

Polynomially Bounded Sequences and Polynomial Sequences

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Summary. In this article, we formalize polynomially bounded sequences that plays an important role in computational complexity theory. Class P is a fundamental computational complexity class that contains all polynomial-time decision problems [11], [12]. It takes polynomially bounded amount of computation time to solve polynomial-time decision problems by the deterministic Turing machine. Moreover we formalize polynomial sequences [5].

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The notation and terminology used in this paper have been introduced in the following articles: [26], [18], [16], [17], [6], [22], [10], [7], [8], [24], [14], [1], [2], [3], [13], [20], [27], [28], [21], [25], and [9].

1. Preliminaries

Now we state the proposition:

(1) Let us consider natural numbers m, k. If $1 \leq m$, then $1 \leq m^k$.

Let us consider natural numbers m, n. Now we state the propositions:

- (2) $m \leqslant m^{n+1}$.
- (3) If $2 \leq m$, then $n+1 \leq m^n$.

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- (4) Let us consider a natural number k. Then $2 \cdot k \leq 2^k$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 2 \cdot \$_1 \leq 2^{\$_1}$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (25)], [24, (5)], [1, (14)], (2). For every natural number $n, \mathcal{P}[n]$ from [1, Sch. 2]. \Box
- (5) Let us consider natural numbers k, n. If $k \leq n$, then $n + k \leq 2^n$. PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \$_1 + k + k \leq 2^{\$_1+k}$. $2 \cdot k \leq 2^k$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (27), (25), (24)]. For every natural number $n, \mathcal{P}[n]$ from [1, Sch. 2]. \Box
- (6) Let us consider natural numbers k, m. If $2 \cdot k + 1 \leq m$, then $2^k \leq 2^m/_m$. The theorem is a consequence of (5).
- (7) Let us consider real numbers a, b, c. If 1 < a and $0 < b \leq c$, then $\log_a b \leq \log_a c$.

Let us consider a natural number n and a real number a. Now we state the propositions:

- (8) If 1 < a, then $a^n < a^{n+1}$.
- (9) If $1 \leq a$, then $a^n \leq a^{n+1}$.
- (10) There exists a partial function g from \mathbb{R} to \mathbb{R} such that
 - (i) dom $g = [0, +\infty)$, and
 - (ii) for every real number x such that $x \in [0, +\infty)$ holds $g(x) = \log_2 x$, and
 - (iii) g is differentiable on $]0, +\infty[$, and
 - (iv) for every real number x such that $x \in [0, +\infty)$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and 0 < g'(x).

PROOF: Set $g = \log_2 e \cdot (\text{the function ln})$. For every real number d such that $d \in [0, +\infty[$ holds $g(d) = \log_2 d$ by [20, (56)]. For every real number x such that $x \in [0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and 0 < g'(x) by [23, (18)], [22, (15)], [20, (57)], [23, (11)]. \Box

- (11) There exists a partial function f from \mathbb{R} to \mathbb{R} such that
 - (i) $]e, +\infty[= \text{dom } f, \text{ and }]$
 - (ii) for every real number x such that $x \in \text{dom } f$ holds $f(x) = x/_{\log_2 x}$, and
 - (iii) f is differentiable on $]e, +\infty[$, and
 - (iv) for every real number x_0 such that $x_0 \in]e, +\infty[$ holds $0 \leq f'(x_0)$, and
 - (v) f is non-decreasing.

PROOF: Consider g being a partial function from \mathbb{R} to \mathbb{R} such that dom $g = [0, +\infty[$ and for every real number x such that $x \in [0, +\infty[$ holds $g(x) = \log_2 x$ and g is differentiable on $]0, +\infty[$ and for every real number x such that $x \in [0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and 0 < g'(x). Set $g_0 = g \upharpoonright e_+ \infty[$. For every object x such that $x \in e_+ \infty[$ holds $x \in [0, +\infty[$ by [23, (11)]. Set $f = \operatorname{id}_{\Omega_{\mathbb{R}}}/g_0$. $g_0^{-1}(\{0\}) = \emptyset$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in e_+ \infty[$ holds $f(x) = x/\log_2 x$ by [7, (49)]. For every real number x such that $x \in e_+ \infty[$ holds f is differentiable in x and $f'(x) = \log_2 x - \log_2 e/(\log_2 x)^2$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in e_+ \infty[$ holds f is differentiable in x and $f'(x) = \log_2 x - \log_2 e/(\log_2 x)^2$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in e_+ \infty[$ holds $0 \leq f'(x)$ by [20, (57)], [23, (11)]. \Box

- (12) Let us consider real numbers x, y. If $e < x \le y$, then $x/\log_2 x \le y/\log_2 y$. The theorem is a consequence of (11).
- (13) Let us consider a natural number k. Suppose e < k. Then there exists a natural number N such that for every natural number n such that $N \le n$ holds $2^k \le n/\log_2 n$. The theorem is a consequence of (12) and (6).

Let us consider a natural number x. Let us assume that 1 < x.

- (14) There exists a natural number N such that for every natural number n such that $N \leq n$ holds $4 < n/\log_r n$.
- (15) There exist natural numbers N, c such that for every natural number n such that $N \leq n$ holds $n^x \leq c \cdot x^n$.
- (16) Let us consider a natural number x. Suppose 1 < x. Then there exist no natural numbers N, c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$.

PROOF: Consider N being a natural number such that there exists a natural number c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$. $N \neq 0$ by [20, (42), (24)]. Consider c being a natural number such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$. There exists an element n of N such that $N \leq n$ and 0 < n - (x/4) by [24, (6), (3)]. Consider n being an element of N such that $N \leq n$ and 0 < n - (x/4). 0 < c by [20, (34)]. For every natural number k such that $1 \leq k$ holds $2^{k \cdot n} \leq c \cdot (k \cdot n)^x$. For every natural number k such that $1 \leq k$ holds $k \cdot n \leq \log_2 c + x \cdot \log_2 k + x \cdot \log_2 n$ by [20, (34)], (7), [20, (55), (52), (53)]. Consider Z being an element of N such that for every natural number k such that $Z \leq k$ holds $4 < k/{\log_2 k}$. There exists a natural number k such that $Z \leq k$ and $\log_2 c + x \cdot \log_2 n/_{n-(x/4)} \leq k$ by [24, (6), (3)]. There exists a natural number k such that $Z \leq k$ and $\log_2 c + x \cdot \log_2 n/_{n-(x/4)} \leq k$ and 1 < k by [1, (11)]. Consider k being a natural number such that $Z \leq k$ and 1 < k and $\log_2 c + x \cdot \log_2 n/_{n-(x/4)} \leq k$. \Box

- (17) Let us consider natural numbers a, b. If $a \leq b$, then $\{n^a\}_{n \in \mathbb{N}} \in O(\{n^b\}_{n \in \mathbb{N}})$.
- (18) Let us consider a natural number x. Suppose 1 < x. Then there exist no natural numbers N, c such that for every natural number n such that $N \leq n$ holds $x^n \leq c \cdot n^x$. PROOF: There exist natural numbers N, c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$ by [24, (7)]. \Box
- (19) Let us consider a non negative real number a, and a natural number n. If $1 \leq n$, then $0 < \{n^a\}_{n \in \mathbb{N}}(n)$.

2. Polynomially Bounded Sequences

Let p be a sequence of real numbers. We say that p is polynomially bounded if and only if

- (Def. 1) there exists a natural number k such that $p \in O(\{n^k\}_{n \in \mathbb{N}})$. Now we state the propositions:
 - (20) Let us consider a sequence f of real numbers. Suppose f is not polynomially bounded. Let us consider a natural number k. Then $f \notin O(\{n^k\}_{n \in \mathbb{N}})$.
 - (21) Let us consider a sequence f of real numbers. Suppose for every natural number $k, f \notin O(\{n^k\}_{n \in \mathbb{N}})$. Then f is not polynomially bounded.
 - (22) Let us consider a positive real number a. Then $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ is positive. Let us consider a real number a. Now we state the propositions:
 - (23) If $1 \leq a$, then $\{a^{1 \cdot n+0}\}_{n \in \mathbb{N}}$ is non-decreasing. The theorem is a consequence of (9).
 - (24) If 1 < a, then $\{a^{1 \cdot n+0}\}_{n \in \mathbb{N}}$ is increasing. The theorem is a consequence of (8).
 - (25) Let us consider a natural number a. If 1 < a, then $\{a^{1 \cdot n+0}\}_{n \in \mathbb{N}}$ is not polynomially bounded.

PROOF: Consider k being a natural number such that $\{a^{1\cdot n+0}\}_{n\in\mathbb{N}} \in O(\{n^k\}_{n\in\mathbb{N}})$. Reconsider $f = \{n^k\}_{n\in\mathbb{N}}$ as an eventually positive sequence of real numbers. Reconsider $t = \{a^{1\cdot n+0}\}_{n\in\mathbb{N}}$ as an eventually nonnegative sequence of real numbers. $t \in O(f)$ and for every element n of \mathbb{N} such that $1 \leq n$ holds 0 < f(n). Consider c being a real number such that c > 0 and for every element n of \mathbb{N} such that $n \geq 1$ holds $(\{a^{1\cdot n+0}\}_{n\in\mathbb{N}})(n) \leq c \cdot \{n^k\}_{n\in\mathbb{N}}(n)$. For every natural number n such that $n \geq 1$ holds $2^n \leq c \cdot n^k$ by [24, (7)]. There exist natural numbers N, b such that for every natural number n such that $N \leq n$ holds $2^n \leq b \cdot n^k$ by [24, (3)]. \Box

3. Polynomial Sequences

Now we state the proposition:

- (26) Let us consider a finite 0-sequence x of \mathbb{R} , and a sequence y of real numbers. Then
 - (i) $x \cdot y$ is a finite transfinite sequence of elements of \mathbb{R} , and
 - (ii) $\operatorname{dom}(x \cdot y) = \operatorname{dom} x$, and
 - (iii) for every object i such that $i \in \text{dom } x$ holds $(x \cdot y)(i) = x(i) \cdot y(i)$.

Let x be a finite 0-sequence of \mathbb{R} and y be a sequence of real numbers. Observe that the functor $x \cdot y$ yields a finite 0-sequence of \mathbb{R} . Now we state the proposition:

(27) Let us consider a finite 0-sequence d of \mathbb{R} , and natural numbers x, i. Suppose $i \in \text{dom } d$. Then $(d \cdot \{x^{1 \cdot n+0}\}_{n \in \mathbb{N}})(i) = d(i) \cdot x^i$. The theorem is a consequence of (26).

Let c be a finite 0-sequence of \mathbb{R} . The functor $\operatorname{Seq}_{\operatorname{poly}}(c)$ yielding a sequence of real numbers is defined by

(Def. 2) for every natural number x, $it(x) = \sum (c \cdot \{x^{1 \cdot n+0}\}_{n \in \mathbb{N}})$.

Let us consider a finite 0-sequence d of \mathbb{R} and a natural number k. Now we state the propositions:

- (28) Suppose len d = k + 1. Then there exists a real number a and there exists a finite 0-sequence d_1 of \mathbb{R} and there exists a sequence y of real numbers such that len $d_1 = k$ and $d_1 = d \upharpoonright k$ and a = d(k) and $d = d_1 \land \langle a \rangle$ and $\operatorname{Seq}_{\operatorname{poly}}(d) = \operatorname{Seq}_{\operatorname{poly}}(d_1) + y$ and for every natural number $i, y(i) = a \cdot i^k$. PROOF: Consider a being a real number, d_1 being a finite 0-sequence of \mathbb{R} such that len $d_1 = k$ and $d_1 = d \upharpoonright k$ and a = d(k) and $d = d_1 \land \langle a \rangle$. Define $\mathcal{F}(\text{natural number}) = a \cdot \$_1^k$. Consider y being a sequence of real numbers such that for every natural number $x, y(x) = \mathcal{F}(x)$ from [15, Sch. 1]. For every element x of \mathbb{N} , $(\operatorname{Seq}_{\operatorname{poly}}(d))(x) = (\operatorname{Seq}_{\operatorname{poly}}(d_1) + y)(x)$ by (26), [1, (13), (44)], (27). \Box
- (29) If len d = 1, then there exists a real number a such that a = d(0) and for every natural number x, $(\text{Seq}_{\text{poly}}(d))(x) = a$. The theorem is a consequence of (26).
- (30) If len d = 1 and d is non-negative yielding, then $\operatorname{Seq}_{\operatorname{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}})$. The theorem is a consequence of (29).
- (31) Let us consider a natural number k, a real number a, and a sequence y of real numbers. Suppose $0 \le a$ and for every natural number i, $y(i) = a \cdot i^k$. Then $y \in O(\{n^k\}_{n \in \mathbb{N}})$.

- (32) Let us consider natural numbers k, n. If $k \leq n$, then $O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^n\}_{n \in \mathbb{N}})$. PROOF: Consider i being a natural number such that n = k + i. Define $\mathcal{P}[\text{natural number}] \equiv O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^{(k+\mathfrak{s}_1)}\}_{n \in \mathbb{N}})$. For every natural number x such that $\mathcal{P}[x]$ holds $\mathcal{P}[x+1]$. For every natural number $x, \mathcal{P}[x]$ from [1, Sch. 2]. \Box
- (33) Let us consider a natural number k, and a non-negative yielding finite 0-sequence c of \mathbb{R} . Suppose len c = k + 1. Then $\operatorname{Seq}_{\operatorname{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$. PROOF: Define $\mathcal{P}[\operatorname{natural number}] \equiv$ for every non-negative yielding finite 0-sequence c of \mathbb{R} such that len $c = \$_1 + 1$ holds $\operatorname{Seq}_{\operatorname{poly}}(c) \in O(\{n^{\$_1}\}_{n \in \mathbb{N}})$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (28), [7, (47)], [1, (13), (39)]. For every natural number $k, \mathcal{P}[k]$ from $[1, \operatorname{Sch. 2}]$. \Box
- (34) Let us consider a natural number k, and a finite 0-sequence c of \mathbb{R} . Then there exists a finite 0-sequence d of \mathbb{R} such that
 - (i) $\operatorname{len} d = \operatorname{len} c$, and
 - (ii) for every natural number i such that $i \in \text{dom } d$ holds d(i) = |c(i)|.

PROOF: Define $\mathcal{F}(\text{natural number}) = |c(\$_1)| (\in \mathbb{R})$. Consider d being a finite 0-sequence of \mathbb{R} such that len d = len c and for every natural number j such that $j \in \text{len } c$ holds $d(j) = \mathcal{F}(j)$ from [18, Sch. 1]. \Box

(35) Let us consider a finite 0-sequence c of \mathbb{R} , and a finite 0-sequence d of \mathbb{R} . Suppose len d = len c and for every natural number i such that $i \in \text{dom } d$ holds d(i) = |c(i)|. Let us consider a natural number n. Then $(\text{Seq}_{\text{poly}}(c))(n) \leq (\text{Seq}_{\text{poly}}(d))(n)$.

PROOF: dom $(d \cdot \{x^{1 \cdot n + 0}\}_{n \in \mathbb{N}})$ = dom d. For every natural number i such that $i \in \text{dom}(c \cdot \{x^{1 \cdot n + 0}\}_{n \in \mathbb{N}})$ holds $(c \cdot \{x^{1 \cdot n + 0}\}_{n \in \mathbb{N}})(i) \leq (d \cdot \{x^{1 \cdot n + 0}\}_{n \in \mathbb{N}})(i)$ by (26), (27), [19, (4)]. \Box

(36) Let us consider a natural number k, and a finite 0-sequence c of \mathbb{R} . Suppose len c = k + 1 and $\operatorname{Seq}_{\operatorname{poly}}(c)$ is eventually nonnegative. Then $\operatorname{Seq}_{\operatorname{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}}).$

PROOF: Consider d being a finite 0-sequence of \mathbb{R} such that len d = len cand for every natural number i such that $i \in \text{dom } d$ holds d(i) = |c(i)|. For every natural number i such that $i \in \text{dom } d$ holds $0 \leq d(i)$ by [6, (46)]. For every real number r such that $r \in \text{rng } d$ holds $0 \leq r$. Seq_{poly} $(d) \in$ $O(\{n^k\}_{n \in \mathbb{N}})$. Consider t being an element of $\mathbb{R}^{\mathbb{N}}$ such that Seq_{poly}(d) = tand there exists a real number c and there exists an element N of \mathbb{N} such that c > 0 and for every element n of \mathbb{N} such that $n \geq N$ holds $t(n) \leq$ $c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \geq 0$. Consider N_1 being a natural number such that for every natural number n such that $N_1 \leq n$ holds $0 \leq (\text{Seq}_{\text{poly}}(c))(n)$. Consider a being a real number, N_2 being an element of \mathbb{N} such that a > 0and for every element n of \mathbb{N} such that $n \ge N_2$ holds $t(n) \le a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \ge 0$. Set $N = N_1 + N_2$. For every element n of \mathbb{N} such that $n \ge N$ holds $(\operatorname{Seq}_{\operatorname{poly}}(c))(n) \le a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $(\operatorname{Seq}_{\operatorname{poly}}(c))(n) \ge 0$ by [1, (11)], (35). \Box

- (37) Let us consider natural numbers k, n. If 0 < n, then $n \cdot \{n^k\}_{n \in \mathbb{N}}(n) = \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$.
- (38) Let us consider a finite 0-sequence c of \mathbb{R} . Suppose len c = 0. Let us consider a natural number x. Then $(\text{Seq}_{polv}(c))(x) = 0$.
- (39) Let us consider an eventually nonnegative sequence f of real numbers, and a natural number k. Suppose $f \in O(\{n^k\}_{n \in \mathbb{N}})$. Then there exists a natural number N such that for every natural number n such that $N \leq n$ holds $f(n) \leq \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$. The theorem is a consequence of (37).
- (40) Let us consider a finite 0-sequence c of \mathbb{R} . Then there exists a finite 0-sequence a_1 of \mathbb{R} such that
 - (i) $a_1 = |c|$, and
 - (ii) for every natural number n, $(\text{Seq}_{\text{poly}}(c))(n) \leq (\text{Seq}_{\text{poly}}(a_1))(n)$.

PROOF: Reconsider $a_1 = |c|$ as a finite 0-sequence of \mathbb{R} . Set $m_1 = c \cdot \{n^{1 \cdot n+0}\}_{n \in \mathbb{N}}$. Set $m_2 = a_1 \cdot \{n^{1 \cdot n+0}\}_{n \in \mathbb{N}}$. For every natural number x such that $x \in \text{dom } m_1$ holds $m_1(x) \leq m_2(x)$ by [19, (4)]. \Box

- (41) Let us consider finite 0-sequences c, a_1 of \mathbb{R} . Suppose $a_1 = |c|$. Let us consider a natural number n. Then $|(\operatorname{Seq}_{\operatorname{poly}}(c))(n)| \leq (\operatorname{Seq}_{\operatorname{poly}}(a_1))(n)$. PROOF: Define $\mathcal{P}[$ natural number $] \equiv$ for every finite 0-sequences c, a_1 of \mathbb{R} such that len $c = \$_1$ and $a_1 = |c|$ for every natural number x, $|(\operatorname{Seq}_{\operatorname{poly}}(c))(x)| \leq (\operatorname{Seq}_{\operatorname{poly}}(a_1))(x)$. $\mathcal{P}[0]$ by (26), [6, (44)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (28), [7, (47)], [15, (7)], [6, (56), (65)]. For every natural number $n, \mathcal{P}[n]$ from [1, Sch. 2]. \Box
- (42) Let us consider a real number a. Suppose 0 < a. Let us consider a natural number k, and a non-negative yielding finite 0-sequence d of \mathbb{R} . Suppose len d = k. Then there exists a natural number N such that for every natural number x such that $N \leq x$ for every natural number i such that $i \in \text{dom } d$ holds $d(i) \cdot x^i \cdot k \leq a \cdot x^k$.

PROOF: For every natural number *i* such that $i \in \text{dom } d$ holds $0 \leq d(i)$ by [7, (3)]. \Box

(43) Let us consider a natural number k, a finite 0-sequence d of \mathbb{R} , a real number a, and a sequence y of real numbers. Suppose 0 < a and len d = k and for every natural number $x, y(x) = a \cdot x^k$. Then there exists a natural number N such that for every natural number x such that $N \leq x$ holds

 $|(\text{Seq}_{\text{poly}}(d))(x)| \leq y(x)$. The theorem is a consequence of (38), (42), (26), (27), and (41).

(44) Let us consider a natural number k, and a finite 0-sequence d of \mathbb{R} . Suppose len d = k+1 and 0 < d(k). Then $\operatorname{Seq}_{\operatorname{poly}}(d)$ is eventually nonnegative. PROOF: Consider a being a real number, d_1 being a finite 0-sequence of \mathbb{R} , y being a sequence of real numbers such that len $d_1 = k$ and $d_1 = d \upharpoonright k$ and a = d(k) and $d = d_1 \land \langle a \rangle$ and $\operatorname{Seq}_{\operatorname{poly}}(d) = \operatorname{Seq}_{\operatorname{poly}}(d_1) + y$ and for every natural number i, $y(i) = a \cdot i^k$. Consider N being a natural number such that for every natural number i such that $N \leq i$ holds $|(\operatorname{Seq}_{\operatorname{poly}}(d_1))(i)| \leq y(i)$. For every natural number i such that $N \leq i$ holds $0 \leq (\operatorname{Seq}_{\operatorname{poly}}(d))(i)$ by [19, (4)], [15, (7)]. \Box

Let us consider a natural number k and a finite 0-sequence c of \mathbb{R} .

Let us assume that len c = k+1 and 0 < c(k). Now we state the propositions:

- (45) Seq_{poly}(c) $\in O(\{n^k\}_{n \in \mathbb{N}}).$
- (46) $\operatorname{Seq}_{\operatorname{poly}}(c)$ is polynomially bounded. The theorem is a consequence of (36) and (44).

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