, or

(ii) there exists an element P of $\mathcal{R}^n$ such that $L = \{P\}$.

The theorem is a consequence of (3).

(7) L is a line or a point.

Let us assume that L is a line. Now we state the propositions:

(8) There exists no element P of $\mathcal{R}^n$ such that $L = \{P\}$.

(9) L is not a point.

2. Betweenness

In the sequel A, B, C denote points of $\mathcal{E}^2_{\mathrm{T}}$.

Now we state the propositions:

(10) If $C \in \mathcal{L}(A, B)$, then $|A - B| = |A - C| + |C - B|$.

(11) If $|A - B| = |A - C| + |C - B|$, then $C \in \mathcal{L}(A, B)$. The theorem is a consequence of (10).

(12) Let us consider points p, q_1, q_2 of $\mathcal{E}^2_{\mathrm{T}}$. Then $p \in \mathcal{L}(q_1, q_2)$ if and only if $\rho(q_1, p) + \rho(p, q_2) = \rho(q_1, q_2)$. The theorem is a consequence of (11).

Let us consider elements p, q, r of $\mathcal{E}^2$.

Let us assume that p, q, r are mutually different and $p = A$ and $q = B$ and $r = C$. Now we state the propositions:

(13) $A \in \mathcal{L}(B, C)$ if and only if p is between q and r. The theorem is a consequence of (12) and (11).

(14) $A \in \mathcal{L}(B, C)$ if and only if p is between q and r. The theorem is a consequence of (13).

3. Real Plane

From now on x, y, z, y_1, y_2 denote elements of $\mathcal{R}^2$, L, L_1, L_2 denote elements of Lines($\mathcal{R}^2$), D, E, F denote points of $\mathcal{E}^2_{\mathrm{T}}$, and b, c, d, r, s denote real numbers.

Now we state the propositions:

(15) Let us consider elements O, O_1, O_2 of $\mathcal{R}^2$. Suppose $O = [0, 0]$ and $O_1 = [1, 0]$ and $O_2 = [0, 1]$. Then $\mathcal{R}^2 = \mathrm{Plane}(O, O_1, O_2)$.

(16) $\mathcal{R}^2$ is an element of Planes($\mathcal{R}^2$). The theorem is a consequence of (15).

(17) (i) $[1, 0] \neq [0, 1]$, and

(ii) $[1, 0] \neq [0, 0]$, and

(iii) $[0, 1] \neq [0, 0]$.

(18) There exists x such that $x \notin L$. The theorem is a consequence of (6) and (17).

(19) There exists L such that

(i) L is a point, and

(ii) L misses L_1.

The theorem is a consequence of (18) and (3).

Let us assume that $L_1 \nparallel L_2$. Now we state the propositions:

(20) (i) there exists x such that $L_1 = \{x\}$ or $L_2 = \{x\}$, or

(ii) L_1 is a line and L_2 is a line and there exists x such that $L_1 \cap L_2 = \{x\}$.

The theorem is a consequence of (3) and (16).

(21) (i) L_1 is a point, or

(ii) L_2 is a point, or

(iii) L_1 is a line and L_2 is a line and $L_1 \cap L_2$ is a point.

Now we state the proposition:

(22) If $L_1 \cap L_2$ is a point and $A \in L_1 \cap L_2$, then $L_1 \cap L_2 = \{A\}$.

4. The Midpoint of a Segment

Let A, B be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor half-length(A, B) yielding a real number is defined by the term

(Def. 2) $(\frac{1}{2}) \cdot |A - B|$.

Now we state the propositions:

(23) half-length$(A, B) =$ half-length(B, A).

(24) half-length$(A, A) = 0$.

(25) $|A - (\frac{1}{2}) \cdot (A + B)| = (\frac{1}{2}) \cdot |A - B|$.

(26) There exists C such that

(i) $C \in \mathcal{L}(A, B)$, and

(ii) $|A - C| = (\frac{1}{2}) \cdot |A - B|$.

The theorem is a consequence of (25).

(27) If $|A - B| = |A - C|$ and B, $C \in \mathcal{L}(A, D)$, then $B = C$. The theorem is a consequence of (1).

Let A, B be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor SegMidpoint(A, B) yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by

(Def. 3) there exists C such that $C \in \mathcal{L}(A, B)$ and *it* $= C$ and $|A - C| =$ half-length(A, B).

Now we state the propositions:

(28) SegMidpoint$(A, B) \in \mathcal{L}(A, B)$.

(29) SegMidpoint$(A, B) = (\frac{1}{2}) \cdot (A + B)$. The theorem is a consequence of (25).

(30) SegMidpoint$(A, B) =$ SegMidpoint(B, A). The theorem is a consequence of (29).

(31) SegMidpoint$(A, A) = A$. The theorem is a consequence of (29).

(32) If SegMidpoint$(A, B) = A$, then $A = B$. The theorem is a consequence of (29).

(33) If SegMidpoint$(A, B) = B$, then $A = B$. The theorem is a consequence of (30) and (32).

Let us assume that $C \in \mathcal{L}(A, B)$ and $|A - C| = |B - C|$. Now we state the propositions:

(34) half-length$(A, B) = |A - C|$. The theorem is a consequence of (10).

(35) $C =$ SegMidpoint(A, B). The theorem is a consequence of (34).

Now we state the propositions:

(36) $|A -$ SegMidpoint$(A, B)| = |$SegMidpoint$(A, B) - B|$. The theorem is a consequence of (29) and (25).

(37) If $A \neq B$ and r is positive and $r \neq 1$ and $|A - C| = r \cdot |A - B|$, then A, B, C are mutually different.

(38) If $C \in \mathcal{L}(A, B)$ and $|A - C| = (\frac{1}{2}) \cdot |A - B|$, then $|B - C| = (\frac{1}{2}) \cdot |A - B|$. The theorem is a consequence of (10).

5. Perpendicularity

Now we state the propositions:

(39) L_1 and L_2 are coplanar. The theorem is a consequence of (15).

(40) If $L_1 \perp L_2$, then L_1 meets L_2.

(41) If L_1 is a line and L_2 is a line and L_1 misses L_2, then $L_1 \parallel L_2$.

(42) Suppose $L_1 \neq L_2$ and L_1 meets L_2. Then

(i) there exists x such that $L_1 = \{x\}$ or $L_2 = \{x\}$, or

(ii) L_1 is a line and L_2 is a line and there exists x such that $L_1 \cap L_2 = \{x\}$.

The theorem is a consequence of (20).

Let us assume that $L_1 \perp L_2$. Now we state the propositions:

(43) There exists x such that $L_1 \cap L_2 = \{x\}$. The theorem is a consequence of (39), (8), and (42).

(44) $L_1 \cap L_2$ is a point.

Now we state the propositions:

(45) If $L_1 \perp L_2$, then $L_1 \nparallel L_2$. The theorem is a consequence of (39).

(46) If L_1 is a line and L_2 is a line and $L_1 \parallel L_2$, then $L_1 \not\perp L_2$.

Now we state the propositions:

(47) If L_1 is a line, then there exists L_2 such that $x \in L_2$ and $L_1 \perp L_2$. The theorem is a consequence of (18).

(48) If $L_1 \perp L_2$ and $L_1 = \mathrm{Line}(A, B)$ and $L_2 = \mathrm{Line}(C, D)$, then $|(B-A, D-C)| = 0$. The theorem is a consequence of (1).

(49) If L is a line and A, $B \in L$ and $A \neq B$, then $L = \mathrm{Line}(A, B)$. The theorem is a consequence of (4).

Let us assume that $L_1 \perp L_2$ and $C \in L_1 \cap L_2$ and $A \in L_1$ and $B \in L_2$ and $A \neq C$ and $B \neq C$. Now we state the propositions:

(50) (i) $\measuredangle(A, C, B) = \frac{\pi}{2}$, or

(ii) $\measuredangle(A, C, B) = \frac{3 \cdot \pi}{2}$.

The theorem is a consequence of (49) and (48).

(51) A, B, C form a triangle.

Proof: $A \notin \mathrm{Line}(B, C)$ by [5, (67)], (43), (49). $\square$

6. The Perpendicular Bisector of a Segment

Now we state the proposition:

(52) Suppose $A \neq B$ and $L_1 = \mathrm{Line}(A, B)$ and $C \in \mathcal{L}(A, B)$ and $|A - C| = (\frac{1}{2}) \cdot |A - B|$. Then there exists L_2 such that

(i) $C \in L_2$, and

(ii) $L_1 \perp L_2$.

The theorem is a consequence of (4) and (47).

Let A, B be elements of $\mathcal{E}^2_{\mathrm{T}}$. Assume $A \neq B$. The functor $\mathrm{PerpBisec}(A, B)$ yielding an element of $\mathrm{Lines}(\mathcal{R}^2)$ is defined by

(Def. 4) there exist elements L_1, L_2 of $\mathrm{Lines}(\mathcal{R}^2)$ such that $it = L_2$ and $L_1 = \mathrm{Line}(A, B)$ and $L_1 \perp L_2$ and $L_1 \cap L_2 = \{\mathrm{SegMidpoint}(A, B)\}$.

Let us assume that $A \neq B$. Now we state the propositions:

(53) $\mathrm{PerpBisec}(A, B)$ is a line.

(54) $\mathrm{PerpBisec}(A, B) = \mathrm{PerpBisec}(B, A)$. The theorem is a consequence of (43), (16), and (30).

(55) Suppose $A \neq B$ and $L_1 = \mathrm{Line}(A, B)$ and $C \in \mathcal{L}(A, B)$ and $|A - C| = (\frac{1}{2}) \cdot |A - B|$ and $C \in L_2$ and $L_1 \perp L_2$ and $D \in L_2$. Then $|D - A| = |D - B|$. The theorem is a consequence of (38), (37), and (50).

(56) If $A \neq B$ and $C \in \mathrm{PerpBisec}(A, B)$, then $|C - A| = |C - B|$. The theorem is a consequence of (28) and (55).

(57) If $C \in \mathrm{Line}(A, B)$ and $|A - C| = |B - C|$, then $C \in \mathcal{L}(A, B)$. The theorem is a consequence of (4), (3), and (2).

(58) If $A \neq B$, then $\mathrm{SegMidpoint}(A, B) \in \mathrm{PerpBisec}(A, B)$.

(59) If $A \neq B$ and $L_1 = \mathrm{Line}(A, B)$ and $L_1 \perp L_2$ and $\mathrm{SegMidpoint}(A, B) \in L_2$, then $L_2 = \mathrm{PerpBisec}(A, B)$. The theorem is a consequence of (16).

(60) If $A \neq B$ and $|C - A| = |C - B|$, then $C \in \mathrm{PerpBisec}(A, B)$. The theorem is a consequence of (47), (43), (50), (57), (35), (58), and (59).

7. The Circumcircle of a Triangle

Let us assume that A, B, C form a triangle. Now we state the propositions:

(61) $\mathrm{PerpBisec}(A, B) \cap \mathrm{PerpBisec}(B, C)$ is a point. The theorem is a consequence of (16), (8), and (20).

(62) There exists D such that

(i) $\mathrm{PerpBisec}(A, B) \cap \mathrm{PerpBisec}(B, C) = \{D\}$, and

(ii) $\mathrm{PerpBisec}(B, C) \cap \mathrm{PerpBisec}(C, A) = \{D\}$, and

(iii) $\mathrm{PerpBisec}(C, A) \cap \mathrm{PerpBisec}(A, B) = \{D\}$, and

(iv) $|D - A| = |D - B|$, and

(v) $|D - A| = |D - C|$, and

(vi) $|D - B| = |D - C|$.

The theorem is a consequence of (61), (56), and (60).

Let A, B, C be points of $\mathcal{E}^2_{\mathrm{T}}$. Assume A, B, C form a triangle. The functor Circumcenter $\triangle(A, B, C)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by

(Def. 5) $\mathrm{PerpBisec}(A, B) \cap \mathrm{PerpBisec}(B, C) = \{it\}$ and $\mathrm{PerpBisec}(B, C) \cap \mathrm{PerpBisec}(C, A) = \{it\}$ and $\mathrm{PerpBisec}(C, A) \cap \mathrm{PerpBisec}(A, B) = \{it\}$.

Assume A, B, C form a triangle. The functor RadCircumCirc $\triangle(A, B, C)$ yielding a real number is defined by the term

(Def. 6) $|\,\mathrm{Circumcenter} \triangle(A, B, C) - A|$.

(63) If A, B, C form a triangle, then there exists a and there exists b and there exists r such that A, B, $C \in \mathrm{circle}(a, b, r)$. The theorem is a consequence of (62).

(64) Suppose A, B, C form a triangle and A, B, $C \in \mathrm{circle}(a, b, r)$. Then

(i) $[a, b] = \mathrm{Circumcenter} \triangle(A, B, C)$, and

(ii) $r = |\,\mathrm{Circumcenter} \triangle(A, B, C) - A|$.

The theorem is a consequence of (60), (22), and (61).

Let us assume that A, B, C form a triangle. Now we state the propositions:

(65) RadCircumCirc $\triangle(A, B, C) > 0$. The theorem is a consequence of (63) and (64).

(66) (i) $|\,\mathrm{Circumcenter} \triangle(A, B, C) - A| = |\,\mathrm{Circumcenter} \triangle(A, B, C) - B|$, and

(ii) $|\,\mathrm{Circumcenter} \triangle(A, B, C) - A| = |\,\mathrm{Circumcenter} \triangle(A, B, C) - C|$, and

(iii) $|\,\mathrm{Circumcenter} \triangle(A, B, C) - B| = |\,\mathrm{Circumcenter} \triangle(A, B, C) - C|$.

The theorem is a consequence of (62).

(67) (i) $\mathrm{RadCircumCirc} \triangle(A, B, C) = |\,\mathrm{Circumcenter} \triangle(A, B, C) - B|$, and

(ii) $\mathrm{RadCircumCirc} \triangle(A, B, C) = |\,\mathrm{Circumcenter} \triangle(A, B, C) - C|$.

The theorem is a consequence of (66).

(68) If A, B, C form a triangle and A, B, $C \in \mathrm{circle}(a, b, r)$ and A, B, $C \in \mathrm{circle}(c, d, s)$, then $a = c$ and $b = d$ and $r = s$. The theorem is a consequence of (64).

(69) If $r \neq s$, then $\mathrm{circle}(a, b, r)$ misses $\mathrm{circle}(a, b, s)$.

8. Extended Law of Sines

Now we state the propositions:

(70) Suppose A, B, C form a triangle and A, B, $C \in \mathrm{circle}(a, b, r)$ and A, B, D form a triangle and A, B, $D \in \mathrm{circle}(a, b, r)$ and $C \neq D$. Then

(i) $\varnothing_{\circlearrowleft}(A, B, C) = \varnothing_{\circlearrowleft}(D, B, C)$, or

(ii) $\varnothing_{\circlearrowleft}(A, B, C) = -\varnothing_{\circlearrowleft}(D, B, C)$.

Proof: D, B, C form a triangle by [6, (20), (11)], [2, (68)], [6, (30)]. □

(71) Suppose A, B, C form a triangle and A, B, $C \in \mathrm{circle}(a, b, r)$. Then

(i) $\varnothing_{\circlearrowleft}(A, B, C) = 2 \cdot r$, or

(ii) $\varnothing_{\circlearrowleft}(A, B, C) = -2 \cdot r$.

The theorem is a consequence of (70).

(72) If A, B, C form a triangle and $0 < \measuredangle(C, B, A) < \pi$, then $\varnothing_{\circlearrowleft}(A, B, C) > 0$.

(73) If A, B, C form a triangle and $\pi < \measuredangle(C, B, A) < 2 \cdot \pi$, then $\varnothing_{\circlearrowleft}(A, B, C) < 0$.

(74) Suppose A, B, C form a triangle and $0 < \measuredangle(C, B, A) < \pi$ and A, B, $C \in \mathrm{circle}(a, b, r)$. Then $\varnothing_{\circlearrowleft}(A, B, C) = 2 \cdot r$. The theorem is a consequence of (71) and (72).

(75) Suppose A, B, C form a triangle and $\pi < \measuredangle(C, B, A) < 2 \cdot \pi$ and A, B, $C \in \mathrm{circle}(a, b, r)$. Then $\varnothing_{\circlearrowleft}(A, B, C) = -2 \cdot r$. The theorem is a consequence of (71) and (73).

(76) Suppose A, B, C form a triangle and $0 < \measuredangle(C, B, A) < \pi$ and A, B, $C \in \mathrm{circle}(a, b, r)$. Then

(i) $|A - B| = 2 \cdot r \cdot \sin \measuredangle(A, C, B)$, and

(ii) $|B - C| = 2 \cdot r \cdot \sin \measuredangle(B, A, C)$, and

(iii) $|C - A| = 2 \cdot r \cdot \sin \measuredangle(C, B, A)$.

The theorem is a consequence of (74).

(77) Suppose A, B, C form a triangle and $\pi < \measuredangle(C, B, A) < 2 \cdot \pi$ and A, B, $C \in \mathrm{circle}(a, b, r)$. Then

(i) $|A - B| = -2 \cdot r \cdot \sin \measuredangle(A, C, B)$, and

(ii) $|B - C| = -2 \cdot r \cdot \sin \measuredangle(B, A, C)$, and

(iii) $|C - A| = -2 \cdot r \cdot \sin \measuredangle(C, B, A)$.

The theorem is a consequence of (75).

(78) EXTENDED LAW OF SINES:
Suppose A, B, C form a triangle and $0 < \measuredangle(C,B,A) < \pi$ and A, B, $C \in \mathrm{circle}(a,b,r)$. Then

(i) $\frac{|A-B|}{\sin \measuredangle(A,C,B)} = 2 \cdot r$, and

(ii) $\frac{|B-C|}{\sin \measuredangle(B,A,C)} = 2 \cdot r$, and

(iii) $\frac{|C-A|}{\sin \measuredangle(C,B,A)} = 2 \cdot r$.

The theorem is a consequence of (76).

(79) Suppose A, B, C form a triangle and $\pi < \measuredangle(C,B,A) < 2 \cdot \pi$ and A, B, $C \in \mathrm{circle}(a,b,r)$. Then

(i) $\frac{|A-B|}{\sin \measuredangle(A,C,B)} = -2 \cdot r$, and

(ii) $\frac{|B-C|}{\sin \measuredangle(B,A,C)} = -2 \cdot r$, and

(iii) $\frac{|C-A|}{\sin \measuredangle(C,B,A)} = -2 \cdot r$.

The theorem is a consequence of (77).

9. THE CENTROID OF A TRIANGLE

Now we state the proposition:

(80) Suppose A, B, C form a triangle and $D = (1 - (\frac{1}{2})) \cdot B + (\frac{1}{2}) \cdot C$ and $E = (1-(\frac{1}{2})) \cdot C + (\frac{1}{2}) \cdot A$ and $F = (1-(\frac{1}{2})) \cdot A + (\frac{1}{2}) \cdot B$. Then $\mathrm{Line}(A,D)$, $\mathrm{Line}(B,E)$, $\mathrm{Line}(C,F)$ are concurrent.

Let A, B, C be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{Median}\,\triangle(A,B,C)$ yielding an element of $\mathrm{Lines}(\mathcal{R}^2)$ is defined by the term

(Def. 7) $\mathrm{Line}(A, \mathrm{SegMidpoint}(B,C))$.

(81) $\mathrm{Median}\,\triangle(A,A,A) = \{A\}$. The theorem is a consequence of (4), (3), and (31).

(82) $\mathrm{Median}\,\triangle(A,A,B) = \mathrm{Line}(A,B)$. The theorem is a consequence of (28), (32), (4), (3), and (81).

(83) $\mathrm{Median}\,\triangle(A,B,A) = \mathrm{Line}(A,B)$. The theorem is a consequence of (28), (33), (4), (3), and (81).

(84) $\mathrm{Median}\,\triangle(B,A,A) = \mathrm{Line}(A,B)$.

Let us assume that A, B, C form a triangle. Now we state the propositions:

(85) $\mathrm{Median}\,\triangle(A,B,C)$ is a line. The theorem is a consequence of (6) and (28).

(86) There exists D such that

(i) $D \in \mathrm{Median} \triangle(A, B, C)$, and

(ii) $D \in \mathrm{Median} \triangle(B, C, A)$, and

(iii) $D \in \mathrm{Median} \triangle(C, A, B)$.

The theorem is a consequence of (29), (80), and (4).

(87) There exists D such that

(i) $\mathrm{Median} \triangle(A, B, C) \cap \mathrm{Median} \triangle(B, C, A) = \{D\}$, and

(ii) $\mathrm{Median} \triangle(B, C, A) \cap \mathrm{Median} \triangle(C, A, B) = \{D\}$, and

(iii) $\mathrm{Median} \triangle(C, A, B) \cap \mathrm{Median} \triangle(A, B, C) = \{D\}$.

The theorem is a consequence of (86), (4), (85), (28), (32), (5), (8), and (20).

Let A, B, C be points of $\mathcal{E}^2_{\mathrm{T}}$. Assume A, B, C form a triangle. The functor $\mathrm{Centroid} \triangle(A, B, C)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by

(Def. 8) $\mathrm{Median} \triangle(A, B, C) \cap \mathrm{Median} \triangle(B, C, A) = \{it\}$ and $\mathrm{Median} \triangle(B, C, A) \cap \mathrm{Median} \triangle(C, A, B) = \{it\}$ and $\mathrm{Median} \triangle(C, A, B) \cap \mathrm{Median} \triangle(A, B, C) = \{it\}$.

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