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The Formal Construction of Fuzzy Numbers

Adam Grabowski
Institute of Informatics
University of Białystok
Akademicka 2, 15-267 Białystok
Poland

Summary. In this article, we continue the development of the theory of fuzzy sets [23], started with [14] with the future aim to provide the formalization of fuzzy numbers [8] in terms reflecting the current state of the Mizar Mathematical Library. Note that in order to have more usable approach in [14], we revised that article as well; some of the ideas were described in [12]. As we can actually understand fuzzy sets just as their membership functions (via the equality of membership function and their set-theoretic counterpart), all the calculations are much simpler. To test our newly proposed approach, we give the notions of (normal) triangular and trapezoidal fuzzy sets as the examples of concrete fuzzy objects. Also α-cuts, the core of a fuzzy set, and normalized fuzzy sets were defined. Main technical obstacle was to prove continuity of the glued maps, and in fact we did this not through its topological counterpart, but extensively reusing properties of the real line (with loss of generality of the approach, though), because we aim at formalizing fuzzy numbers in our future submissions, as well as merging with rough set approach as introduced in [13] and [11]. Our base for formalization was [9] and [10].

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The notation and terminology used in this paper have been introduced in the following articles: [16], [3], [4], [5], [14], [2], [19], [1], [6], [17], [21], [22], [20], and [7].

1. Preliminaries: Affine Maps

Now we state the proposition:

(1) Let us consider real numbers a, b. Suppose $a \leqslant b$. Then $\mathbb{R} \setminus]a, b[\neq \emptyset$.

From now on a, b, c, x denote real numbers.

Now we state the propositions:

(2) $(\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))(a) = 0$.

(3) If $b - a \neq 0$, then $(\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))(b) = 1$.

(4) If $c - b \neq 0$, then $(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))(b) = 1$.

(5) $(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))(c) = 0$.

(6) If $b-a \neq 0$ and $(\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))(x) = 1$, then $x = b$. The theorem is a consequence of (3).

(7) If $c-b \neq 0$ and $(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))(x) = 1$, then $x = b$. The theorem is a consequence of (4).

(8) $\mathrm{rng}(\mathrm{AffineMap}(0, a)) = \{a\}$.

(9) Let us consider a non empty subset C of $\mathbb{R}$. Then $\mathrm{rng}((\mathrm{AffineMap}(0, a)) \restriction C) = \{a\}$.

PROOF: Set $f = (\mathrm{AffineMap}(0, a)) \restriction C$. $\mathrm{rng}\, f \subseteq \{a\}$ by [3, (49)]. □

(10) If $b - a > 0$, then $\mathrm{rng}((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})) \restriction [a, b]) = [0, 1]$.

PROOF: Set $f = \mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})$. Set $g = f \restriction [a, b]$. $\mathrm{rng}\, g \subseteq [0, 1]$ by [21, (57)], [3, (47)], (2), [16, (53)]. □

Let us assume that $c - b > 0$. Now we state the propositions:

(11) $\mathrm{rng}((\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})) \restriction]b, c]) = [0, 1[$.

PROOF: Set $f = \mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})$. Set $g = f \restriction]b, c]$. $\mathrm{rng}\, g \subseteq [0, 1[$ by [21, (57)], [3, (47)], (4), [16, (52), (54)]. □

(12) $\mathrm{rng}((\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})) \restriction [b, c]) = [0, 1]$.

PROOF: Set $f = \mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})$. Set $g = f \restriction [b, c]$. $\mathrm{rng}\, g \subseteq [0, 1]$ by [21, (57)], [3, (47)], (4), [16, (54)]. □

Now we state the propositions:

(13) $(\mathrm{AffineMap}(0, 0))(x) \neq 1$.

(14) $(\mathrm{AffineMap}(0, 1))(b) = 1$.

(15) Let us consider a real number a. Then $(\mathrm{AffineMap}(0, b))(a) = b$.

2. Towards Development of Fuzzy Numbers

In the sequel C denotes a non empty set.

Let C be a non empty set.

A fuzzy set of C is a membership function of C. Let F be a fuzzy set of C. We say that F is normalized if and only if

(Def. 1) there exists an element x of C such that $F(x) = 1$.

We introduce F is normal as a synonym of F is normalized.

We introduce F is subnormal as an antonym for F is normal.

We say that F is strictly normalized if and only if

(Def. 2) there exists an element x of C such that $F(x) = 1$ and for every element y of C such that $F(y) = 1$ holds $y = x$.

One can verify that every fuzzy set of C which is strictly normalized is also normalized.

Let F be a fuzzy set of C and α be a real number. The functor α-cut(F) yielding a subset of C is defined by the term

(Def. 3) $\{x$, where x is an element of $C : F(x) \geqslant \alpha\}$.

Now we state the proposition:

(16) Let us consider a fuzzy set F of C, and a real number α. Then α-cut$(F) = F^{-1}([\alpha, 1])$.

PROOF: α-cut$(F) \subseteq F^{-1}([\alpha, 1])$ by [6, (4)]. $\square$

Let us consider C. Let us note that $\mathrm{UMF}\, C$ is normalized and there exists a fuzzy set of C which is normalized.

Let F be a fuzzy set of C. The functor $\mathrm{Core}\, F$ yielding a subset of C is defined by the term

(Def. 4) $\{x$, where x is an element of $C : F(x) = 1\}$.

Now we state the propositions:

(17) $\mathrm{Core}\,\mathrm{UMF}\, C = C$.

(18) $\mathrm{Core}\,\mathrm{EMF}\, C = \emptyset$.

Let us consider C. One can check that $\mathrm{Core}\,\mathrm{EMF}\, C$ is empty.

Let us consider a fuzzy set F of C. Now we state the propositions:

(19) $\mathrm{Core}\, F = F^{-1}(\{1\})$.

(20) $\mathrm{Core}\, F = 1$-cut(F). The theorem is a consequence of (16) and (19).

3. Convexity and the Height of a Fuzzy Set

Let F be a fuzzy set of $\mathbb{R}$. We say that F is convex if and only if

(Def. 5) for every real numbers x_1, x_2 and for every real number l such that $0 \leqslant l \leqslant 1$ holds $F(l \cdot x_1 + (1 - l) \cdot x_2) \geqslant \min(F(x_1), F(x_2))$.

Observe that UMF $\mathbb{R}$ is convex and EMF $\mathbb{R}$ is convex.

Let C be a non empty set and F be a fuzzy set of C. The functor height F yielding an extended real is defined by the term

(Def. 6) $\sup \operatorname{rng} F$.

Now we state the propositions:

(21) Let us consider a fuzzy set F of C. Then $0 \leqslant \operatorname{height} F \leqslant 1$.
PROOF: 0 is a lower bound of $\operatorname{rng} F$ by [15, (1)]. 1 is a upper bound of $\operatorname{rng} F$ by [15, (1)]. □

(22) Let us consider a fuzzy set F of C. If F is normalized, then $\operatorname{height} F = 1$. The theorem is a consequence of (21).

4. Pasting aka Glueing Lemmas

Let us consider partial functions f, g from $\mathbb{R}$ to $\mathbb{R}$. Now we state the proposition:

(23) Suppose f is continuous and g is continuous and there exists an object x such that $\operatorname{dom} f \cap \operatorname{dom} g = \{x\}$ and for every object x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$. Then there exists a partial function h from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) $h = f{+}\cdot g$, and

(ii) for every real number x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds h is continuous in x.

PROOF: Reconsider $h = f{+}\cdot g$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real number x_1 such that $x_1 \in \operatorname{dom} h$ and $|x_1 - x| < s$ holds $|h(x_1) - h(x)| < r$ by [21, (57)], [16, (3)], [5, (12)], [3, (47)]. □

Let us assume that f is continuous and non empty and g is continuous and non empty and there exist real numbers a, b, c such that $\operatorname{dom} f = [a, b]$ and $\operatorname{dom} g = [b, c]$ and $f \approx g$. Now we state the propositions:

(24) There exists a partial function h from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) $h = f{+}\cdot g$, and

(ii) for every real number x such that $x \in \operatorname{dom} h$ holds h is continuous in x.

(25) $f+\cdot g$ is continuous. The theorem is a consequence of (24).

Now we state the proposition:

(26) Suppose g is not empty and $f = (\text{AffineMap}(0,0))\restriction(\mathbb{R}\setminus]a,b[)$ and $\operatorname{dom} g = [a,b]$ and $g(a) = 0$ and $g(b) = 0$. Then $f \approx g$.
PROOF: For every object x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$ by [18, (1)], [3, (47)], (15). □

Let us assume that g is continuous and non empty and $f = (\text{AffineMap}(0,0))\restriction(\mathbb{R} \setminus]a,b[)$ and $\operatorname{dom} g = [a,b]$ and $g(a) = 0$ and $g(b) = 0$. Now we state the propositions:

(27) There exists a partial function h from $\mathbb{R}$ to $\mathbb{R}$ such that

(i) $h = f+\cdot g$, and

(ii) for every real number x such that $x \in \operatorname{dom} h$ holds h is continuous in x.

The theorem is a consequence of (26).

(28) $f+\cdot g$ is continuous. The theorem is a consequence of (27).

Note that there exists a subset of $\mathbb{R}$ which is non trivial, closed interval, and closed.

5. Triangular and Trapezoidal Fuzzy Sets

Let a, b, c be real numbers. Assume $a < b$ and $b < c$.

The functor TriangularFS(a,b,c) yielding a fuzzy set of $\mathbb{R}$ is defined by the term

(Def. 7) $((\text{AffineMap}(0,0))\restriction(\mathbb{R} \setminus]a,c[)+\cdot$
$(\text{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))\restriction[a,b])+\cdot$
$(\text{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))\restriction[b,c]$.

Let us consider real numbers a, b, c. Let us assume that $a < b < c$. Now we state the propositions:

(29) TriangularFS(a,b,c) is strictly normalized.
PROOF: Set $F = \text{TriangularFS}(a,b,c)$. Reconsider $b_1 = b$ as an element of $\mathbb{R}$. For every element y of $\mathbb{R}$ such that $F(y) = 1$ holds $y = b_1$ by [21, (57)], [5, (11), (13)], [3, (49)]. □

(30) TriangularFS(a,b,c) is continuous.
PROOF: Set $f_1 = \text{AffineMap}(0,0)$. Set $f = f_1\restriction(\mathbb{R} \setminus]a,c[)$. Set $g_1 = \text{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})$. Reconsider $g = g_1\restriction[a,b]$ as a partial function from

$\mathbb{R}$ to $\mathbb{R}$. Set $h_1 = \mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})$. Reconsider $h = h_1 \restriction [b, c]$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. For every object x such that $x \in \mathrm{dom}\, g \cap \mathrm{dom}\, h$ holds $g(x) = h(x)$ by [3, (49)], (4), (3). Set $\mathfrak{h} = g{+}{\cdot}h$. Consider h_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $h_2 = f{+}{\cdot}\mathfrak{h}$ and for every real number x such that $x \in \mathrm{dom}\, h_2$ holds h_2 is continuous in x. □

Let a, b, c, d be real numbers. Assume $a < b$ and $b < c$ and $c < d$. The functor TrapezoidalFS(a, b, c, d) yielding a fuzzy set of $\mathbb{R}$ is defined by the term

(Def. 8) $(((\mathrm{AffineMap}(0, 0)) \restriction (\mathbb{R} \setminus]a, d[) {+}{\cdot}$
$(\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})) \restriction [a, b]) {+}{\cdot}$
$(\mathrm{AffineMap}(0, 1)) \restriction [b, c]) {+}{\cdot} (\mathrm{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c})) \restriction [c, d]$.

Let us consider real numbers a, b, c, d. Let us assume that $a < b < c < d$. Now we state the propositions:

(31) TrapezoidalFS(a, b, c, d) is normalized. The theorem is a consequence of (4).

(32) TrapezoidalFS(a, b, c, d) is continuous.

Proof: Set $f_1 = \mathrm{AffineMap}(0, 0)$. Set $f = f_1 \restriction (\mathbb{R} \setminus]a, d[)$. Set $g_1 = \mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})$. Reconsider $g = g_1 \restriction [a, b]$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. Set $h_1 = \mathrm{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c})$. Reconsider $h = h_1 \restriction [c, d]$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. Set $i_1 = \mathrm{AffineMap}(0, 1)$. Reconsider $i = i_1 \restriction [b, c]$ as a partial function from $\mathbb{R}$ to $\mathbb{R}$. For every object x such that $x \in \mathrm{dom}\, g \cap \mathrm{dom}\, i$ holds $g(x) = i(x)$ by [3, (49)], (15), (3). Set $\mathfrak{h} = g{+}{\cdot}i$. $\mathfrak{h}$ is continuous. For every object x such that $x \in \mathrm{dom}\, \mathfrak{h} \cap \mathrm{dom}\, h$ holds $\mathfrak{h}(x) = h(x)$ by [5, (13)], [3, (49)], (15). Set $g_2 = \mathfrak{h}{+}{\cdot}h$. Consider h_2 being a partial function from $\mathbb{R}$ to $\mathbb{R}$ such that $h_2 = f{+}{\cdot}g_2$ and for every real number x such that $x \in \mathrm{dom}\, h_2$ holds h_2 is continuous in x. □

Let F be a fuzzy set of $\mathbb{R}$. We say that F is triangular if and only if

(Def. 9) there exist real numbers a, b, c such that $F = \mathrm{TriangularFS}(a, b, c)$.

We say that F is trapezoidal if and only if

(Def. 10) there exist real numbers a, b, c, d such that $F = \mathrm{TrapezoidalFS}(a, b, c, d)$.

One can verify that there exists a fuzzy set of $\mathbb{R}$ which is triangular and there exists a fuzzy set of $\mathbb{R}$ which is trapezoidal.

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