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# Formalization of Generalized Almost Distributive Lattices

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**Summary.** Almost Distributive Lattices (ADL) are structures defined by Swamy and Rao [14] as a common abstraction of some generalizations of the Boolean algebra. In our paper, we deal with a certain further generalization of ADLs, namely the Generalized Almost Distributive Lattices (GADL). Our main aim was to give the formal counterpart of this structure and we succeeded formalizing all items from the Section 3 of Rao et al.'s paper [13]. Essentially among GADLs we can find structures which are neither $\vee$-commutative nor $\wedge$-commutative (resp., $\wedge$-commutative); consequently not all forms of absorption identities hold.

We characterized some necessary and sufficient conditions for commutativity and distributivity, we also defined the class of GADLs with zero element. We tried to use as much attributes and cluster registrations as possible, hence many identities are expressed in terms of adjectives; also some generalizations of well-known notions from lattice theory [11] formalized within the Mizar Mathematical Library were proposed. Finally, some important examples from Rao's paper were introduced. We construct the example of GADL which is not an ADL. Mechanization of proofs in this specific area could be a good starting point towards further generalization of lattice theory [10] with the help of automated theorem provers [8].

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The notation and terminology used in this paper have been introduced in the




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following articles: [3], [15], [4], [5], [22], [16], [17], [6], [2], [19], [21], [9], [18], [1], and [7].

## 1. Preliminaries

Now we state the proposition:

(1) Let us consider a non empty 1-sorted structure $L$ and a total binary relation $R$ on the carrier of $L$. Then $R$ is reflexive if and only if for every element $x$ of $L$, $\langle x, x\rangle \in R$.
PROOF: If $R$ is reflexive, then for every element $x$ of $L$, $\langle x, x\rangle \in R$. For every object $x$ such that $x \in \operatorname{field} R$ holds $\langle x, x\rangle \in R$ by [20, (8)]. $\square$

One can check that every non empty lattice structure which is trivial is also distributive.

## 2. Almost Distributive Lattices

Let $L$ be a non empty lattice structure. We say that $L$ is right distributive over $\sqcup$ if and only if

(Def. 1) for every elements $x$, $y$, $z$ of $L$, $(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$.

We say that $L$ is right $\sqcup$-absorbing if and only if

(Def. 2) for every elements $x$, $y$ of $L$, $(x \sqcup y) \sqcap y = y$.

We say that $L$ is left $\sqcup$-absorbing if and only if

(Def. 3) for every elements $x$, $y$ of $L$, $(x \sqcup y) \sqcap x = x$.

Let us note that every non empty lattice structure which is trivial is also right distributive over $\sqcup$, right $\sqcup$-absorbing, left $\sqcup$-absorbing, and quasi-meet-absorbing and every non empty lattice structure which is trivial is also lattice-like. There exists a lattice which is trivial and there exists a non empty lattice structure which is right distributive over $\sqcup$, distributive, right $\sqcup$-absorbing, left $\sqcup$-absorbing, and quasi-meet-absorbing.

An almost distributive lattice is a right distributive over $\sqcup$, distributive, right $\sqcup$-absorbing, left $\sqcup$-absorbing, quasi-meet-absorbing, non empty lattice structure.

## 3. Properties of Almost Distributive Lattices

From now on $L$ denotes an almost distributive lattice and $x$, $y$, $z$ denote elements of $L$.

Now we state the propositions:

(2) $x \sqcup y = x$ if and only if $x \sqcap y = y$.

(3) $x \sqcup x = x$.

(4) $x \sqcap x = x$. The theorem is a consequence of (18).

(5) $(x \sqcap y) \sqcup y = y$. The theorem is a consequence of (19).

(6) $x \sqcup y = y$ if and only if $x \sqcap y = x$. The theorem is a consequence of (19) and (5).

(7) $x \sqcap (x \sqcup y) = x$. The theorem is a consequence of (19).

(8) $x \sqcup (y \sqcap x) = x$. The theorem is a consequence of (19).

(9) (i) $x \sqsubseteq x \sqcup y$, and

(ii) $x \sqcap y \sqsubseteq y$.

The theorem is a consequence of (7) and (5).

(10) $x \sqsubseteq y$ if and only if $x \sqcap y = x$.

(11) $x \sqcap (y \sqcap x) = y \sqcap x$. The theorem is a consequence of (5).

(12) $(x \sqcap y) \sqcup x = x$ if and only if $x \sqcap (y \sqcup x) = x$. The theorem is a consequence of (19).

(13) $(y \sqcap x) \sqcup y = y$ if and only if $y \sqcap (x \sqcup y) = y$.

(14) If $(x \sqcap y) \sqcup x = x$, then $x \sqcap y = y \sqcap x$. The theorem is a consequence of (31).

(15) If $x \sqcap (y \sqcup x) = x$, then $x \sqcup y = y \sqcup x$. The theorem is a consequence of (7).

(16) If there exists an element $z$ of $L$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$, then $x \sqcup y = y \sqcup x$. The theorem is a consequence of (19), (6), and (15).

(17) If $x \sqsubseteq y$, then $x \sqcup y = y \sqcup x$. The theorem is a consequence of (18) and (16).

## 4. Generalization of Almost Distributive Lattices

Let $L$ be a non empty lattice structure. We say that $L$ is left distributive over $\sqcap$ if and only if

(Def. 4) for every elements $x$, $y$, $z$ of $L$, $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

We say that $L$ is $\sqcup$-right-absorbing if and only if

(Def. 5) for every elements $x$, $y$ of $L$, $x \sqcap (y \sqcup x) = x$.

Let us note that every non empty lattice structure which is trivial is also meet-associative, distributive, left distributive over $\sqcap$, and left $\sqcup$-absorbing and there exists a non empty lattice structure which is meet-associative, distributive, left distributive over $\sqcap$, join-absorbing, left $\sqcup$-absorbing, and meet-absorbing.

A generalized almost distributive lattice is a meet-associative, distributive, left distributive over $\sqcap$, join-absorbing, left $\sqcup$-absorbing, meet-absorbing, non empty lattice structure. From now on $L$ denotes a generalized almost distributive lattice and $x$, $y$, $z$ denote elements of $L$.

Now we state the propositions:

(18) $x \sqcup x = x$.

(19) $x \sqcap x = x$. The theorem is a consequence of (18).

(20) $x \sqcup (x \sqcap y) = x$. The theorem is a consequence of (18).

(21) $x \sqcup (y \sqcap x) = x$. The theorem is a consequence of (18).

(22) If $x \sqcap y = y$, then $x \sqcup y = x$.

(23) $x \sqcup y = y$ if and only if $x \sqcap y = x$.

## 5. Order Properties of the Generated Relation on GADLs

Now we state the propositions:

(24) $x \sqsubseteq x$. The theorem is a consequence of (19).

(25) If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$.

Let $L$ be a non empty lattice structure. The functor $\leqslant_L$ yielding a binary relation is defined by the term

(Def. 6) $\{\langle a, b\rangle$, where $a$, $b$ are elements of $L : a \sqsubseteq b\}$.

Now we state the proposition:

(26) (i) dom $\leqslant_L =$ the carrier of $L$, and

(ii) rng $\leqslant_L =$ the carrier of $L$, and

(iii) field $\leqslant_L =$ the carrier of $L$.

The theorem is a consequence of (24).

Let us consider $L$. Observe that the functor $\leqslant_L$ yields a binary relation on the carrier of $L$. One can check that $\leqslant_L$ is total as a binary relation on the carrier of $L$.

Now we state the proposition:

(27) $\langle x, y\rangle \in \leqslant_L$ if and only if $x \sqsubseteq y$.

Let $L$ be a non empty lattice structure. The functor $\Theta_L$ yielding a binary relation is defined by the term

(Def. 7) $\{\langle a, b\rangle$, where $a$, $b$ are elements of $L : a \sqcap b = b\}$.

Now we state the proposition:

(28) (i) dom $\Theta_L =$ the carrier of $L$, and

(ii) rng $\Theta_L =$ the carrier of $L$, and

(iii) field $\Theta_L =$ the carrier of $L$.

The theorem is a consequence of (19).

Let us consider $L$. Let us note that the functor $\Theta_L$ yields a binary relation on the carrier of $L$. One can verify that $\Theta_L$ is total as a binary relation on the carrier of $L$.

Now we state the proposition:

(29) $\langle x, y \rangle \in \Theta_L$ if and only if $x \sqcap y = y$.

Let us consider $L$. Let us note that $\leqslant_L$ is reflexive and $\leqslant_L$ is transitive and $\Theta_L$ is reflexive and $\Theta_L$ is transitive.

## 6. Formalization of [13] paper

Now we state the propositions:

(30) $x \sqcup (x \sqcup y) = x \sqcup y$.

(31) $x \sqcap (y \sqcap x) = y \sqcap x$.

(32) $y \sqcap (x \sqcap y) = x \sqcap y$.

Let us consider $L$. Let $a$, $b$ be elements of $L$. We say that there exists the least upper bound of $a$ and $b$ if and only if

(Def. 8) there exists an element $c$ of $L$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$ and for every element $x$ of $L$ such that $a \sqsubseteq x$ and $b \sqsubseteq x$ holds $c \sqsubseteq x$.

We say that there exists the greatest lower bound of $a$ and $b$ if and only if

(Def. 9) there exists an element $c$ of $L$ such that $c \sqsubseteq a$ and $c \sqsubseteq b$ and for every element $x$ of $L$ such that $x \sqsubseteq a$ and $x \sqsubseteq b$ holds $x \sqsubseteq c$.

Assume there exists the least upper bound of $a$ and $b$. The functor $\mathrm{lub}\{a, b\}$ yielding an element of $L$ is defined by

(Def. 10) $a \sqsubseteq \mathit{it}$ and $b \sqsubseteq \mathit{it}$ and for every element $x$ of $L$ such that $a \sqsubseteq x$ and $b \sqsubseteq x$ holds $\mathit{it} \sqsubseteq x$.

Assume there exists the greatest lower bound of $a$ and $b$. The functor $\mathrm{glb}\{a, b\}$ yielding an element of $L$ is defined by

(Def. 11) $\mathit{it} \sqsubseteq a$ and $\mathit{it} \sqsubseteq b$ and for every element $x$ of $L$ such that $x \sqsubseteq a$ and $x \sqsubseteq b$ holds $x \sqsubseteq \mathit{it}$.

Now we state the propositions:

(33) $(x \sqcap y) \sqcup x = x$ if and only if $x \sqcap (y \sqcup x) = x$.

(34) $(x \sqcap y) \sqcup x = x$ if and only if $(y \sqcap x) \sqcup y = y$.

(35) $(x \sqcap y) \sqcup x = x$ if and only if $y \sqcap (x \sqcup y) = y$.

(36) $(x \sqcap y) \sqcup x = x$ if and only if $x \sqcap y = y \sqcap x$.

(37) $(x \sqcap y) \sqcup x = x$ if and only if $x \sqcup y = y \sqcup x$.

(38) $x \sqsubseteq y$ if and only if $x \sqcap y = x$.

(39) $x \sqcup y = y \sqcup x$ if and only if $y \sqsubseteq x \sqcup y$.

(40) $x \sqcup y = y \sqcup x$ if and only if there exists $z$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$.

(41) $x \sqcup y = y \sqcup x$ if and only if there exists the least upper bound of $x$ and $y$ and $x \sqcup y = \text{lub}\{x, y\}$.

(42) $x \sqcup y = y \sqcup x$ if and only if $x \sqsubseteq y \sqcup x$.

(43) $x \sqcup y = y \sqcup x$ if and only if there exists the least upper bound of $x$ and $y$ and $y \sqcup x = \text{lub}\{x, y\}$.

(44) If $x \sqcap y \sqsubseteq x$, then there exists $z$ such that $z \sqsubseteq x$ and $z \sqsubseteq y$.

(45) $x \sqcap y = y \sqcap x$ if and only if $y \sqcap x \sqsubseteq y$.

(46) $x \sqcap y = y \sqcap x$ if and only if there exists the greatest lower bound of $x$ and $y$ and $y \sqcap x = \text{glb}\{x, y\}$. The theorem is a consequence of (45).

(47) $x \sqcap y = y \sqcap x$ if and only if $x \sqcap y \sqsubseteq x$.

(48) $x \sqcap y = y \sqcap x$ if and only if there exists the greatest lower bound of $x$ and $y$ and $x \sqcap y = \text{glb}\{x, y\}$.

(49) $(x \sqcap y) \sqcap z = (y \sqcap x) \sqcap z$. The theorem is a consequence of (31).

Let $L$ be a generalized almost distributive lattice. The functor $\langle L, \leqslant_L \rangle$ yielding a strict relational structure is defined by the term

(Def. 12) $\langle$the carrier of $L, \leqslant_L\rangle$.

Note that $\langle L, \leqslant_L \rangle$ is reflexive and transitive.

Now we state the propositions:

(50) Let us consider elements $a$, $b$ of $L$ and elements $x$, $y$ of $\langle L, \leqslant_L \rangle$. If $a = x$ and $b = y$, then $x \leqslant y$ iff $a \sqsubseteq b$.

(51) $L$ is join-commutative if and only if $L$ is lattice-like and distributive.

(52) $L$ is join-commutative if and only if $\langle L, \leqslant_L \rangle$ is directed. The theorem is a consequence of (27).

(53) $L$ is join-commutative if and only if $L$ is $\sqcup$-right-absorbing.

(54) $L$ is join-commutative if and only if $L$ is meet-commutative.

(55) $L$ is join-commutative if and only if $\Theta_L$ is antisymmetric.

PROOF: If $L$ is join-commutative, then $\Theta_L$ is antisymmetric by (29), [12, (31)]. For every elements $x$, $y$ of $L$, $x \sqcap y = y \sqcap x$ by (49), (19), [12, (31)]. □

(56) $L$ is join-commutative if and only if $\Theta_L$ is a partial-order. The theorem is a consequence of (55).

Let $L$ be a join-commutative generalized almost distributive lattice. Let us note that $\Theta_L$ is antisymmetric and every generalized almost distributive lattice which is join-commutative is also $\sqcup$-right-absorbing and every generalized almost distributive lattice which is $\sqcup$-right-absorbing is also join-commutative. Every generalized almost distributive lattice which is join-commutative is also meet-commutative and every generalized almost distributive lattice which is meet-commutative is also join-commutative.

Now we state the propositions:

(57) Suppose for every elements $a$, $b$, $c$ of $L$, $(a \sqcup b) \sqcap c = (b \sqcup a) \sqcap c$. Let us consider elements $a$, $b$, $c$ of $L$. Then $(a \sqcup b) \sqcap c = (a \sqcap c) \sqcup (b \sqcap c)$. The theorem is a consequence of (30).

(58) If for every elements $a$, $b$, $c$ of $L$, $(a \sqcup b) \sqcap c = (a \sqcap c) \sqcup (b \sqcap c)$, then for every elements $a$, $b$ of $L$, $(a \sqcup b) \sqcap b = b$. The theorem is a consequence of (19).

(59) If for every elements $a$, $b$ of $L$, $(a \sqcup b) \sqcap b = b$, then for every elements $a$, $b$, $c$ of $L$, $(a \sqcup b) \sqcap c = (b \sqcup a) \sqcap c$. The theorem is a consequence of (31) and (19).

## 7. Generalized Almost Distributive Lattices with Zero

Let us consider $L$. We say that $L$ has zero if and only if

(Def. 13) there exists an element $x$ of $L$ such that for every element $a$ of $L$, $x \sqcap a = x$.

One can check that every non empty generalized almost distributive lattice which is trivial has also zero and there exists a non empty generalized almost distributive lattice which has zero.

Let us consider $L$. Assume $L$ has zero. The functor $0_L$ yielding an element of $L$ is defined by

(Def. 14) for every element $a$ of $L$, $\mathit{it} \sqcap a = \mathit{it}$.

From now on $L$ denotes a generalized almost distributive lattice with zero and $x$, $y$ denote elements of $L$.

Now we state the propositions:

(60) $x \sqcup 0_L = x$. The theorem is a consequence of (49) and (37).

(61) $0_L \sqcup x = x$.

(62) $x \sqcap 0_L = 0_L$. The theorem is a consequence of (49).

(63) $x \sqcap y = 0_L$ if and only if $y \sqcap x = 0_L$. The theorem is a consequence of (19) and (49).

(64) If $x \sqcap y = 0_L$, then $x \sqcup y = y \sqcup x$. The theorem is a consequence of (63) and (37).

## 8. Constructing Examples of Almost Distributive Lattices

Let $x$, $y$ be elements of $\{1, 2, 3\}$. The functors: $x \sqcap_{\rm GAD} y$ and $x \sqcup_{\rm GAD} y$ yielding elements of $\{1, 2, 3\}$ are defined by terms

(Def. 15) $\begin{cases} 1, & \textbf{if } y = 1 \text{ or } y = 2 \text{ and } (x = 1 \text{ or } x = 3), \\ 2, & \textbf{if } x = 2 \text{ and } y = 2, \\ 3, & \textbf{if } y = 3, \end{cases}$

(Def. 16) $\begin{cases} 1, & \textbf{if } x = 1 \text{ and } (y = 1 \text{ or } y = 3), \\ 2, & \textbf{if } x = 2 \text{ or } x = 1 \text{ and } y = 2, \\ 3, & \textbf{if } x = 3, \end{cases}$

respectively. The functors: $\cup_{\rm GAD}$ and $\cap_{\rm GAD}$ yielding binary operations on $\{1, 2, 3\}$ are defined by conditions

(Def. 17) for every elements $x$, $y$ of $\{1, 2, 3\}$, $\cup_{\rm GAD}(x, y) = x \sqcup_{\rm GAD} y$,

(Def. 18) for every elements $x$, $y$ of $\{1, 2, 3\}$, $\cap_{\rm GAD}(x, y) = x \sqcap_{\rm GAD} y$,

respectively. Now we state the proposition:

(65) There exists a non empty lattice structure $L$ such that

- (i) for every element $x$ of $L$, $x = 1$ or $x = 2$ or $x = 3$, and
- (ii) for every elements $x$, $y$ of $L$, ($x \sqcap y = 1$ iff $y = 1$ or $y = 2$ and ($x = 1$ or $x = 3$)) and ($x \sqcap y = 2$ iff $x = 2$ and $y = 2$) and ($x \sqcap y = 3$ iff $y = 3$), and
- (iii) for every elements $x$, $y$ of $L$, ($x \sqcup y = 1$ iff $x = 1$ and ($y = 1$ or $y = 3$)) and ($x \sqcup y = 2$ iff $x = 2$ or $x = 1$ and $y = 2$) and ($x \sqcup y = 3$ iff $x = 3$), and
- (iv) $L$ is a generalized almost distributive lattice, and
- (v) $L$ is not an almost distributive lattice.

Let $x$, $y$ be elements of $\{1, 2, 3\}$. The functors: $x \sqcap_{\rm GADL} y$ and $x \sqcup_{\rm GADL} y$ yielding elements of $\{1, 2, 3\}$ are defined by terms

(Def. 19) $\begin{cases} 1, & \textbf{if } x = 1 \text{ and } y = 1, \\ 2, & \textbf{if } y = 2 \text{ or } y = 1 \text{ and } (x = 2 \text{ or } x = 3), \\ 3, & \textbf{if } y = 3, \end{cases}$

(Def. 20) $\begin{cases} 1, & \textbf{if } x = 1 \text{ or } x = 2 \text{ and } y = 1, \\ 2, & \textbf{if } x = 2 \text{ and } (y = 2 \text{ or } y = 3), \\ 3, & \textbf{if } x = 3, \end{cases}$

respectively. The functors: $\cup_{\rm GADL}$ and $\cap_{\rm GADL}$ yielding binary operations on $\{1, 2, 3\}$ are defined by conditions

(Def. 21) for every elements $x$, $y$ of $\{1, 2, 3\}$, $\cup_{\rm GADL}(x, y) = x \sqcup_{\rm GADL} y$,

(Def. 22) for every elements $x$, $y$ of $\{1, 2, 3\}$, $\cap_{\rm GADL}(x, y) = x \sqcap_{\rm GADL} y$, respectively. Now we state the proposition:

(66) There exists a non empty lattice structure $L$ such that

(i) for every element $x$ of $L$, $x = 1$ or $x = 2$ or $x = 3$, and

(ii) for every elements $x$, $y$ of $L$, ($x \sqcap y = 1$ iff $x = 1$ and $y = 1$) and ($x \sqcap y = 2$ iff $y = 2$ or $y = 1$ and ($x = 2$ or $x = 3$)) and ($x \sqcap y = 3$ iff $y = 3$), and

(iii) for every elements $x$, $y$ of $L$, ($x \sqcup y = 1$ iff $x = 1$ or $x = 2$ and $y = 1$) and ($x \sqcup y = 2$ iff $x = 2$ and ($y = 2$ or $y = 3$)) and ($x \sqcup y = 3$ iff $x = 3$), and

(iv) $L$ is a generalized almost distributive lattice.

Let $L$ be a non empty lattice structure.

A sublattice structure of $L$ is a lattice structure and is defined by

(Def. 23) the carrier of *it* $\subseteq$ the carrier of $L$ and the join operation of *it* = (the join operation of $L$) $\upharpoonright$ (the carrier of *it*) and the meet operation of *it* = (the meet operation of $L$) $\upharpoonright$ (the carrier of *it*).

Let us note that there exists a sublattice structure of $L$ which is strict.

Let $S$ be a subset of $L$. We say that $S$ is meet-closed if and only if

(Def. 24) for every elements $p$, $q$ of $L$ such that $p, q \in S$ holds $p \sqcap q \in S$.

We say that $S$ is join-closed if and only if

(Def. 25) for every elements $p$, $q$ of $L$ such that $p, q \in S$ holds $p \sqcup q \in S$.

One can verify that there exists a subset of $L$ which is meet-closed, join-closed, and non empty.

A closed subset of $L$ is a meet-closed, join-closed subset of $L$. Let $P$ be a closed subset of $L$. The functor $\mathbb{L}_P^L$ yielding a strict sublattice structure of $L$ is defined by

(Def. 26) the carrier of *it* $= P$.

Let $S$ be a non empty closed subset of $L$. Note that $\mathbb{L}_S^L$ is non empty and there exists a sublattice structure of $L$ which is non empty.

Let us consider a non empty lattice structure $L$, a non empty sublattice structure $S$ of $L$, elements $x_1$, $x_2$ of $L$, and elements $y_1$, $y_2$ of $S$.

Let us assume that $x_1 = y_1$ and $x_2 = y_2$. Now we state the propositions:

(67) $x_1 \sqcup x_2 = y_1 \sqcup y_2$.

(68) $x_1 \sqcap x_2 = y_1 \sqcap y_2$.

Now we state the propositions:

(69) Let us consider a non empty lattice structure $L$ and a non empty closed subset $S$ of $L$. Then

(i) if $L$ is meet-associative, then $\mathbb{L}_S^L$ is meet-associative, and

(ii) if $L$ is meet-absorbing, then $\mathbb{L}_S^L$ is meet-absorbing, and

(iii) if $L$ is meet-commutative, then $\mathbb{L}_S^L$ is meet-commutative, and

(iv) if $L$ is join-associative, then $\mathbb{L}_S^L$ is join-associative, and

(v) if $L$ is join-absorbing, then $\mathbb{L}_S^L$ is join-absorbing, and

(vi) if $L$ is join-commutative, then $\mathbb{L}_S^L$ is join-commutative, and

(vii) if $L$ is left $\sqcup$-absorbing, then $\mathbb{L}_S^L$ is left $\sqcup$-absorbing, and

(viii) if $L$ is distributive, then $\mathbb{L}_S^L$ is distributive, and

(ix) if $L$ is left distributive over $\sqcap$, then $\mathbb{L}_S^L$ is left distributive over $\sqcap$.

The theorem is a consequence of (68) and (67).

(70) Let us consider an element $a$ of $L$ and a set $X$. Suppose $X = \{x \sqcap a$, where $x$ is an element of $L\}$. Then

(i) $X = \{x$, where $x$ is an element of $L : x \sqsubseteq a\}$, and

(ii) $X$ is a closed subset of $L$.

(71) Let us consider an element $a$ of $L$, a non empty closed subset $S$ of $L$, and an element $b$ of $\mathbb{L}_S^L$. Suppose $b = a$ and $S = \{x \sqcap a$, where $x$ is an element of $L\}$. Then

(i) $\mathbb{L}_S^L$ is lattice-like and distributive, and

(ii) for every element $c$ of $\mathbb{L}_S^L$, $b \sqcup c = b$ and $c \sqcup b = b$ and $c \sqsubseteq b$.

The theorem is a consequence of (68), (49), (69), (51), (67), and (21).

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