Algebraic Approach to Algorithmic Logic

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Summary. We introduce algorithmic logic – an algebraic approach according to [25]. It is done in three stages: propositional calculus, quantifier calculus with equality, and finally proper algorithmic logic. For each stage appropriate signature and theory are defined. Propositional calculus and quantifier calculus with equality are explored according to [24]. A language is introduced with language signature including free variables, substitution, and equality. Algorithmic logic requires a bialgebra structure which is an extension of language signature and program algebra. While-if algebra of generator set and algebraic signature is bialgebra with appropriate properties and is used as basic type of algebraic logic.

The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [15], [3], [4], [5], [12], [16], [22], [26], [29], [17], [18], [13], [6], [34], [27], [31], [9], [20], [32], [35], [33], [23], [14], and [21].

1. Algorithmic Langugage Signature

From now on $\mathcal{X}$, Y denote sets, x, y, z denote objects, and i, j, n denote natural numbers.

Let f be a non empty yielding function. One can check that $\bigcup f$ is non empty.

Let I be a set, f be a many sorted set indexed by I, and i be a set. Let us consider x. One can verify that the functor $f +\cdot (i, x)$ yields a many sorted set indexed by I. Let f be a non-empty many sorted set indexed by I and x be a non empty set. One can check that $f +\cdot (i, x)$ is non-empty.

Let Σ be a non empty, non void many sorted signature. Let us observe that there exists a strict free variable algebra over Σ which is non-empty.

Let f, g be functions. We say that g is f-tolerating if and only if

(Def. 1) $f \approx g$.

Now we state the propositions:

(1)   Let us consider functions f, g. Then g is f-tolerating if and only if for every x such that $x \in \operatorname{dom} f$ and $x \in \operatorname{dom} g$ holds $f(x) = g(x)$.

(2)   Let us consider sets I, Ω, a many sorted set f indexed by I, and a many sorted set g indexed by Ω. Then g is f-tolerating if and only if for every x such that $x \in I$ and $x \in \Omega$ holds $f(x) = g(x)$. The theorem is a consequence of (1).

(3)   Let us consider functions f, g. Then $f \approx g + \cdot f$.

Let $\mathcal{X}$, Y be functions. Note that $Y + \cdot \mathcal{X}$ is $\mathcal{X}$-tolerating.

Let $\mathcal{X}$ be a function, Ω be a set, and Y be a many sorted set indexed by Ω. Observe that $Y + \cdot \mathcal{X} \restriction \Omega$ is $\mathcal{X}$-tolerating and there exists a many sorted set indexed by Ω which is $\mathcal{X}$-tolerating.

Let $\mathcal{X}$ be a non-empty function. Note that there exists a non-empty many sorted set indexed by Ω which is $\mathcal{X}$-tolerating.

Let I be a non empty set and $\mathcal{X}$ be a non empty yielding many sorted set indexed by I. One can verify that $\bigcup \mathcal{X}$ is non empty.

Now we state the propositions:

(4)   Let us consider a non empty, non void many sorted signature Σ, an operation symbol o of Σ, a sort symbol r of Σ, and an algebra $\mathfrak{T}$ over Σ. If o is of type $\emptyset \to r$, then $\emptyset \in \operatorname{Args}(o, \mathfrak{T})$.

(5)   Let us consider a non empty, non void many sorted signature Σ, an operation symbol o of Σ, sort symbols σ, r of Σ, and an algebra $\mathfrak{T}$ over Σ. Suppose o is of type $\langle \sigma \rangle \to r$ and $x \in$ (the sorts of $\mathfrak{T}$)(σ). Then $\langle x \rangle \in \operatorname{Args}(o, \mathfrak{T})$.

(6)   Let us consider a non empty, non void many sorted signature Σ, an operation symbol o of Σ, sort symbols σ_1, σ_2, r of Σ, and an algebra $\mathfrak{T}$ over Σ. Suppose o is of type $\langle \sigma_1, \sigma_2 \rangle \to r$ and $x \in$ (the sorts of $\mathfrak{T}$)(σ_1) and $y \in$ (the sorts of $\mathfrak{T}$)(σ_2). Then $\langle x, y \rangle \in \operatorname{Args}(o, \mathfrak{T})$.

(7)   Let us consider a non empty, non void many sorted signature Σ, an operation symbol o of Σ, sort symbols σ_1, σ_2, σ_3, r of Σ, and an algebra $\mathfrak{T}$ over Σ. Suppose o is of type $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \to r$ and $x \in$ (the sorts of $\mathfrak{T}$)(σ_1) and $y \in$ (the sorts of $\mathfrak{T}$)(σ_2) and $z \in$ (the sorts of $\mathfrak{T}$)(σ_3). Then $\langle x, y, z \rangle \in \operatorname{Args}(o, \mathfrak{T})$.

Let Σ, E be signatures. We say that E is Σ-extension if and only if

(Def. 2) Σ is a subsignature of E.

Let Σ be a signature. Let us observe that every extension of Σ is Σ-extension. Now we state the propositions:

(8)   Let us consider non empty signatures Σ, E. Suppose E is Σ-extension. Then every sort symbol of Σ is a sort symbol of E.

(9)   Let us consider non void signatures Σ, E. Suppose E is Σ-extension. Let us consider an operation symbol o of Σ, a set a, an element r of Σ, and an element r_1 of E. If $r = r_1$ and o is of type $a \to r$, then o is of type $a \to r_1$.

Let $\mathcal{X}$ be a function and Ω, Y be sets. The functor $\mathcal{X}$ extended by(Y, Ω) yielding a many sorted set indexed by Ω is defined by the term

(Def. 3) $(\Omega \longmapsto Y) + \cdot \mathcal{X} {\restriction} \Omega$.

One can check that $\mathcal{X}$ extended by(Y, Ω) is $\mathcal{X}$-tolerating.

We consider PC language signatures which extend connectives signatures and are systems

$$\langle \text{a carrier, a carrier', an arity, a result sort,}$$

$$\text{a formula sort, connectives} \rangle$$

where the carrier is a set, the carrier' is a set, the arity is a function from the carrier' into (the carrier)*, the result sort is a function from the carrier' into the carrier, the formula sort is an element of the carrier, the connectives constitute a finite sequence of elements of the carrier'.

Let $\mathcal{X}$ be a set. We consider QC language signatures over $\mathcal{X}$ which extend PC language signatures and are systems

$$\langle \text{a carrier, a carrier', an arity, a result sort,}$$

$$\text{a formula sort, connectives, a quantifier sort, quantifiers} \rangle$$

where the carrier is a set, the carrier' is a set, the arity is a function from the carrier' into (the carrier)*, the result sort is a function from the carrier' into the carrier, the formula sort is an element of the carrier, the connectives constitute a finite sequence of elements of the carrier', the quantifier sort is a set, the quantifiers constitute a function from (the quantifier sort) $\times \mathcal{X}$ into the carrier'.

We consider algorithmic language signatures of $\mathcal{X}$ which extend QC language signatures over $\mathcal{X}$ and are systems

$$\langle \text{a carrier, a carrier', an arity, a result sort,}$$

$$\text{a formula sort, a program sort, connectives, a quantifier sort, quantifiers} \rangle$$

where the carrier is a set, the carrier' is a set, the arity is a function from the carrier' into (the carrier)*, the result sort is a function from the carrier' into the carrier, the formula sort and the program sort are elements of the carrier, the connectives constitute a finite sequence of elements of the carrier', the quantifier sort is a set, the quantifiers constitute a function from (the quantifier sort) $\times \mathcal{X}$ into the carrier'.

Let n be a natural number and L be a PC language signature. We say that L is PC correct w.r.t. n if and only if

(Def. 4) len(the connectives of L) $\geqslant n+5$ and (the connectives of L)$\upharpoonright\{n, n+1, n+2, n+3, n+4, n+5\}$ is one-to-one and (the connectives of L)(n) is of type $\langle$the formula sort of $L\rangle \to$ the formula sort of L and (the connectives of L)$(n+5)$ is of type $\emptyset \to$ the formula sort of L and (the connectives of L)$(n+1)$ is of type $\langle$the formula sort of L, the formula sort of $L\rangle \to$ the formula sort of L and ... and (the connectives of L)$(n+4)$ is of type $\langle$the formula sort of L, the formula sort of $L\rangle \to$ the formula sort of L.

Let us consider $\mathcal{X}$. Let L be a QC language signature over $\mathcal{X}$. We say that L is QC correct if and only if

(Def. 5) the quantifier sort of $L = \{1, 2\}$ and the quantifiers of L is one-to-one and rng(the quantifiers of L) misses rng(the connectives of L) and for every objects q, x such that $q \in$ the quantifier sort of L and $x \in \mathcal{X}$ holds (the quantifiers of L)(q, x) is of type $\langle$the formula sort of $L\rangle \to$ the formula sort of L.

Let n be a natural number, $\mathcal{X}$ be a set, and L be an algorithmic language signature of $\mathcal{X}$. We say that L is AL correct w.r.t. n if and only if

(Def. 6) the program sort of $L \neq$ the formula sort of L and len(the connectives of L) $\geqslant n+8$ and (the connectives of L)$(n+6)$ is of type $\langle$the program sort of L, the formula sort of $L\rangle \to$ the formula sort of L and ... and (the connectives of L)$(n+8)$ is of type $\langle$the program sort of L, the formula sort of $L\rangle \to$ the formula sort of L.

Let us consider n. Let us note that every PC language signature which is PC correct w.r.t. n is also non void.

Let $\mathcal{X}$, Y be sets. Assume $Y \subseteq \mathcal{X}$. The functor incl$(Y, \mathcal{X})$ yielding a function from Y into $\mathcal{X}$ is defined by the term

(Def. 7) id$_Y$.

Let n be a non empty, natural number and $\mathcal{X}$ be a set. Let us observe that there exists a QC language signature over $\mathcal{X}$ which is non void, non empty, PC correct w.r.t. n, and QC correct and there exists a PC language signature which is non void, non empty, and PC correct w.r.t. n.

Let $\mathcal{X}$ be a set. Note that there exists a strict algorithmic language signature

of $\mathcal{X}$ which is non void and non empty and every set which is ordinal is also non pair.

(10) Let us consider an ordinal number a and natural numbers n_1, n_2. If $n_1 \neq n_2$, then $a + n_1 \neq a + n_2$.

Let R be a non empty binary relation. Observe that every element of R is pair.

(11) Let us consider a non empty, natural number n, a non empty set $\mathcal{X}$, and a signature Ω. Then there exists a strict, non void, non empty algorithmic language signature Σ of $\mathcal{X}$ such that

(i) Σ is PC correct w.r.t. n, QC correct, AL correct w.r.t. n, and Ω-extension, and

(ii) for every i such that $i = 0$ or ... or $i = 8$ holds (the connectives of Σ)$(n + i) = \sup$(the carrier' of Ω) $+ i$, and

(iii) for every element x of $\mathcal{X}$, (the quantifiers of Σ)$(1, x) = \langle$the carrier' of Ω, 1, $x\rangle$ and (the quantifiers of Σ)$(2, x) = \langle$the carrier' of Ω, 2, $x\rangle$, and

(iv) the formula sort of $\Sigma = \sup$(the carrier of Ω), and

(v) the program sort of $\Sigma = \sup$(the carrier of Ω) $+ 1$, and

(vi) the carrier of $\Sigma = ($the carrier of $\Omega) \cup \{$the formula sort of Σ, the program sort of $\Sigma\}$, and

(vii) for every ordinal number w such that $w = \sup$(the carrier' of Ω) holds the carrier' of $\Sigma = (($the carrier' of $\Omega) \cup \{w+0, w+1, w+2, w+3, w+4, w+5, w+6, w+7, w+8\}) \cup (\{$the carrier' of $\Omega\} \times \{1, 2\} \times \mathcal{X})$.

PROOF: Set $w = \sup$(the carrier' of Ω). Set $u = \sup$(the carrier' of Ω). Set $O_1 = \{w+0, w+1, w+2, w+3, w+4, w+5, w+6, w+7, w+8\} \cup (\{$the carrier' of $\Omega\} \times \{1, 2\} \times \mathcal{X})$. Set $O = ($the carrier' of $\Omega) \cup O_1$. Set $a = (((\{w+1, w+2, w+3, w+4\} \longmapsto \langle u+0, u+0\rangle) \cup (\{w+0\} \cup (\{$the carrier' of $\Omega\} \times \{1, 2\} \times \mathcal{X}) \longmapsto \langle u+0\rangle)) \cup (\{w+5\} \longmapsto \emptyset)) \cup (\{w+6, w+7, w+8\} \longmapsto \langle u+1, u+0\rangle)$. Set $r = O_1 \longmapsto u+0$. Set $r_2 = ($the result sort of $\Omega)+\cdot r$. $\{w+0, w+1, w+2, w+3, w+4, w+5, w+6, w+7, w+8\}$ misses $\{$the carrier' of $\Omega\} \times \{1, 2\} \times \mathcal{X}$ by [30, (68)]. $\{w+1, w+2, w+3, w+4\}$ misses $\{w+0\} \cup (\{$the carrier' of $\Omega\} \times \{1, 2\} \times \mathcal{X})$ by [11, (21)], [30, (68)]. $\{w+0, w+1, w+2, w+3, w+4\} \cup (\{$the carrier' of $\Omega\} \times \{1, 2\} \times \mathcal{X})$ misses $\{w+5\}$ by [11, (21)], [30, (68)]. $\{w+0, w+1, w+2, w+3, w+4, w+5\} \cup (\{$the carrier' of $\Omega\} \times \{1, 2\} \times \mathcal{X})$ misses $\{w+6, w+7, w+8\}$ by [11, (21)], [30, (68)]. O_1 misses the carrier' of Ω by [10, (19), (27)], [30, (68)], [11, (22)]. Reconsider $a_1 = ($the arity of $\Omega) \cup a$ as a function. Set $\gamma = ($the carrier of $\Omega) \cup \{u+0, u+1\}$. Reconsider $0_0 = u+0$, $0_1 = u+1$ as

an element of γ. Reconsider $o_0 = w+0$, $o_1 = w+1$, $o_2 = w+2$, $o_3 = w+3$, $o_4 = w+4$, $o_5 = w+5$, $o_6 = w+6$, $o_7 = w+7$, $o_8 = w+8$ as an element of O. Set $p = $ the $(n -' 1$ **qua** natural number)-element finite sequence of elements of $\{w+0, w+1, w+2, w+3, w+4, w+5, w+6, w+7, w+8\}$. Set $c' = \langle o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \rangle ^\frown \langle o_8 \rangle$. Reconsider $c = p ^\frown c'$ as a finite sequence of elements of O. If $i = 0$ or ... or $i = 8$, then $c(n+i) = w+i$ by [2, (29)]. Define $\mathcal{Q}(\text{object}) = \langle$the carrier' of Ω, $\$_{11}$, $\$_{12}\rangle$. Consider q being a function such that $\mathrm{dom}\, q = \{1,2\} \times \mathcal{X}$ and for every x such that $x \in \{1,2\} \times \mathcal{X}$ holds $q(x) = \mathcal{Q}(x)$ from [17, Sch. 3]. $\mathrm{rng}\, q \subseteq O$ by [30, (69)]. Set $L = \langle \gamma, O, a_1, r_2, 0_0, 0_1, c, \{1,2\}, q \rangle$. Set $N = \{n, n+1, n+2, n+3, n+4, n+5\}$. (The connectives of $L){\upharpoonright}N$ is one-to-one by [35, (57)], [17, (49)], [11, (21)]. (the connectives of $L)(n)$ is of type $\langle$the formula sort of $L\rangle \to$ the formula sort of L by [21, (106)], [17, (1)], [21, (87)], [19, (13)]. (the connectives of $L)(n+5)$ is of type $\emptyset \to$ the formula sort of L by [21, (106)], [17, (1)], [21, (87)], [19, (13)]. (the connectives of $L)(n+1)$ is of type $\langle$the formula sort of L, the formula sort of $L\rangle \to$ the formula sort of L and ... and (the connectives of $L)(n+4)$ is of type $\langle$the formula sort of L, the formula sort of $L\rangle \to$ the formula sort of L by [21, (106)], [17, (1)], [19, (13)]. The quantifiers of L is one-to-one. rng(the quantifiers of $L) \subseteq \{$the carrier' of $\Omega\} \times \{1,2\} \times \mathcal{X}$ by [30, (10), (69)]. (the connectives of $L)(n+6)$ is of type $\langle$the program sort of L, the formula sort of $L\rangle \to$ the formula sort of L and ... and (the connectives of $L)(n+8)$ is of type $\langle$the program sort of L, the formula sort of $L\rangle \to$ the formula sort of L by [21, (106)], [17, (1)], [19, (13)]. L is Ω-extension by [35, (53)], [19, (33)], [35, (65)], [17, (102)]. $\square$

Let n be a non empty, natural number, $\mathcal{X}$ be a non empty set, and Ω be a signature. Let us observe that there exists a non void, non empty, strict algorithmic language signature of $\mathcal{X}$ which is Ω-extension, PC correct w.r.t. n, QC correct, and AL correct w.r.t. n and there exists a non void, non empty, strict algorithmic language signature of $\mathcal{X}$ which is PC correct w.r.t. n, QC correct, and AL correct w.r.t. n.

2. LANGUAGE

Let Ω be a non empty, non void signature and $\mathfrak{T}$ be an algebra over Ω.

A variable set in $\mathfrak{T}$ is a set and is defined by

(Def. 8)   there exists a generator set G of $\mathfrak{T}$ such that $it = \bigcup G$.

Let $\mathcal{X}$ be a generator set of $\mathfrak{T}$. Note that the functor $\bigcup \mathcal{X}$ yields a variable set in $\mathfrak{T}$. Now we state the proposition:

(12) Let us consider a non empty, non void signature Ω, an algebra $\mathfrak{T}$ over Ω, and a variable set $\mathcal{X}$ in $\mathfrak{T}$. Then $\mathcal{X} \subseteq \bigcup(\text{the sorts of } \mathfrak{T})$.

Let Σ be a non empty, non void signature, $\mathcal{X}$ be a many sorted set indexed by the carrier of Σ, and $\mathfrak{T}$ be a free variable algebra over Σ. We say that $\mathfrak{T}$ is $\mathcal{X}$-vf-yielding if and only if

(Def. 9) the free variables of $\mathfrak{T}$ is a double many sorted set of the sorts of $\mathfrak{T}$ and $\mathcal{X}$.

Let Ω be a non empty set, Q be a many sorted set indexed by Ω, Y be a set, and f be a function from $\bigcup Q \times Y$ into $\bigcup Q$. We say that f is sort-preserving if and only if

(Def. 10) for every element j of Ω, $f°(Q(j) \times Y) \subseteq Q(j)$.

One can verify that there exists a function from $\bigcup Q \times Y$ into $\bigcup Q$ which is sort-preserving.

Let Ω be a non empty, non void signature and $\mathcal{X}$ be a many sorted set indexed by the carrier of Ω. We consider algebras over Ω, $\mathcal{X}$ with substitution which extend algebras over Ω and are systems

$$\langle \text{sorts, a characteristics, a substitution operation} \rangle$$

where the sorts constitute a many sorted set indexed by the carrier of Ω, the characteristics is a many sorted function from $(\text{the sorts})^{\#} \cdot (\text{the arity of } \Omega)$ into $(\text{the sorts}) \cdot (\text{the result sort of } \Omega)$, the substitution operation is a sort-preserving function from $\bigcup(\text{the sorts}) \times \bigcup[\![\mathcal{X}, \text{the sorts}]\!]$ into $\bigcup(\text{the sorts})$.

Now we state the proposition:

(13) Let us consider a set I, a many sorted set $\mathcal{X}$ indexed by I, a many sorted subset Σ of $\mathcal{X}$, and x. Then $\Sigma(x)$ is a subset of $\mathcal{X}(x)$.

Let Ω be a non empty, non void signature, $\mathcal{X}$ be a non empty yielding many sorted set indexed by the carrier of Ω, and Q be an algebra over Ω, $\mathcal{X}$ with substitution. Assume $\mathcal{X}$ is a many sorted subset of the sorts of Q. Let x be an element of $\bigcup \mathcal{X}$. The functor $^{@}(x, Q)$ yielding an element of $\bigcup(\text{the sorts of } Q)$ is defined by the term

(Def. 11) x.

Assume $\mathcal{X}$ is a many sorted subset of the sorts of Q. Let j be a sort symbol of Ω. Assume $(\text{the sorts of } Q)(j) \neq \emptyset$. Let α be an element of Q from j and y be an element of $\bigcup \mathcal{X}$. Given sort symbol a of Ω such that $x, y \in \mathcal{X}(a)$. The functor $\alpha_{(x \leftarrow y)}$ yielding an element of Q from j is defined by the term

(Def. 12) $(\text{the substitution operation of } Q)(\langle \alpha, \langle x, y \rangle \rangle)$.

Assume $(\text{the sorts of } Q)(j) \neq \emptyset$. Let τ be an element of $\bigcup(\text{the sorts of } Q)$. Given sort symbol a of Ω such that $x \in \mathcal{X}(a)$ and $\tau \in (\text{the sorts of } Q)(a)$. The functor $\alpha_{(x \leftarrow \tau)}$ yielding an element of Q from j is defined by the term

(Def. 13)   (the substitution operation of Q)($\langle \alpha, \langle x, \tau \rangle \rangle$).

Let Ω be a non empty, non void many sorted signature and $\mathcal{X}$ be a many sorted set indexed by the carrier of Ω. One can verify that there exists an algebra over Ω, $\mathcal{X}$ with substitution which is non-empty.

Let Ω be a non empty, non void signature, $\mathcal{X}$ be a non empty yielding many sorted set indexed by the carrier of Ω, Q be a non-empty algebra over Ω, $\mathcal{X}$ with substitution, o be an operation symbol of Ω, p be an element of $\mathrm{Args}(o, Q)$, x be an element of $\bigcup \mathcal{X}$, and y be an element of $\bigcup$(the sorts of Q). The functor $p_{(x \leftarrow y)}$ yielding an element of $\mathrm{Args}(o, Q)$ is defined by

(Def. 14)   for every natural number i such that $i \in \mathrm{dom}\,\mathrm{Arity}(o)$ there exists a sort symbol j of Ω such that $j = \mathrm{Arity}(o)(i)$ and there exists an element α of Q from j such that $\alpha = p(i)$ and $it(i) = \alpha_{(x \leftarrow y)}$.

Let I be a non empty set, $\mathcal{X}$ be a non-empty many sorted set indexed by I, Σ be a non-empty many sorted subset of $\mathcal{X}$, x be an element of I, and z be an element of $\Sigma(x)$. The functor $^{@}z$ yielding an element of $\mathcal{X}(x)$ is defined by the term

(Def. 15) z.

Let Ω be a non empty, non void signature, $\mathcal{X}$ be a non empty yielding many sorted set indexed by the carrier of Ω, and Q be a non-empty algebra over Ω, $\mathcal{X}$ with substitution. We say that Q is substitution correct (1) if and only if

(Def. 16)   for every element x of $\bigcup \mathcal{X}$ and for every sort symbol a of Ω such that $x \in \mathcal{X}(a)$ holds for every sort symbol j of Ω and for every element α of Q from j, $\alpha_{(x \leftarrow x)} = \alpha$ and for every element y of $\bigcup$(the sorts of Q) such that $y \in$ (the sorts of Q)(a) for every operation symbol o of Ω for every element p of $\mathrm{Args}(o, Q)$ for every element α of Q from the result sort of o such that $\alpha = (\mathrm{Den}(o, Q))(p)$ holds if there exists no QC language signature Σ over $\bigcup \mathcal{X}$ such that $\Omega = \Sigma$ and there exists an element z of $\bigcup \mathcal{X}$ and there exists an element q of $\{1, 2\}$ such that $o =$ (the quantifiers of Σ)(q, z), then $\alpha_{(x \leftarrow y)} = (\mathrm{Den}(o, Q))(p_{(x \leftarrow y)})$.

We say that Q is substitution correct (2) if and only if

(Def. 17)   for every sort symbol j of Ω and for every element q of Q from j and for every element τ of Q from j and for every element x of $\bigcup \mathcal{X}$ such that $\tau = x \in \mathcal{X}(j)$ holds $\tau_{(x \leftarrow q)} = q$.

Now we state the proposition:

(14)   Let us consider a non empty, non void signature Ω, a non empty yielding many sorted set $\mathcal{X}$ indexed by the carrier of Ω, and an algebra Q over Ω, $\mathcal{X}$ with substitution. Suppose $\mathcal{X}$ is a many sorted subset of the sorts of Q. Let us consider a sort symbol a of Ω. Suppose (the sorts of Q)(a) $\neq \emptyset$. Let us consider an element α of Q from a, elements x, y of $\bigcup \mathcal{X}$, and an

element τ of $\bigcup$(the sorts of Q). Suppose $y = \tau$. Let us consider a sort symbol j of Ω. If $x, y \in \mathcal{X}(j)$, then $\alpha_{(x \leftarrow y)} = \alpha_{(x \leftarrow \tau)}$. The theorem is a consequence of (13).

Let Ω be a non void signature. Let us note that every signature which is Ω-extension is also non void and non empty.

Let Ω be a signature. Let us note that there exists a non empty, non void signature which is Ω-extension.

Let $\mathcal{X}$ be a non empty set. Let us note that there exists a non empty, non void QC language signature over $\mathcal{X}$ which is Ω-extension.

Let n be a non empty, natural number. Note that there exists a non empty, non void, PC correct w.r.t. n, QC correct QC language signature over $\mathcal{X}$ which is Ω-extension.

Let n be a non empty natural number and Σ be an Ω-extension, PC correct w.r.t. n, feasible algorithmic language signature of $\mathcal{X}$. We say that Σ is essential if and only if

(Def. 18)   (the connectives of Σ)$^\circ(n+9 \backslash n)$ misses the carrier' of Ω and rng(the quantifiers of Σ) misses the carrier' of Ω and {the formula sort of Σ, the program sort of Σ} misses the carrier of Ω.

Let n be a non empty, natural number. Observe that there exists an Ω-extension, PC correct w.r.t. n, QC correct, AL correct w.r.t. n, non void, non empty, strict algorithmic language signature of $\mathcal{X}$ which is essential.

Let Ω be a non empty signature, Σ be an Ω-extension, non empty signature, and $\mathcal{X}$ be a non empty yielding many sorted set indexed by the carrier of Ω. Note that there exists a non empty yielding many sorted set indexed by the carrier of Σ which is $\mathcal{X}$-tolerating.

Let Σ be a non empty signature, $\mathfrak{T}$ be an algebra over Ω, and Q be an algebra over Σ. We say that Q is $\mathfrak{T}$-extension if and only if

(Def. 19) $Q{\restriction}\Omega =$ the algebra of $\mathfrak{T}$.

Now we state the propositions:

(15)   Let us consider a non empty, non void signature Ω, an Ω-extension signature Σ, an algebra $\mathfrak{T}$ over Ω, and algebras Q_1, Q_2 over Σ. Suppose the algebra of $Q_1 =$ the algebra of Q_2. If Q_1 is $\mathfrak{T}$-extension, then Q_2 is $\mathfrak{T}$-extension.

(16)   Let us consider a non empty, non void signature Ω, an Ω-extension signature Σ, an algebra $\mathfrak{T}$ over Ω, and an algebra Q over Σ. Suppose Q is $\mathfrak{T}$-extension. Suppose $x \in$ the carrier of Ω. Then (the sorts of $\mathfrak{T}$)$(x) =$ (the sorts of Q)(x).

(17)   Let us consider a non empty, non void signature Ω, an Ω-extension signature Σ, an algebra $\mathfrak{T}$ over Ω, and a set I. Suppose $I \subseteq$ (the carrier of

$\Sigma) \setminus$ (the carrier of Ω). Let us consider a many sorted set $\mathcal{X}$ indexed by I. Then there exists an algebra Q over Σ such that

(i) Q is $\mathfrak{T}$-extension, and

(ii) (the sorts of $Q)\!\restriction\! I = \mathcal{X}$.

PROOF: Set $\mathfrak{U} =$ (the many sorted set indexed by the carrier of $\Sigma +\cdot$ (the sorts of $\mathfrak{T}$))$+\cdot \mathcal{X}$. Set $\gamma =$ the many sorted function from $\mathfrak{U}^{\#} \cdot$ (the arity of Σ) into $\mathfrak{U} \cdot$ (the result sort of Σ)$+\cdot$(the characteristics of $\mathfrak{T}$). γ is a many sorted function from $\mathfrak{U}^{\#} \cdot$ (the arity of Σ) into $\mathfrak{U}\cdot$ (the result sort of Σ) by [19, (13)], [6, (12)], [17, (49)], [13, (11)]. Q is $\mathfrak{T}$-extension by [35, (65)], [19, (72), (23)]. $\square$

(18) Let us consider a non empty, non void signature Ω, an Ω-extension signature Σ, a non-empty algebra $\mathfrak{T}$ over Ω, and a set I. Suppose $I \subseteq$ (the carrier of $\Sigma) \setminus$ (the carrier of Ω). Let us consider a non-empty many sorted set $\mathcal{X}$ indexed by I. Then there exists a non-empty algebra Q over Σ such that

(i) Q is $\mathfrak{T}$-extension, and

(ii) (the sorts of $Q)\!\restriction\! I = \mathcal{X}$.

The theorem is a consequence of (17) and (16).

Let Ω be a non empty, non void signature, Σ be an Ω-extension signature, and $\mathfrak{T}$ be an algebra over Ω. One can verify that there exists an algebra over Σ which is $\mathfrak{T}$-extension.

Let $\mathfrak{T}$ be a non-empty algebra over Ω. One can check that there exists a non-empty algebra over Σ which is $\mathfrak{T}$-extension.

Now we state the propositions:

(19) Let us consider a set I and an object a. Then $\pi_1(I \boxtimes \{a\})$ is one-to-one.

(20) Let us consider signatures Σ_1, Σ_2, E_1, E_2. Suppose the many sorted signature of $\Sigma_1 =$ the many sorted signature of Σ_2 and the many sorted signature of $E_1 =$ the many sorted signature of E_2 and E_1 is an extension of Σ_1. Then E_2 is an extension of Σ_2.

Let I be a set and a be an object. Observe that $\pi_1(I \boxtimes \{a\})$ is one-to-one.

Let a, b, c be non empty sets, g be a function from a into b, and f be a function from b into c. Note that the functor $f \cdot g$ yields a function from a into c. Now we state the propositions:

(21) Let us consider a one-to-one function f. If $\mathcal{X}$ misses Y, then $f^{\circ}\mathcal{X}$ misses $f^{\circ}Y$.

(22) Let us consider a non empty, natural number n, a set $\mathcal{X}$, and a non empty signature Ω. Then there exists a non empty, non void, PC correct w.r.t. n, QC correct QC language signature Q over $\mathcal{X}$ such that

(i) the carrier of Q misses the carrier of Ω, and

(ii) the carrier' of Q misses the carrier' of Ω.

PROOF: Set $Q =$ the non empty, non void, PC correct w.r.t. n, QC correct QC language signature over $\mathcal{X}$. Reconsider $\alpha =$ (the carrier of $Q) \times \{$the carrier of $\Omega\}$ as a non empty set. Reconsider $\beta =$ (the carrier' of $Q) \times \{$the carrier' of $\Omega\}$ as a non empty set. Reconsider $f = \pi_1(($the carrier of $Q) \boxtimes \{$the carrier of $\Omega\})$ as a function from α into the carrier of Q. Reconsider $g = \pi_1(($the carrier' of $Q) \boxtimes \{$the carrier' of $\Omega\})$ as a function from β into the carrier' of Q. Reconsider $f_1 = f^{-1}$ as a function from the carrier of Q into α. Reconsider $g_1 = g^{-1}$ as a function from the carrier' of Q into β. Define $\mathcal{F}(\text{object}) = f_1 \cdot \$_1 (\in ($the carrier of $Q)^*)$. Consider f_2 being a function such that $\operatorname{dom} f_2 = ($the carrier of $Q)^*$ and for every object p such that $p \in ($the carrier of $Q)^*$ holds $f_2(p) = \mathcal{F}(p)$ from [17, Sch. 3]. $\operatorname{rng} f_2 \subseteq \alpha^*$. Reconsider $A_6 = f_2 \cdot ($the arity of $Q) \cdot g$ as a function from β into α^*. Reconsider $r_3 = f_1 \cdot ($the result sort of $Q) \cdot g$ as a function from β into α. Reconsider $f_3 = f_1(($the formula sort of $Q))$ as an element of α. Reconsider $c_2 = g_1 \cdot ($the connectives of $Q)$ as a finite sequence of elements of β. Reconsider $q_5 = g_1 \cdot ($the quantifiers of $Q)$ as a function from (the quantifier sort of $Q) \times \mathcal{X}$ into β. Set $Q_3 = \langle(\alpha \text{ \textbf{qua} non empty set}), (\beta \text{ \textbf{qua} non empty set}), A_6, r_3, f_3, c_2, ($the quantifier sort of Q \textbf{qua} set$), q_5\rangle$. Q_3 is PC correct w.r.t. n by [35, (27), (83)], [8, (12), (13)]. Q_3 is QC correct by [28, (52)], [21], [21, (87)], [18, (5)]. The carrier of Q_3 misses the carrier of Ω. The carrier' of Q_3 misses the carrier' of Ω. $\square$

Let Ω be a non empty signature. Let us note that there exists a non empty, non void signature which is Ω-extension.

Let $\mathcal{X}$ be a set. Observe that there exists a non empty, non void QC language signature over $\mathcal{X}$ which is Ω-extension.

Let n be a non empty, natural number. Observe that there exists a non empty, non void, PC correct w.r.t. n, QC correct QC language signature over $\mathcal{X}$ which is Ω-extension.

Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, Σ be an Ω-extension, non empty, non void QC language signature over $\bigcup \mathcal{X}$, and Y be an $\mathcal{X}$-tolerating many sorted set indexed by the carrier of Σ. We consider language structures over $\mathfrak{T}$, Y and Σ which extend free variable algebras over Σ and algebras over Σ, Y with substitution and are systems

$$\langle \text{sorts, a characteristics, free variables, a substitution operation,}$$

$$\text{an equality}\rangle$$

where the sorts constitute a many sorted set indexed by the carrier of Σ, the characteristics is a many sorted function from (the sorts)$^{\#} \cdot$ (the arity of Σ) into (the sorts)$\cdot$(the result sort of Σ), the free variables constitute a double many sorted set of the sorts and the sorts, the substitution operation is a sort-preserving function from $\bigcup$(the sorts) $\times \bigcup[\![Y, \text{the sorts}]\!]$ into $\bigcup$(the sorts), the equality is a function from $\bigcup[\![$the sorts of $\mathfrak{T}$, the sorts of $\mathfrak{T}]\!]$ into (the sorts)((the formula sort of Σ)).

Let Σ be a non empty PC language signature and L be an algebra over Σ. We say that L is a language if and only if

(Def. 20)   (the sorts of L)((the formula sort of Σ)) is not empty.

One can check that every algebra over Σ which is non-empty is also a language and there exists an algebra over Σ which is a language.

Now we state the proposition:

(23)   Let us consider a non void signature Ω, an Ω-extension, non void signature Σ, and algebras A_1, A_2 over Σ. Suppose the algebra of $A_1 =$ the algebra of A_2. Then $A_1 {\restriction} \Omega = A_2 {\restriction} \Omega$.

Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, and Σ be an Ω-extension, non empty, non void QC language signature over $\bigcup \mathcal{X}$. Observe that there exists a non empty yielding many sorted set indexed by the carrier of Σ which is $\mathcal{X}$-tolerating.

Let Y be an $\mathcal{X}$-tolerating, non empty yielding many sorted set indexed by the carrier of Σ. Let us observe that there exists a language structure over $\mathfrak{T}$, Y and Σ which is non-empty, language, and $\mathfrak{T}$-extension.

Let L be a non-empty language structure over $\mathfrak{T}$, Y and Σ. We say that L is substitution correct (3) if and only if

(Def. 21)   for every sort symbols σ, σ_1 of Σ and for every element τ of L from σ and for every element τ_1 of L from σ_1 and for every element y of $\bigcup Y$ such that $y \in Y(\sigma)$ holds $y \notin (\text{vf}\,\tau_1)(\sigma)$ and if $y \notin (\text{vf}\,\tau_1)(\sigma)$, then $\tau_{1(y \leftarrow \tau)} = \tau_1$ and if $\tau_1 = y \in Y(\sigma_1)$, then $\tau_{1(y \leftarrow \tau)} = \tau$ and for every element x of $\bigcup Y$ such that $x \in Y(\sigma)$ holds $\tau_{1(x \leftarrow y)(y \leftarrow x)} = \tau_1$.

Let Ω be a non empty signature, Σ be an Ω-extension, non empty signature, $\mathcal{X}$ be a non empty yielding many sorted set indexed by the carrier of Ω, and Y be a set. Let us note that $\mathcal{X}$ extended by$(Y, (\text{the carrier of } \Sigma))$ is non empty yielding.

Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, and Σ be an Ω-extension, non empty, non void QC language signature over $\bigcup \mathcal{X}$. Let us observe that there exists a language structure over $\mathfrak{T}$, $\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma))$ and Σ

which is ($\mathcal{X}$ extended by($\emptyset$, (the carrier of Σ)))-vf-yielding, non-empty, language, and $\mathfrak{T}$-extension.

Let $\mathcal{X}$ be a set, Σ be a non empty QC language signature over $\mathcal{X}$, and L be language algebra over Σ. Note that (the sorts of L)((the formula sort of Σ)) is non empty.

Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, Σ be an Ω-extension, non empty, non void QC language signature over $\bigcup \mathcal{X}$, and Y be an $\mathcal{X}$-tolerating, non empty yielding many sorted set indexed by the carrier of Σ.

A language over Y and Σ is language, $\mathfrak{T}$-extension language structure over $\mathfrak{T}$, Y and Σ. Let Σ be a non empty PC language signature and L be language algebra over Σ.

A formula of L is an element of (the sorts of L)((the formula sort of Σ)). Let n be a non empty, natural number and Σ be a non void, non empty, PC correct w.r.t. n PC language signature. The functor true_L yielding a formula of L is defined by the term

(Def. 22) $(\text{Den}((\text{the connectives of } \Sigma)(n+5)(\in (\text{the carrier' of } \Sigma)), L))(\emptyset)$.

Let α be a formula of L. The functor $\neg \alpha$ yielding a formula of L is defined by the term

(Def. 23) $(\text{Den}((\text{the connectives of } \Sigma)(n)(\in (\text{the carrier' of } \Sigma)), L))(\langle \alpha \rangle)$.

Let β be a formula of L. The functors: $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \Rightarrow \beta$, and $\alpha \Leftrightarrow \beta$ yielding formulae of L are defined by terms,

(Def. 24) $(\text{Den}((\text{the connectives of } \Sigma)(n+1)(\in (\text{the carrier' of } \Sigma)), L))(\langle \alpha, \beta \rangle)$,

(Def. 25) $(\text{Den}((\text{the connectives of } \Sigma)(n+2)(\in (\text{the carrier' of } \Sigma)), L))(\langle \alpha, \beta \rangle)$,

(Def. 26) $(\text{Den}((\text{the connectives of } \Sigma)(n+3)(\in (\text{the carrier' of } \Sigma)), L))(\langle \alpha, \beta \rangle)$,

(Def. 27) $(\text{Den}((\text{the connectives of } \Sigma)(n+4)(\in (\text{the carrier' of } \Sigma)), L))(\langle \alpha, \beta \rangle)$,

respectively. Let Ω be a non empty, non void signature and $\mathfrak{T}$ be a non-empty algebra over Ω. Let us note that there exists a variable set in $\mathfrak{T}$ which is non empty.

Let n be a non empty, natural number, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, Σ be an Ω-extension, non void, non empty, PC correct w.r.t. n, QC correct QC language signature over $\bigcup \mathcal{X}$, Y be an $\mathcal{X}$-tolerating, non empty yielding many sorted set indexed by the carrier of Σ, L be a language over Y and Σ, α be a formula of L, and x be an element of $\bigcup \mathcal{X}$. The functors: $\forall_x \alpha$ and $\exists_x \alpha$ yielding formulae of L are defined by terms,

(Def. 28) $(\text{Den}((\text{the quantifiers of } \Sigma)(1, x)(\in (\text{the carrier' of } \Sigma)), L))(\langle \alpha \rangle)$,

(Def. 29) $(\text{Den}((\text{the quantifiers of } \Sigma)(2, x)(\in (\text{the carrier' of } \Sigma)), L))(\langle \alpha \rangle)$,

respectively. Let x, y be elements of $\bigcup \mathcal{X}$. The functors: $\forall_{x,y} \alpha$ and $\exists_{x,y} \alpha$ yielding formulae of L are defined by terms,

(Def. 30) $\quad \forall_x (\forall_y \alpha)$,

(Def. 31) $\quad \exists_x (\exists_y \alpha)$,

respectively. Let x, y, z be elements of $\bigcup \mathcal{X}$. The functors: $\forall_{x,y,z} \alpha$ and $\exists_{x,y,z} \alpha$ yielding formulae of L are defined by terms,

(Def. 32) $\quad \forall_{x,y} (\forall_z \alpha)$,

(Def. 33) $\quad \exists_{x,y} (\exists_z \alpha)$,

respectively. Let τ_1, τ_2 be objects. Given sort symbol a of Ω such that τ_1, $\tau_2 \in$ (the sorts of $\mathfrak{T}$)(a). The functor $\tau_1 =_L \tau_2$ yielding a formula of L is defined by the term

(Def. 34) (the equality of L)(τ_1, τ_2).

Let L be a non-empty language over Y and Σ. We say that L is vf-QC-correct if and only if

(Def. 35) for every formulae α, β of L, $\mathrm{vf}\, \neg \alpha = \mathrm{vf}\, \alpha$ and $\mathrm{vf}(\alpha \wedge \beta) = \mathrm{vf}\, \alpha \cup \mathrm{vf}\, \beta$ and $\mathrm{vf}(\alpha \vee \beta) = \mathrm{vf}\, \alpha \cup \mathrm{vf}\, \beta$ and $\mathrm{vf}(\alpha \Rightarrow \beta) = \mathrm{vf}\, \alpha \cup \mathrm{vf}\, \beta$ and $\mathrm{vf}(\alpha \Leftrightarrow \beta) = \mathrm{vf}\, \alpha \cup \mathrm{vf}\, \beta$ and $\mathrm{vf}\, \mathrm{true}_L = \mathbf{0}.$(the carrier of Σ) and for every element x of $\bigcup \mathcal{X}$ and for every sort symbol a of Σ such that $x \in \mathcal{X}(a)$ holds $\mathrm{vf}\, \forall_x \alpha = \mathrm{vf}\, \alpha \setminus a$ -singleton x and $\mathrm{vf}\, \exists_x \alpha = \mathrm{vf}\, \alpha \setminus a$ -singleton x.

Let L be a non-empty, $\mathfrak{T}$-extension language over Y and Σ. We say that L is vf finite if and only if

(Def. 36) for every sort symbol σ of Σ and for every element τ of L from σ, $\mathrm{vf}\, \tau$ is finite-yielding.

We say that L is substitution for $\forall$ and $\exists$ if and only if

(Def. 37) for every formula α of L and for every element x of $\bigcup \mathcal{X}$ and for every sort symbols σ, σ_1 of Σ and for every element τ of L from σ such that $x \in \mathcal{X}(\sigma_1)$ for every element y of $\bigcup Y$ such that $y \in Y(\sigma)$ holds if $x = y$, then $(\forall_x \alpha)_{(y \leftarrow \tau)} = \forall_x \alpha$ and $(\exists_x \alpha)_{(y \leftarrow \tau)} = \exists_x \alpha$ and if $x \neq y$ and $x \in (\mathrm{vf}\, \tau)(\sigma_1)$, then there exists an element z of $\bigcup \mathcal{X}$ and there exist elements x_0, z_0 of $\bigcup Y$ such that $x = x_0$ and $z_0 = z =$ the element of $(\mathcal{X}(\sigma_1) \setminus (\mathrm{vf}\, \tau)(\sigma_1)) \setminus (\mathrm{vf}\, \alpha)(\sigma_1)$ and $(\forall_x \alpha)_{(y \leftarrow \tau)} = \forall_z (\alpha_{(x_0 \leftarrow z_0)(y \leftarrow \tau)})$ and $(\exists_x \alpha)_{(y \leftarrow \tau)} = \exists_z (\alpha_{(x_0 \leftarrow z_0)(y \leftarrow \tau)})$ and if $x \neq y$ and $x \notin (\mathrm{vf}\, \tau)(\sigma)$, then $(\forall_x \alpha)_{(y \leftarrow \tau)} = \forall_x (\alpha_{(y \leftarrow \tau)})$ and $(\exists_x \alpha)_{(y \leftarrow \tau)} = \exists_x (\alpha_{(y \leftarrow \tau)})$.

Now we state the propositions:

(24) Let us consider a non void signature Ω, an algebra $\mathfrak{T}$ over Ω, a many sorted subset $\mathcal{X}$ of the sorts of $\mathfrak{T}$, an Ω-extension, non void signature Σ, and a $\mathfrak{T}$-extension algebra Q over Σ. Then $\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma))$

is a many sorted subset of the sorts of Q. The theorem is a consequence of (16).

(25) Let us consider a non void signature Ω, an algebra $\mathfrak{T}$ over Ω, a many sorted subset $\mathcal{X}$ of the sorts of $\mathfrak{T}$, and an Ω-extension, non void signature Σ. Then $\bigcup(\mathcal{X}$ extended by$(\emptyset, ($the carrier of $\Sigma))) = \bigcup \mathcal{X}$.

(26) Let us consider a non empty, natural number n, a non empty set $\mathcal{X}$, a non empty, non void, PC correct w.r.t. n, QC correct QC language signature Σ over $\mathcal{X}$, and language algebra Q over Σ. Then

(i) $\emptyset \in \mathrm{Args}(($the connectives of $\Sigma)(n+5)(\in ($the carrier' of $\Sigma)), Q)$, and

(ii) for every formula α of Q, $\langle \alpha \rangle \in \mathrm{Args}(($the connectives of $\Sigma)(n)(\in ($the carrier' of $\Sigma)), Q)$ and for every formula β of Q, $\langle \alpha, \beta \rangle \in \mathrm{Args}(($the connectives of $\Sigma)(n + 1)(\in ($the carrier' of $\Sigma)), Q)$ and ... and $\langle \alpha, \beta \rangle \in \mathrm{Args}(($the connectives of $\Sigma)(n+4)(\in ($the carrier' of $\Sigma)), Q)$ and for every element x of $\mathcal{X}$, $\langle \alpha \rangle \in \mathrm{Args}(($the quantifiers of $\Sigma)(1, x)(\in ($the carrier' of $\Sigma)), Q)$ and $\langle \alpha \rangle \in \mathrm{Args}(($the quantifiers of $\Sigma)(2, x)(\in ($the carrier' of $\Sigma)), Q)$.

The theorem is a consequence of (4), (5), and (6).

(27) Let us consider a non empty, natural number n, a non empty, non void signature Ω, a non-empty algebra $\mathfrak{T}$ over Ω, a non empty yielding generator set $\mathcal{X}$ of $\mathfrak{T}$, an Ω-extension, non void, non empty, PC correct w.r.t. n, QC correct QC language signature Σ over $\bigcup \mathcal{X}$, an $\mathcal{X}$-tolerating, non empty yielding many sorted set Y indexed by the carrier of Σ, a non-empty language L over Y and Σ, an element x of $\bigcup Y$, an element τ of $\bigcup ($the sorts of $L)$, and a sort symbol σ of Σ. Suppose $x \in Y(\sigma)$ and $\tau \in ($the sorts of $L)(\sigma)$. Then

(i) for every element a of $\mathrm{Args}(($the connectives of $\Sigma)(n+5)(\in ($the carrier' of $\Sigma)), L)$ such that $a = \emptyset$ holds $a_{(x \leftarrow \tau)} = \emptyset$, and

(ii) for every formula α of L, for every element a of $\mathrm{Args}(($the connectives of $\Sigma)(n)(\in ($the carrier' of $\Sigma)), L)$ such that $\langle \alpha \rangle = a$ holds $a_{(x \leftarrow \tau)} = \langle \alpha_{(x \leftarrow \tau)} \rangle$ and for every formula β of L, for every element a of $\mathrm{Args}(($the connectives of $\Sigma)(n+1)(\in ($the carrier' of $\Sigma)), L)$ such that $\langle \alpha, \beta \rangle = a$ holds $a_{(x \leftarrow \tau)} = \langle \alpha_{(x \leftarrow \tau)}, \beta_{(x \leftarrow \tau)} \rangle$ and ... and for every element a of $\mathrm{Args}(($the connectives of $\Sigma)(n+4)(\in ($the carrier' of $\Sigma)), L)$ such that $\langle \alpha, \beta \rangle = a$ holds $a_{(x \leftarrow \tau)} = \langle \alpha_{(x \leftarrow \tau)}, \beta_{(x \leftarrow \tau)} \rangle$ and for every element z of $\bigcup \mathcal{X}$, for every element a of $\mathrm{Args}(($the quantifiers of $\Sigma)(1, z)(\in ($the carrier' of $\Sigma)), L)$ such that $\langle \alpha \rangle = a$ holds $a_{(x \leftarrow \tau)} = \langle \alpha_{(x \leftarrow \tau)} \rangle$ and for every element a of $\mathrm{Args}(($the quantifiers of $\Sigma)(2, z)(\in ($the carrier' of $\Sigma)), L)$ such that $\langle \alpha \rangle = a$ holds $a_{(x \leftarrow \tau)} = \langle \alpha_{(x \leftarrow \tau)} \rangle$.

PROOF: for every element a of Args((the connectives of Σ)$(n+1)(\in$ (the carrier' of Σ)), L) such that $\langle \alpha, \beta \rangle = a$ holds $a_{(x \leftarrow \tau)} = \langle \alpha_{(x \leftarrow \tau)}, \beta_{(x \leftarrow \tau)} \rangle$ and ... and for every element a of Args((the connectives of Σ)$(n + 4)(\in$ (the carrier' of Σ)), L) such that $\langle \alpha, \beta \rangle = a$ holds $a_{(x \leftarrow \tau)} = \langle \alpha_{(x \leftarrow \tau)}, \beta_{(x \leftarrow \tau)} \rangle$ by [7, (2)], [12, (44), (89), (1)]. Set $o =$ (the quantifiers of Σ)$(2, z)(\in$ (the carrier' of Σ)). Consider j being a sort symbol of Σ such that $j = \mathrm{Arity}(o)(1)$ and there exists an element α of L from j such that $\alpha = a(1)$ and $a_{(x \leftarrow \tau)}(1) = \alpha_{(x \leftarrow \tau)}$. $\square$

(28)   Let us consider a non empty, natural number n, a non empty, non void signature Ω, a non-empty algebra $\mathfrak{T}$ over Ω, a non empty yielding generator set $\mathcal{X}$ of $\mathfrak{T}$, an Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct QC language signature Σ over $\bigcup \mathcal{X}$, an $\mathcal{X}$-tolerating, non empty yielding many sorted set Y indexed by the carrier of Σ, and a non-empty language L over Y and Σ. Suppose L is substitution correct (1) and Y is a many sorted subset of the sorts of L. Let us consider elements x, y of $\bigcup Y$ and a sort symbol a of Σ. Suppose $x, y \in Y(a)$. Let us consider a formula α of L. Then

  (i) $(\neg \alpha)_{(x \leftarrow y)} = \neg(\alpha_{(x \leftarrow y)})$, and

  (ii) for every formula β of L, $(\alpha \wedge \beta)_{(x \leftarrow y)} = \alpha_{(x \leftarrow y)} \wedge \beta_{(x \leftarrow y)}$ and $(\alpha \vee \beta)_{(x \leftarrow y)} = \alpha_{(x \leftarrow y)} \vee \beta_{(x \leftarrow y)}$ and $(\alpha \Rightarrow \beta)_{(x \leftarrow y)} = \alpha_{(x \leftarrow y)} \Rightarrow \beta_{(x \leftarrow y)}$ and $(\alpha \Leftrightarrow \beta)_{(x \leftarrow y)} = \alpha_{(x \leftarrow y)} \Leftrightarrow \beta_{(x \leftarrow y)}$ and $(\mathrm{true}_L)_{(x \leftarrow y)} = \mathrm{true}_L$.

The theorem is a consequence of (26), (27), and (14).

3. ALGORITHMIC THEORY

Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, Σ be an Ω-extension, non void, non empty QC language signature over $\bigcup \mathcal{X}$, and Y be an $\mathcal{X}$-tolerating many sorted set indexed by the carrier of Σ. We consider bialgebra structures over Σ, Y which extend language structures over $\mathfrak{T}$, Y and Σ and program algebra structures of Ω, $\mathfrak{T}$, and $\mathcal{X}$ and are systems

$$\langle \mathrm{sorts, a\ characteristics, free\ variables, a\ substitution\ operation,}$$

$$\mathrm{an\ equality, a\ carrier, a\ characteristic, assignments} \rangle$$

where the sorts constitute a many sorted set indexed by the carrier of Σ, the characteristics is a many sorted function from (the sorts)$^{\#} \cdot$ (the arity of Σ) into (the sorts)$\cdot$(the result sort of Σ), the free variables constitute a double many sorted set of the sorts and the sorts, the substitution operation is a sort-preserving

function from $\bigcup$(the sorts) $\times$ $\bigcup[\![Y, \text{the sorts}]\!]$ into $\bigcup$(the sorts), the equality is a function from $\bigcup[\![\text{the sorts of } \mathfrak{T}, \text{the sorts of } \mathfrak{T}]\!]$ into (the sorts)((the formula sort of Σ)), the carrier is a set, the characteristic is a finite sequence of operational functions of the carrier, the assignments constitute a function from $\bigcup[\![\mathcal{X}, \text{the sorts of } \mathfrak{T}]\!]$ into the carrier.

Let us note that there exists a non void, non empty algorithmic language signature of $\bigcup \mathcal{X}$ which is Ω-extension.

Let Σ be an Ω-extension, non void, non empty algorithmic language signature of $\bigcup \mathcal{X}$, Y be an $\mathcal{X}$-tolerating many sorted set indexed by the carrier of Σ, and L be a bialgebra structure over Σ, Y. We say that L is AL correct if and only if

(Def. 38)   the carrier of L = (the sorts of L)((the program sort of Σ)).

Let Σ be a 1-sorted structure. We introduce Σ is 1s-empty as a synonym of Σ is empty.

Let Σ be a universal algebra structure. We introduce Σ is ua-non-empty as a synonym of Σ is non-empty.

Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, Σ be an Ω-extension, non void, non empty algorithmic language signature of $\bigcup \mathcal{X}$, and Y be an $\mathcal{X}$-tolerating many sorted set indexed by the carrier of Σ. Let us observe that there exists a strict bialgebra structure over Σ, Y which is non 1s-empty.

Let n be a non empty natural number and Σ be an essential, Ω-extension, PC correct w.r.t. n, non void, non empty, feasible algorithmic language signature of $\bigcup \mathcal{X}$. One can check that there exists a non 1s-empty, strict bialgebra structure over Σ, Y which is non-empty, language, AL correct, quasi total, partial, ua-non-empty, and $\mathfrak{T}$-extension and has empty-instruction, catenation, if-instruction, and while-instruction.

Let us consider pre-if-while algebras $\mathfrak{U}_1$, $\mathfrak{U}_2$.

Let us assume that the universal algebra structure of $\mathfrak{U}_1$ = the universal algebra structure of $\mathfrak{U}_2$. Now we state the propositions:

(29)   (i) $\text{EmptyIns}_{\mathfrak{U}_1} = \text{EmptyIns}_{\mathfrak{U}_2}$, and

(ii) for every elements I_1, J_1 of $\mathfrak{U}_1$ and for every elements I_2, J_2 of $\mathfrak{U}_2$ such that $I_1 = I_2$ and $J_1 = J_2$ holds $I_1; J_1 = I_2; J_2$ and while I_1 do $J_1 = $ while I_2 do J_2 and for every element C_1 of $\mathfrak{U}_1$ and for every element C_2 of $\mathfrak{U}_2$ such that $C_1 = C_2$ holds if C_1 then I_1 else $J_1 = $ if C_2 then I_2 else J_2.

(30) $\text{ElementaryInstructions}_{\mathfrak{U}_1} = \text{ElementaryInstructions}_{\mathfrak{U}_2}$.

Now we state the propositions:

(31)   Let us consider universal algebras $\mathfrak{U}_1$, $\mathfrak{U}_2$. Suppose the universal algebra structure of $\mathfrak{U}_1$ = the universal algebra structure of $\mathfrak{U}_2$. Let us consider a

subset Σ_1 of $\mathfrak{U}_1$ and a subset Σ_2 of $\mathfrak{U}_2$. Suppose $\Sigma_1 = \Sigma_2$. Let us consider an operation o_1 of $\mathfrak{U}_1$ and an operation o_2 of $\mathfrak{U}_2$. Suppose $o_1 = o_2$. If Σ_1 is closed on o_1, then Σ_2 is closed on o_2.

(32)   Let us consider universal algebras $\mathfrak{U}_1$, $\mathfrak{U}_2$. Suppose the universal algebra structure of $\mathfrak{U}_1$ = the universal algebra structure of $\mathfrak{U}_2$. Let us consider a subset Σ_1 of $\mathfrak{U}_1$ and a subset Σ_2 of $\mathfrak{U}_2$. Suppose $\Sigma_1 = \Sigma_2$. If Σ_1 is operations closed, then Σ_2 is operations closed.

Let us consider universal algebras $\mathfrak{U}_1$, $\mathfrak{U}_2$.

Let us assume that the universal algebra structure of $\mathfrak{U}_1$ = the universal algebra structure of $\mathfrak{U}_2$. Now we state the propositions:

(33)   Every generator set of $\mathfrak{U}_1$ is a generator set of $\mathfrak{U}_2$. The theorem is a consequence of (32).

(34)   signature $\mathfrak{U}_1$ = signature $\mathfrak{U}_2$.

Let n be a non empty, natural number, Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, and Σ be an essential, Ω-extension, non void, non empty, PC correct w.r.t. n, QC correct algorithmic language signature of $\bigcup \mathcal{X}$. Let us observe that there exists a non-empty, quasi total, partial, ua-non-empty, a language, non 1s-empty, $\mathfrak{T}$-extension bialgebra structure over Σ, $\mathcal{X}$ extended by $(\emptyset, ($the carrier of $\Sigma))$ with empty-instruction, catenation, if-instruction, and while-instruction which is AL correct, vf-QC-correct, vf-correct, vf finite, substitution correct (1), substitution for $\forall$ and $\exists$, non degenerated, well founded, and E.C.I.W.-strict.

Let Σ be an essential, Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct, AL correct w.r.t. n algorithmic language signature of $\bigcup \mathcal{X}$.

An if-while algebra of $\mathcal{X}$ over Σ is an AL correct, vf-QC-correct, vf-correct, substitution correct (1), substitution for $\forall$ and $\exists$, non degenerated, well founded, E.C.I.W.-strict, non-empty, quasi total, partial, ua-non-empty, a language, non 1s-empty, $\mathfrak{T}$-extension bialgebra structure over Σ, $\mathcal{X}$ extended by $(\emptyset, ($the carrier of $\Sigma))$ with empty-instruction, catenation, if-instruction, and while-instruction. Let n be a non empty natural number, L be an if-while algebra of $\mathcal{X}$ over Σ, K be a formula of L, and P be an algorithm of L. The functors: $P \cdot K$, $\mathbb{U}(P, K)$, and $\mathbb{m}(P, K)$ yielding formulae of L are defined by terms,

(Def. 39) $(\mathrm{Den}(($the connectives of $\Sigma)(n + 6)(\in ($the carrier' of $\Sigma)), L))(\langle P, K \rangle)$,

(Def. 40) $(\mathrm{Den}(($the connectives of $\Sigma)(n + 7)(\in ($the carrier' of $\Sigma)), L))(\langle P, K \rangle)$,

(Def. 41) $(\mathrm{Den}(($the connectives of $\Sigma)(n + 8)(\in ($the carrier' of $\Sigma)), L))(\langle P, K \rangle)$,

respectively. Let Σ be a non empty, non void, PC correct w.r.t. n PC language signature, L be language algebra over Σ, and F be a subset of (the sorts of $L)(($the formula sort of $\Sigma))$. We say that F is PC-closed if and only if

(Def. 42) for every formulae α, β, γ of L, $\alpha \Rightarrow (\beta \Rightarrow \alpha)$, $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$, $(\neg\alpha \Rightarrow \neg\beta) \Rightarrow (\beta \Rightarrow \alpha)$, $\alpha \Rightarrow (\alpha \vee \beta)$, $\alpha \Rightarrow (\beta \vee \alpha)$, $(\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \vee \beta) \Rightarrow \gamma))$, $(\alpha \wedge \beta) \Rightarrow \alpha$, $(\alpha \wedge \beta) \Rightarrow \beta$, $\alpha \Rightarrow (\beta \Rightarrow (\alpha \wedge \beta))$, $(\alpha \wedge \neg\alpha) \Rightarrow \beta$, $(\alpha \Rightarrow \beta) \Rightarrow ((\alpha \Rightarrow \neg\beta) \Rightarrow \neg\alpha)$, $\alpha \vee \neg\alpha$, $(\alpha \Leftrightarrow \beta) \Rightarrow (\alpha \Rightarrow \beta)$, $(\alpha \Leftrightarrow \beta) \Rightarrow (\beta \Rightarrow \alpha)$, $((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \Rightarrow (\alpha \Leftrightarrow \beta)$, true_L, $(\text{true}_L \wedge \alpha) \Leftrightarrow \alpha$, $(\text{true}_L \vee \alpha) \Leftrightarrow \text{true}_L \in F$ and if α, $\alpha \Rightarrow \beta \in F$, then $\beta \in F$.

Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, Σ be an Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct QC language signature over $\bigcup \mathcal{X}$, and L be a non-empty language over $\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma))$ and Σ. We say that F is QC closed if and only if

(Def. 43) for every elements α, β of (the sorts of L)((the formula sort of Σ)) and for every element x of $\bigcup \mathcal{X}$, for every sort symbol a of Ω, for every element τ of $\bigcup$(the sorts of L) such that $x \in (\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma)))(a)$ and $\tau \in (\text{the sorts of } L)(a)$ for every element y of $\bigcup(\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma)))$ such that $x = y$ holds $\forall_x \alpha \Rightarrow \alpha_{(y \leftarrow \tau)} \in F$ and if $x \in \mathcal{X}(a)$ and $x \notin (\text{vf } \alpha)(a)$, then $\forall_x(\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \forall_x \beta) \in F$ and $\neg \exists_x \alpha \Leftrightarrow \forall_x(\neg\alpha)$, $\exists_x(\neg\alpha) \Leftrightarrow \neg \forall_x \alpha \in F$ and if $\alpha \in F$, then $\forall_x \alpha \in F$.

Let L be a non-empty, $\mathfrak{T}$-extension language over $\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma))$ and Σ. We say that L is substitution eq-correct if and only if

(Def. 44) for every element x_0 of $\bigcup(\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma)))$ and for every sort symbols σ, σ_1 of Σ such that $x_0 \in \mathcal{X}(\sigma)$ for every element τ of L from σ for every elements τ_1, τ_2 of L from σ_1, $(\tau_1 =_L \tau_2)_{(x_0 \leftarrow \tau)} = \tau_{1(x_0 \leftarrow \tau)} =_L \tau_{2(x_0 \leftarrow \tau)}$.

We say that L is vf-eq-correct if and only if

(Def. 45) for every sort symbol σ of Σ, for every elements τ_1, τ_2 of L from σ, $\text{vf}(\tau_1 =_L \tau_2) = \text{vf } \tau_1 \cup \text{vf } \tau_2$ and for every sort symbol σ of Σ and for every element τ of L from σ such that $\tau \in \mathcal{X}(\sigma)$ holds $\text{vf } \tau = \sigma$ -singleton τ.

We say that F has equality if and only if

(Def. 46) for every element τ of $\mathfrak{T}$, $\tau =_L \tau \in F$ and for every sort symbol b of Σ and for every elements τ_1, τ_2 of L from b and for every element x of $\bigcup(\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma)))$ such that $x \in \mathcal{X}(b)$ holds for every sort symbol c of Σ such that $c \in$ the carrier of Ω for every element τ of L from c, $(\tau_1 =_L \tau_2) \Rightarrow (\tau_{(x \leftarrow \tau_1)} =_L \tau_{(x \leftarrow \tau_2)}) \in F$ and for every formula α of L, $(\tau_1 =_L \tau_2) \Rightarrow (\alpha_{(x \leftarrow \tau_1)} \Rightarrow \alpha_{(x \leftarrow \tau_2)}) \in F$.

Let n be a non empty, natural number, $\mathfrak{T}$ be a non-empty free variable algebra over Ω, $\mathcal{X}$ be a non-empty generator set of $\mathfrak{T}$, Σ be an essential, Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct, AL correct

w.r.t. n algorithmic language signature of $\bigcup \mathcal{X}$, L be a non 1s-empty if-while algebra of $\mathcal{X}$ over Σ, and φ be a formula of L. We say that F is AL closed w.r.t φ if and only if

(Def. 47)   for every formulae α, β of L, for every algorithm M of L, $M \cdot (\alpha \wedge \beta) \Leftrightarrow$
$(M \cdot \alpha \wedge (M \cdot \beta))$, $M \cdot (\alpha \vee \beta) \Leftrightarrow (M \cdot \alpha \vee (M \cdot \beta))$, $\mathbb{U}(M, \alpha) \Leftrightarrow (\alpha \vee$
$\mathbb{U}(M, M \cdot \alpha))$, $\mathbb{m}(M, \alpha) \Leftrightarrow (\alpha \wedge \mathbb{m}(M, M \cdot \alpha)) \in F$ and if $\alpha \Rightarrow \beta \in F$,
then $\mathbb{U}(M, \alpha) \Rightarrow \mathbb{U}(M, \beta)$, $\mathbb{m}(M, \alpha) \Rightarrow \mathbb{m}(M, \beta) \in F$ and for every sort
symbol a of Ω and for every element x of $\mathcal{X}(a)$ and for every element x_0 of
$\bigcup(\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma)))$ such that $x = x_0$ for every element
τ of (the sorts of $\mathfrak{T})(a)$ for every element τ_1 of $\bigcup$(the sorts of L) such that
$\tau_1 = \tau$ holds $x :=_L \tau \cdot \alpha \Leftrightarrow \alpha_{(x_0 \leftarrow \tau_1)} \in F$ and for every element y of $\mathcal{X}(a)$ such
that $y \notin (\text{vf } \tau)(a)$ for every element y_0 of $\bigcup(\mathcal{X}$ extended by$(\emptyset, (\text{the carrier}$
of $\Sigma)))$ such that $y = y_0$ holds $x :=_L \tau \cdot \exists_x \alpha \Leftrightarrow \exists_y (x :=_L \tau \cdot (y :=_L (\overset{@}{} x) \cdot$
$\alpha_{(x_0 \leftarrow y_0)})) \in F$ and $x :=_L \tau \cdot \alpha \Rightarrow \exists_x \alpha \in F$ and for every algorithms M,
M_1, M_2 of L, $(M; M_1) \cdot \alpha \Leftrightarrow (M \cdot (M_1 \cdot \alpha))$, if M then M_1 else $M_2 \cdot \alpha \Leftrightarrow$
$((M \cdot \varphi \wedge (M \cdot (M_1 \cdot \alpha))) \vee (M \cdot \neg \varphi \wedge (M \cdot (M_2 \cdot \alpha))))$, while M do $M_1 \cdot \alpha \Leftrightarrow$
$((M \cdot \neg \varphi \wedge \alpha) \vee (M \cdot \varphi \wedge (M \cdot (M_1 \cdot (\text{while } M \text{ do } M_1 \cdot \alpha))))) \in F$.

Let n be a non empty natural number, Σ be a non empty, non void, PC correct w.r.t. n PC language signature, and L be language algebra over Σ. One can check that $\Omega_{(\text{the sorts of } L)((\text{the formula sort of } \Sigma))}$ is PC-closed and every subset of (the sorts of L)((the formula sort of Σ)) which is PC-closed is also non empty and there exists a subset of (the sorts of L)((the formula sort of Σ)) which is PC-closed.

Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, Σ be an Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct QC language signature over $\bigcup \mathcal{X}$, and L be a non-empty language over $\mathcal{X}$ extended by$(\emptyset, (\text{the carrier of } \Sigma))$ and Σ. Let us observe that $\Omega_{(\text{the sorts of } L)((\text{the formula sort of } \Sigma))}$ is QC closed and there exists a subset of (the sorts of L)((the formula sort of Σ)) which is QC closed and PC-closed.

Let L be a non-empty, $\mathfrak{T}$-extension language over $\mathcal{X}$ extended by$(\emptyset, (\text{the carrier}$ of $\Sigma))$ and Σ. Observe that $\Omega_{(\text{the sorts of } L)((\text{the formula sort of } \Sigma))}$ has equality and there exists a subset of (the sorts of L)((the formula sort of Σ)) which is QC closed and PC-closed and has equality.

Let Σ be a non empty, non void, PC correct w.r.t. n PC language signature and L be language algebra over Σ.

A PC theory of L is a PC-closed subset of (the sorts of L)((the formula sort of Σ)). Let Ω be a non empty, non void signature, $\mathfrak{T}$ be a non-empty algebra over Ω, $\mathcal{X}$ be a non empty yielding generator set of $\mathfrak{T}$, Σ be an Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct QC language signature over

$\bigcup \mathcal{X}$, and L be a non-empty language over $\mathcal{X}$ extended by $(\emptyset, (\text{the carrier of } \Sigma))$ and Σ.

A QC theory of L is a QC closed, PC-closed subset of (the sorts of L)((the formula sort of Σ)). Let L be a non-empty, $\mathfrak{T}$-extension language over $\mathcal{X}$ extended by $(\emptyset, (\text{the carrier of } \Sigma))$ and Σ.

A QC theory of L with equality is a QC closed, PC-closed subset of (the sorts of L)((the formula sort of Σ)) with equality. Let n be a non empty, natural number, $\mathfrak{T}$ be a non-empty free variable algebra over Ω, $\mathcal{X}$ be a non-empty generator set of $\mathfrak{T}$, Σ be an essential, Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct, AL correct w.r.t. n algorithmic language signature of $\bigcup \mathcal{X}$, L be a non 1s-empty if-while algebra of $\mathcal{X}$ over Σ, and φ be a formula of L. Observe that there exists a PC-closed, QC closed subset of (the sorts of L)((the formula sort of Σ)) with equality which is AL closed w.r.t φ.

An AL theory of φ and L is a PC-closed, QC closed, AL closed w.r.t φ subset of (the sorts of L)((the formula sort of Σ)) with equality.

4. Propositional Calculus

In the sequel n denotes a non empty natural number, Σ denotes a non empty, non void, PC correct w.r.t. n PC language signature, L denotes language algebra over Σ, F denotes a PC theory of L, and α, β, γ, δ denote formulae of L.

Now we state the propositions:

(35) $\alpha \Rightarrow \alpha \in F$.

(36) $\alpha \wedge \beta \in F$ if and only if $\alpha, \beta \in F$.

(37) $(\alpha \vee \beta) \Rightarrow (\beta \vee \alpha) \in F$.

(38) $(\beta \Rightarrow \gamma) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)) \in F$.

(39) If $\alpha \Rightarrow (\beta \Rightarrow \gamma) \in F$, then $\beta \Rightarrow (\alpha \Rightarrow \gamma) \in F$. The theorem is a consequence of (38).

(40) $(\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)) \in F$. The theorem is a consequence of (38) and (39).

(41) $\alpha \Rightarrow (\beta \Rightarrow (\alpha \Rightarrow \beta)) \in F$.

(42) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow (\beta \Rightarrow (\alpha \Rightarrow \gamma)) \in F$. The theorem is a consequence of (39) and (38).

(43) $\beta \Rightarrow ((\beta \Rightarrow \alpha) \Rightarrow \alpha) \in F$. The theorem is a consequence of (42) and (35).

(44) $\alpha \Leftrightarrow \beta \in F$ if and only if $\alpha \Rightarrow \beta, \beta \Rightarrow \alpha \in F$. The theorem is a consequence of (36).

(45) If $\beta \in F$, then $\alpha \Rightarrow \beta \in F$.

(46)   If $\alpha \Rightarrow \beta$, $\beta \Rightarrow \gamma \in F$, then $\alpha \Rightarrow \gamma \in F$. The theorem is a consequence of (40).

(47)   If $\gamma \Rightarrow (\beta \Rightarrow \alpha)$, $\beta \in F$, then $\gamma \Rightarrow \alpha \in F$. The theorem is a consequence of (42).

(48) $((\alpha \wedge \beta) \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \in F$. The theorem is a consequence of (45), (40), and (42).

(49) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \wedge \beta) \Rightarrow \gamma) \in F$. The theorem is a consequence of (40), (45), (42), (46), and (47).

(50) $(\gamma \Rightarrow \alpha) \Rightarrow ((\gamma \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow (\alpha \wedge \beta))) \in F$. The theorem is a consequence of (45) and (46).

(51) $(\alpha \wedge \beta) \Rightarrow (\beta \wedge \alpha) \in F$. The theorem is a consequence of (50).

(52) $(\alpha \Leftrightarrow \beta) \Rightarrow (\beta \Leftrightarrow \alpha) \in F$. The theorem is a consequence of (51), (40), and (50).

(53) $(\alpha \vee \alpha) \Rightarrow \alpha \in F$. The theorem is a consequence of (35).

(54) $\alpha \Rightarrow (\alpha \wedge \alpha) \in F$. The theorem is a consequence of (50) and (35).

(55)   If $\alpha \Rightarrow \beta$, $\alpha \Rightarrow \gamma \in F$, then $\alpha \Rightarrow (\beta \wedge \gamma) \in F$. The theorem is a consequence of (50).

(56) $((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \Rightarrow (\alpha \wedge (\beta \vee \gamma)) \in F$. The theorem is a consequence of (46) and (50).

(57) $(\alpha \vee (\beta \wedge \gamma)) \Rightarrow ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \in F$. The theorem is a consequence of (50) and (46).

(58) $\alpha \Rightarrow (\neg \alpha \Rightarrow \beta) \in F$. The theorem is a consequence of (48).

(59) $(\alpha \Rightarrow \beta) \Rightarrow (\neg \beta \Rightarrow \neg \alpha) \in F$. The theorem is a consequence of (39) and (46).

(60) $\alpha \Rightarrow \beta \in F$ if and only if $\neg \beta \Rightarrow \neg \alpha \in F$. The theorem is a consequence of (59).

(61)   If $\alpha \Rightarrow \beta$, $\gamma \Rightarrow \delta \in F$, then $(\alpha \vee \gamma) \Rightarrow (\beta \vee \delta) \in F$. The theorem is a consequence of (46).

(62) $(\alpha \Rightarrow \beta) \Rightarrow ((\gamma \vee \alpha) \Rightarrow (\gamma \vee \beta)) \in F$. The theorem is a consequence of (35), (49), (46), and (48).

(63)   If $\alpha \Rightarrow \beta$, $\gamma \Rightarrow \delta$, $\neg \beta \vee \neg \delta \in F$, then $\neg \alpha \vee \neg \gamma \in F$. The theorem is a consequence of (59) and (61).

(64) $(\alpha \vee \beta) \Rightarrow (\neg \alpha \Rightarrow \beta) \in F$. The theorem is a consequence of (58).

(65) $(\alpha \vee \beta) \Rightarrow (\neg \beta \Rightarrow \alpha) \in F$. The theorem is a consequence of (37), (64), and (46).

(66) $\alpha \Rightarrow \neg\neg \alpha \in F$. The theorem is a consequence of (35) and (45).

(67) $\neg\neg \alpha \Rightarrow \alpha \in F$. The theorem is a consequence of (66).

(68) $\alpha \Leftrightarrow \neg\neg\alpha \in F$. The theorem is a consequence of (66), (67), and (36).

(69) $\alpha \Rightarrow \neg\beta \in F$ if and only if $\beta \Rightarrow \neg\alpha \in F$. The theorem is a consequence of (60), (66), and (46).

(70) $\neg\alpha \Rightarrow \beta \in F$ if and only if $\neg\beta \Rightarrow \alpha \in F$. The theorem is a consequence of (60), (67), and (46).

(71) If $\alpha \Rightarrow (\beta \Rightarrow \gamma)$, $\gamma \Rightarrow \delta \in F$, then $\alpha \Rightarrow (\beta \Rightarrow \delta) \in F$. The theorem is a consequence of (49), (46), and (48).

(72) $\neg(\alpha \wedge \beta) \Rightarrow (\neg\alpha \vee \neg\beta) \in F$. The theorem is a consequence of (60), (67), (46), and (50).

(73) $\neg(\alpha \vee \beta) \Rightarrow (\neg\alpha \wedge \neg\beta) \in F$. The theorem is a consequence of (60) and (50).

(74) If $\alpha \Rightarrow \beta$, $\gamma \Rightarrow \delta \in F$, then $(\alpha \wedge \gamma) \Rightarrow (\beta \wedge \delta) \in F$. The theorem is a consequence of (50) and (46).

(75) $(\neg\alpha \vee \neg\beta) \Rightarrow \neg(\alpha \wedge \beta) \in F$. The theorem is a consequence of (60).

(76) $(\neg\alpha \wedge \neg\beta) \Rightarrow \neg(\alpha \vee \beta) \in F$. The theorem is a consequence of (75), (66), (61), (46), and (60).

(77) $(\alpha \vee (\beta \vee \gamma)) \Rightarrow ((\alpha \vee \beta) \vee \gamma) \in F$. The theorem is a consequence of (46), (35), and (61).

(78) $((\alpha \vee \beta) \vee \gamma) \Leftrightarrow (\alpha \vee (\beta \vee \gamma)) \in F$. The theorem is a consequence of (35), (61), (46), (47), and (44).

(79) $(\alpha \wedge (\beta \wedge \gamma)) \Rightarrow ((\alpha \wedge \beta) \wedge \gamma) \in F$. The theorem is a consequence of (50), (35), (74), and (46).

(80) $((\alpha \wedge \beta) \wedge \gamma) \Leftrightarrow (\alpha \wedge (\beta \wedge \gamma)) \in F$. The theorem is a consequence of (50), (46), (35), (74), and (44).

(81) $(\gamma \vee (\alpha \Rightarrow \beta)) \Rightarrow ((\gamma \vee \alpha) \Rightarrow (\gamma \vee \beta)) \in F$. The theorem is a consequence of (45), (42), and (62).

(82) $((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \Rightarrow (\alpha \vee (\beta \wedge \gamma)) \in F$. The theorem is a consequence of (35), (61), (81), (46), and (49).

(83) $(\alpha \wedge (\beta \vee \gamma)) \Rightarrow ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \in F$. The theorem is a consequence of (82), (72), (74), (73), (46), (35), (76), (75), (61), and (60).

(84) $(\alpha \Rightarrow \beta) \Rightarrow (\neg\alpha \vee \beta) \in F$. The theorem is a consequence of (35), (49), (61), (82), (46), (51), (48), and (37).

(85) $(\alpha \Rightarrow \beta) \Rightarrow \neg(\alpha \wedge \neg\beta) \in F$. The theorem is a consequence of (84), (66), (35), (61), (46), and (75).

(86) $(\beta \vee (\neg\gamma \wedge \gamma)) \Rightarrow \beta \in F$. The theorem is a consequence of (51), (35), (46), (61), and (53).

(87) $(\beta \vee (\gamma \wedge \neg\gamma)) \Rightarrow \beta \in F$. The theorem is a consequence of (35), (61), (53), and (46).

(88) $(\alpha \Leftrightarrow \beta) \Rightarrow ((\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)) \in F$. The theorem is a consequence of (84), (46), (50), (83), (51), (86), (37), (87), (61), and (35).

(89) $(\alpha \Leftrightarrow \beta) \Rightarrow ((\alpha \vee \neg\beta) \wedge (\neg\alpha \vee \beta)) \in F$. The theorem is a consequence of (51), (56), (61), (46), (35), (37), and (88).

(90) $\neg(\alpha \wedge \neg\alpha) \in F$. The theorem is a consequence of (35), (66), (76), (74), (46), (60), and (51).

(91) $\alpha \Leftrightarrow \alpha \in F$. The theorem is a consequence of (35) and (36).

(92) If $\alpha \Leftrightarrow \beta \in F$, then $\beta \Leftrightarrow \alpha \in F$. The theorem is a consequence of (36).

(93) If $\alpha \Leftrightarrow \beta$, $\beta \Leftrightarrow \gamma \in F$, then $\alpha \Leftrightarrow \gamma \in F$. The theorem is a consequence of (46) and (36).

(94) If $\alpha \Leftrightarrow \beta$, $\beta \Rightarrow \gamma \in F$, then $\alpha \Rightarrow \gamma \in F$. The theorem is a consequence of (46).

(95) If $\alpha \Rightarrow \beta$, $\beta \Leftrightarrow \gamma \in F$, then $\alpha \Rightarrow \gamma \in F$. The theorem is a consequence of (46).

(96) $\alpha \Leftrightarrow \beta \in F$ if and only if $\neg\alpha \Leftrightarrow \neg\beta \in F$. The theorem is a consequence of (59) and (36).

(97) $\alpha \Leftrightarrow \beta \in F$ if and only if $\neg\neg\alpha \Leftrightarrow \beta \in F$. The theorem is a consequence of (66), (67), (36), (92), and (93).

(98) If $\alpha \Rightarrow (\beta \Rightarrow \gamma)$, $\delta \Rightarrow \beta \in F$, then $\alpha \Rightarrow (\delta \Rightarrow \gamma) \in F$. The theorem is a consequence of (42) and (46).

(99) If $\alpha \Leftrightarrow (\beta \wedge \gamma)$, $\gamma \Leftrightarrow \delta \in F$, then $\alpha \Leftrightarrow (\beta \wedge \delta) \in F$. The theorem is a consequence of (92), (44), (35), (74), (94), and (95).

(100) If $\alpha \Leftrightarrow (\beta \wedge \gamma)$, $\beta \Leftrightarrow \delta \in F$, then $\alpha \Leftrightarrow (\delta \wedge \gamma) \in F$. The theorem is a consequence of (92), (44), (35), (74), (94), and (95).

(101) If $\alpha \Leftrightarrow (\beta \vee \gamma)$, $\gamma \Leftrightarrow \delta \in F$, then $\alpha \Leftrightarrow (\beta \vee \delta) \in F$. The theorem is a consequence of (92), (44), (35), (61), (94), and (95).

(102) If $\alpha \Leftrightarrow (\beta \vee \gamma)$, $\beta \Leftrightarrow \delta \in F$, then $\alpha \Leftrightarrow (\delta \vee \gamma) \in F$. The theorem is a consequence of (92), (44), (35), (61), (94), and (95).

Let us assume that $\alpha \Rightarrow \beta \in F$. Now we state the propositions:

(103) $(\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma) \in F$. The theorem is a consequence of (40).

(104) $(\gamma \Rightarrow \alpha) \Rightarrow (\gamma \Rightarrow \beta) \in F$. The theorem is a consequence of (40) and (39).

Now we state the proposition:

(105) If $\alpha \Rightarrow \beta$, $\gamma \Rightarrow \delta \in F$, then $(\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \delta) \in F$. The theorem is a consequence of (103), (104), and (46).

5. Quantifier Calculus

From now on Ω denotes a non empty, non void signature, $\mathfrak{T}$ denotes a non-empty algebra over Ω, $\mathcal{X}$ denotes a non empty yielding generator set of $\mathfrak{T}$, Σ_1 denotes an Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct QC language signature over $\bigcup \mathcal{X}$, L denotes a non-empty language over $\mathcal{X}$ extended by $(\emptyset, (\text{the carrier of } \Sigma_1))$ and Σ_1, G denotes a QC theory of L, α, β, γ, δ denote formulae of L, x, y, z denote elements of $\bigcup \mathcal{X}$, and x_0, y_0, z_0 denote elements of $\bigcup(\mathcal{X} \text{ extended by } (\emptyset, (\text{the carrier of } \Sigma_1)))$.

Now we state the propositions:

(106)   If L is substitution correct (1), then $\forall_x \alpha \Rightarrow \alpha \in G$. The theorem is a consequence of (1), (16), (24), and (14).

(107) $\exists_x \alpha \Leftrightarrow \neg \forall_x (\neg \alpha) \in G$. The theorem is a consequence of (96) and (97).

(108) $\forall_x \alpha \Leftrightarrow \neg \exists_x (\neg \alpha) \in G$. The theorem is a consequence of (92), (96), and (97).

(109)   If L is substitution correct (1), then $\forall_x (\alpha \Rightarrow \beta) \Rightarrow (\forall_x \alpha \Rightarrow \beta) \in G$. The theorem is a consequence of (1), (13), (24), (25), (16), (14), and (98).

(110)   Let us consider a sort symbol a of Ω. Suppose $x \in \mathcal{X}(a)$ and $x \notin (\text{vf } \alpha)(a)$ and $\forall_x (\alpha \Rightarrow \beta) \in G$. Then $\alpha \Rightarrow \forall_x \beta \in G$.

(111)   Suppose L is substitution correct (1) and vf-QC-correct. Then $\forall_x (\alpha \Rightarrow \beta) \Rightarrow (\forall_x \alpha \Rightarrow \forall_x \beta) \in G$. The theorem is a consequence of (109), (110), and (46).

(112)   Suppose L is substitution correct (1). Let us consider a sort symbol a of Ω. Suppose $x, y \in \mathcal{X}(a)$ and $x_0 = x$ and $y_0 = y$. Then $\alpha_{(x_0 \leftarrow y_0)} \Rightarrow \exists_x \alpha \in G$. The theorem is a consequence of (16), (8), (2), (24), (14), (60), (28), (66), (107), and (46).

(113)   If L is substitution correct (1) and vf-QC-correct, then $\exists_{x,y} \alpha \Leftrightarrow \neg \forall_{x,y} (\neg \alpha) \in G$. The theorem is a consequence of (107), (68), (92), (96), (93), (44), and (111).

(114)   If L is substitution correct (1), then $\alpha \Rightarrow \exists_x \alpha \in G$. The theorem is a consequence of (8), (2), and (112).

Let us consider a sort symbol a of Σ_1.

Let us assume that L is vf-QC-correct. Now we state the propositions:

(115)   If $x \in \mathcal{X}(a)$, then $x \notin (\text{vf } \forall_x \alpha)(a)$.

(116)   If $x \in \mathcal{X}(a)$, then $x \notin (\text{vf } \exists_x \alpha)(a)$.

Now we state the proposition:

(117)   If L is substitution correct (1) and vf-QC-correct and $\alpha \Rightarrow \beta \in G$, then $\forall_x \alpha \Rightarrow \forall_x \beta \in G$. The theorem is a consequence of (111).

Let us assume that L is substitution correct (1) and vf-QC-correct. Now we state the propositions:

(118) $\forall_x(\neg\alpha \Rightarrow \neg\beta) \Rightarrow (\forall_x \beta \Rightarrow \forall_x \alpha) \in G$. The theorem is a consequence of (117), (111), and (46).

(119) $\forall_x(\alpha \Rightarrow \beta) \Rightarrow (\forall_x(\neg\beta) \Rightarrow \forall_x(\neg\alpha)) \in G$. The theorem is a consequence of (59), (117), (111), and (46).

Now we state the propositions:

(120) If L is substitution correct (1) and vf-QC-correct and $\alpha \Leftrightarrow \beta \in G$, then $\forall_x \alpha \Leftrightarrow \forall_x \beta \in G$. The theorem is a consequence of (117), (111), (46), and (50).

(121) $\exists_x(\neg\alpha) \Leftrightarrow \neg\forall_x \alpha \in G$.

Let us assume that L is substitution correct (1) and vf-QC-correct. Now we state the propositions:

(122) Let us consider a sort symbol a of Ω. Suppose $x \in \mathcal{X}(a)$ and $x \notin (\mathrm{vf}\,\beta)(a)$. Then $\forall_x(\alpha \Rightarrow \beta) \Rightarrow (\exists_x \alpha \Rightarrow \beta) \in G$. The theorem is a consequence of (59), (117), (46), and (71).

(123) $\forall_x(\alpha \Rightarrow \beta) \Rightarrow (\exists_x \alpha \Rightarrow \exists_x \beta) \in G$. The theorem is a consequence of (119), (59), (46), (44), (66), (60), (67), and (105).

(124) $\forall_x(\neg\alpha) \Leftrightarrow \neg\exists_x \alpha \in G$. The theorem is a consequence of (66), (123), (67), (44), (93), (92), (96), and (97).

(125) $\forall_{x,y} \alpha \Leftrightarrow \neg\exists_{x,y}(\neg\alpha) \in G$. The theorem is a consequence of (108), (68), (92), (96), (93), (44), and (123).

(126) $\forall_x \alpha \Leftrightarrow \forall_x(\neg\neg\alpha) \in G$. The theorem is a consequence of (67), (66), (117), and (36).

(127) $\forall_x(\alpha \wedge \beta) \Rightarrow (\forall_x \alpha \wedge \forall_x \beta) \in G$. The theorem is a consequence of (117) and (50).

Now we state the propositions:

(128) If L is vf-QC-correct and substitution correct (1), then $(\forall_x \alpha \wedge \forall_x \beta) \Rightarrow \forall_x(\alpha \wedge \beta) \in G$. The theorem is a consequence of (106), (74), (115), and (110).

(129) If L is substitution correct (1) and vf-QC-correct, then $(\forall_x \alpha \vee \forall_x \beta) \Rightarrow \forall_x(\alpha \vee \beta) \in G$. The theorem is a consequence of (106), (61), (115), and (110).

(130) If L is substitution correct (1) and vf-QC-correct and $\alpha \Rightarrow \beta \in G$, then $\exists_x \alpha \Rightarrow \exists_x \beta \in G$. The theorem is a consequence of (59), (117), (107), (94), (92), and (95).

(131) If L is substitution correct (1) and vf-QC-correct and $\alpha \Leftrightarrow \beta \in G$, then $\exists_x \alpha \Leftrightarrow \exists_x \beta \in G$. The theorem is a consequence of (130) and (36).

Let us assume that L is substitution correct (1) and vf-QC-correct. Now we state the propositions:

(132) $\exists_x \alpha \Leftrightarrow \exists_x(\neg\neg\alpha) \in G$. The theorem is a consequence of (68) and (131).

(133) $(\exists_x \alpha \vee \exists_x \beta) \Leftrightarrow \exists_x(\alpha \vee \beta) \in G$. The theorem is a consequence of (127), (60), (75), (46), (73), (117), (107), (92), (95), (44), (61), (76), (72), and (128).

Now we state the proposition:

(134) Suppose L is substitution correct (1). Let us consider a sort symbol a of Ω. Suppose $x \in \mathcal{X}(a)$ and $x \notin (\mathrm{vf}\,\alpha)(a)$. Then $\alpha \Leftrightarrow \forall_x \alpha \in G$. The theorem is a consequence of (109), (35), (110), and (44).

In the sequel a denotes a sort symbol of Ω.

Let us assume that L is substitution correct (1) and vf-QC-correct and $x \in \mathcal{X}(a)$ and $x \notin (\mathrm{vf}\,\alpha)(a)$. Now we state the propositions:

(135) $\forall_x(\alpha \vee \beta) \Rightarrow (\alpha \vee \forall_x \beta) \in G$. The theorem is a consequence of (64), (117), (46), (35), (67), (61), and (84).

(136) $\exists_x(\alpha \wedge \beta) \Rightarrow (\alpha \wedge \exists_x \beta) \in G$. The theorem is a consequence of (129), (60), (75), (117), (107), (44), (46), (74), (73), (35), and (122).

(137) $\exists_x(\alpha \wedge \beta) \Leftrightarrow (\alpha \wedge \exists_x \beta) \in G$. The theorem is a consequence of (135), (60), (124), (35), (44), (61), (75), (69), (46), (72), (107), (111), (70), and (136).

(138) $\exists_x(\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \exists_x \beta) \in G$. The theorem is a consequence of (35), (110), (74), (128), (46), (60), (66), (61), (75), (84), (117), (107), (94), (44), (72), (64), and (105).

Now we state the proposition:

(139) If L is vf-QC-correct, then $\forall_x \alpha \Rightarrow \forall_{x,x} \alpha \in G$. The theorem is a consequence of (8), (115), (35), and (110).

Let us assume that L is vf-QC-correct and substitution correct (1). Now we state the propositions:

(140) $\forall_{x,y} \alpha \Rightarrow \forall_{y,x} \alpha \in G$. The theorem is a consequence of (106), (111), (115), and (110).

(141) $\exists_{x,y} \alpha \Rightarrow \exists_{y,x} \alpha \in G$. The theorem is a consequence of (140), (60), (113), (92), (94), and (95).

(142) $\exists_x(\forall_y \alpha) \Rightarrow \forall_y(\exists_x \alpha) \in G$. The theorem is a consequence of (114), (111), (116), and (122).

Let us assume that L is substitution correct (1) and vf-QC-correct. Now we state the propositions:

(143) $\forall_x(\alpha \wedge \alpha) \Leftrightarrow \forall_x \alpha \in G$. The theorem is a consequence of (54), (111), and (44).

(144) $\forall_x(\alpha \vee \alpha) \Leftrightarrow \forall_x \alpha \in G$. The theorem is a consequence of (53), (111), and (44).

(145) $\exists_x(\alpha \vee \alpha) \Leftrightarrow \exists_x \alpha \in G$. The theorem is a consequence of (53), (44), and (131).

From now on L denotes a non-empty, $\mathfrak{T}$-extension language over $\mathcal{X}$ extended by $(\emptyset, (\text{the carrier of } \Sigma_1))$ and Σ_1, G_1 denotes a QC theory of L with equality, α, β, γ, δ denote formulae of L, σ, σ_1 denote sort symbols of Σ_1, τ denotes elements of L from σ, and τ_1, τ_2, τ_3 denote elements of L from σ_1.

Now we state the proposition:

(146) Suppose L is substitution eq-correct and $x_0 \in \mathcal{X}(\sigma)$ and $\tau_1 =_L \tau_2 \in G_1$. Then $\tau_{1(x_0 \leftarrow \tau)} =_L \tau_{2(x_0 \leftarrow \tau)} \in G_1$. The theorem is a consequence of (1).

Let us assume that L is substitution eq-correct, vf finite, substitution correct (2), and substitution correct (3) and $\sigma_1 \in$ the carrier of Ω and $\mathcal{X}(\sigma_1)$ is infinite. Now we state the propositions:

(147) $(\tau_1 =_L \tau_2) \Rightarrow (\tau_2 =_L \tau_1) \in G_1$. The theorem is a consequence of (1), (25), (16), and (47).

(148) $((\tau_1 =_L \tau_2) \wedge (\tau_2 =_L \tau_3)) \Rightarrow (\tau_1 =_L \tau_3) \in G_1$. The theorem is a consequence of (1), (25), (16), (147), (46), and (49).

Let us assume that L is substitution correct (3), vf finite, substitution correct (2), substitution correct (1), substitution eq-correct, vf-QC-correct, and vf-eq-correct and $x = x_0 \in \mathcal{X}(\sigma)$ and $y = y_0 \in \mathcal{X}(\sigma)$ and $x \neq y \notin (\text{vf } \alpha)(\sigma)$ and $\mathcal{X}(\sigma)$ is infinite. Now we state the propositions:

(149) $\forall_x(\alpha \Leftrightarrow \exists_y((x =_L y) \wedge \alpha_{(x_0 \leftarrow y_0)})) \in G_1$. The theorem is a consequence of (16), (147), (46), (1), (24), (14), (49), (122), (112), (28), (35), (45), (55), and (44).

(150) $\forall_x(\alpha \Leftrightarrow \forall_y((x =_L y) \Rightarrow \alpha_{(x_0 \leftarrow y_0)})) \in G_1$. The theorem is a consequence of (16), (1), (24), (14), (39), (110), (112), (28), (35), (45), (55), (46), (70), (66), (75), (61), (84), (69), (130), (60), (108), (94), and (44).

Now we state the proposition:

(151) Suppose L is substitution correct (1), substitution eq-correct, substitution correct (3), and vf-eq-correct and $x, y \in \mathcal{X}(\sigma)$ and $x \neq y$. Then $\forall_x(\exists_y(x =_L y)) \in G_1$. The theorem is a consequence of (16), (25), (1), (24), (112), and (14).

Let us assume that L is substitution correct (1) and $x = x_0 \in \mathcal{X}(\sigma)$ and $y = y_0 \in \mathcal{X}(\sigma)$. Now we state the propositions:

(152) $(\alpha \wedge (x =_L y)) \Rightarrow \alpha_{(x_0 \leftarrow y_0)} \in G_1$. The theorem is a consequence of (16), (1), (24), (14), (39), and (49).

(153) $(\alpha \wedge \neg(\alpha_{(x_0 \leftarrow y_0)})) \Rightarrow \neg(x =_L y) \in G_1$. The theorem is a consequence of (16), (152), (48), (59), (46), and (49).

6. Algorithmic Logic

From now on n denotes a non empty, natural number, Ω denotes a non empty, non void signature, $\mathfrak{T}$ denotes a non-empty free variable algebra over Ω, $\mathcal{X}$ denotes a non-empty generator set of $\mathfrak{T}$, Σ denotes an essential, Ω-extension, non empty, non void, PC correct w.r.t. n, QC correct, AL correct w.r.t. n algorithmic language signature of $\bigcup \mathcal{X}$, L denotes a non empty if-while algebra of $\mathcal{X}$ over Σ, M, M_1, M_2 denote algorithms of L, α, β, γ, φ denote formulae of L, H denotes an AL theory of φ and L, a denotes a sort symbol of Ω, x, y denote elements of $\mathcal{X}(a)$, and τ denotes an element of $\mathfrak{T}$ from a.

Now we state the propositions:

(154) $M \cdot ((\alpha \wedge \beta) \wedge \gamma) \Leftrightarrow ((M \cdot \alpha \wedge (M \cdot \beta)) \wedge (M \cdot \gamma)) \in H$. The theorem is a consequence of (100).

(155) $M \cdot ((\alpha \vee \beta) \vee \gamma) \Leftrightarrow ((M \cdot \alpha \vee (M \cdot \beta)) \vee (M \cdot \gamma)) \in H$. The theorem is a consequence of (102).

Let us assume that $\alpha \Leftrightarrow \beta \in H$. Now we state the propositions:

(156) $\mathbb{U}(M, \alpha) \Leftrightarrow \mathbb{U}(M, \beta) \in H$. The theorem is a consequence of (44).

(157) $\mathbb{m}(M, \alpha) \Leftrightarrow \mathbb{m}(M, \beta) \in H$. The theorem is a consequence of (44).

Now we state the propositions:

(158) $\mathbb{U}(M, \alpha) \Leftrightarrow ((\alpha \vee (M \cdot \alpha)) \vee \mathbb{U}(M, (M; M) \cdot \alpha)) \in H$. The theorem is a consequence of (101), (78), (92), (93), and (156).

(159) $\mathbb{m}(M, \alpha) \Leftrightarrow ((\alpha \wedge (M \cdot \alpha)) \wedge \mathbb{m}(M, (M; M) \cdot \alpha)) \in H$. The theorem is a consequence of (99), (80), (92), (93), and (157).

(160) Let us consider elements x_0, y_0 of $\bigcup(\mathcal{X}$ extended by$(\emptyset, ($the carrier of $\Sigma)))$. Suppose $x = x_0$ and $y = y_0$. Then $x :=_L (^@y) \cdot \alpha \Leftrightarrow \alpha_{(x_0 \leftarrow y_0)} \in H$. The theorem is a consequence of (8), (16), (24), and (14).

(161) Suppose $M \cdot \varphi$, $M \cdot (M_1 \cdot \alpha) \in H$ or $M \cdot \neg \varphi$, $M \cdot (M_2 \cdot \alpha) \in H$. Then if M then M_1 else $M_2 \cdot \alpha \in H$. The theorem is a consequence of (36) and (44).

(162) Suppose $M \cdot \neg \varphi$, $\alpha \in H$ or $M \cdot \varphi$, $M \cdot (M_1 \cdot (\text{while } M \text{ do } M_1 \cdot \alpha)) \in H$. Then while M do $M_1 \cdot \alpha \in H$. The theorem is a consequence of (36) and (44).

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