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Abstract Reduction Systems and Idea of Knuth-Bendix Completion Algorithm

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Summary. Educational content for abstract reduction systems concerning reduction, convertibility, normal forms, divergence and convergence, Church-Rosser property, term rewriting systems, and the idea of the Knuth-Bendix Completion Algorithm. The theory is based on [1].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [17], [16], [7], [9], [20], [14], [18], [10], [11], [8], [22], [3], [4], [12], [5], [23], [24], [6], [21], [15], and [13].

1. Reduction and Convertibility

We consider ARS's which extend 1-sorted structures and are systems

$$\langle \text{a carrier}, \text{a reduction} \rangle$$

where the carrier is a set, the reduction is a binary relation on the carrier.

Let A be a non empty set and r be a binary relation on A. Observe that $\langle A, r \rangle$ is non empty and there exists an ARS which is non empty and strict.

Let X be an ARS and x, y be elements of X. We say that $x \to y$ if and only if

(Def. 1) $\langle x, y \rangle \in$ the reduction of X.

We introduce $y \leftarrow x$ as a synonym of $x \to y$.

We say that $x \to_{01} y$ if and only if

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(Def. 2) (i) $x = y$, or

(ii) $x \to y$.

One can verify that the predicate is reflexive. We say that $x \to_* y$ if and only if

(Def. 3) The reduction of X reduces x to y.

Let us observe that the predicate is reflexive.

From now on X denotes an ARS and a, b, c, u, v, w, x, y, z denote elements of X.

Now we state the propositions:

(1) If $a \to b$, then X is not empty.

(2) If $x \to y$, then $x \to_* y$.

(3) If $x \to_* y \to_* z$, then $x \to_* z$.

The scheme *Star* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every elements x, y of $\mathcal{X}$ such that $x \to_* y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$

provided

- for every elements x, y of $\mathcal{X}$ such that $x \to y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

The scheme *Star1* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and elements a, b of $\mathcal{X}$ and states that

(Sch. 2) $\mathcal{P}[b]$

provided

- $a \to_* b$ and
- $\mathcal{P}[a]$ and
- for every elements x, y of $\mathcal{X}$ such that $x \to y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

The scheme *StarBack* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 3) For every elements x, y of $\mathcal{X}$ such that $x \to_* y$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x]$

provided

- for every elements x, y of $\mathcal{X}$ such that $x \to y$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x]$.

The scheme *StarBack1* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and elements a, b of $\mathcal{X}$ and states that

(Sch. 4) $\mathcal{P}[a]$

provided

- $a \to_* b$ and
- $\mathcal{P}[b]$ and

- for every elements x, y of $\mathcal{X}$ such that $x \to y$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x]$.

Let X be an ARS and x, y be elements of X. We say that $x \to_{+} y$ if and only if

(Def. 4) There exists an element z of X such that $x \to z \to_{*} y$.

Now we state the proposition:

(4) $x \to_{+} y$ if and only if there exists z such that $x \to_{*} z \to y$. Proof: If $x \to_{+} y$, then there exists z such that $x \to_{*} z \to y$. Define $\mathcal{P}$[element of X] $\equiv$ there exists u such that $\$_1 \to u \to_{*} y$. For every y and z such that $y \to z$ and $\mathcal{P}[z]$ holds $\mathcal{P}[y]$. For every y and z such that $y \to_{*} z$ and $\mathcal{P}[z]$ holds $\mathcal{P}[y]$ from *StarBack*. $\square$

Let us consider X, x, and y. We introduce $y \leftarrow_{01} x$ as a synonym of $x \to_{01} y$ and $y \leftarrow_{*} x$ as a synonym of $x \to_{*} y$ and $y \leftarrow_{+} x$ as a synonym of $x \to_{+} y$.

We say that $x \leftrightarrow y$ if and only if

(Def. 5) (i) $x \to y$, or

(ii) $x \leftarrow y$.

One can check that the predicate is symmetric.

Now we state the proposition:

(5) $x \leftrightarrow y$ if and only if $\langle x, y\rangle \in$ (the reduction of X) $\cup$ (the reduction of $X)^{\smile}$.

Let us consider X, x, and y. We say that $x \leftrightarrow_{01} y$ if and only if

(Def. 6) (i) $x = y$, or

(ii) $x \leftrightarrow y$.

Observe that the predicate is reflexive and symmetric. We say that $x \leftrightarrow_{*} y$ if and only if

(Def. 7) x and y are convertible w.r.t. the reduction of X.

One can check that the predicate is reflexive and symmetric.

Now we state the propositions:

(6) If $x \leftrightarrow y$, then $x \leftrightarrow_{*} y$.

(7) If $x \leftrightarrow_{*} y \leftrightarrow_{*} z$, then $x \leftrightarrow_{*} z$.

The scheme *Star2* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 5) For every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow_{*} y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$

provided

- for every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

The scheme *Star2A* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and elements a, b of $\mathcal{X}$ and states that

(Sch. 6) $\mathcal{P}[b]$

provided

- $a \leftrightarrow_* b$ and
- $\mathcal{P}[a]$ and
- for every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow y$ and $\mathcal{P}[x]$ holds $\mathcal{P}[y]$.

Let us consider X, x, and y. We say that $x \leftrightarrow_+ y$ if and only if

(Def. 8) There exists z such that $x \leftrightarrow z \leftrightarrow_* y$.

One can check that the predicate is symmetric.

Now we state the propositions:

(8) $x \leftrightarrow_+ y$ if and only if there exists z such that $x \leftrightarrow_* z \leftrightarrow y$.

(9) If $x \rightarrow_{01} y$, then $x \rightarrow_* y$.

(10) If $x \rightarrow_+ y$, then $x \rightarrow_* y$. The theorem is a consequence of (2) and (3).

(11) If $x \rightarrow y$, then $x \rightarrow_+ y$.

(12) If $x \rightarrow y \rightarrow z$, then $x \rightarrow_* z$. The theorem is a consequence of (2) and (3).

(13) If $x \rightarrow y \rightarrow_{01} z$, then $x \rightarrow_* z$. The theorem is a consequence of (2), (9), and (3).

(14) If $x \rightarrow y \rightarrow_* z$, then $x \rightarrow_* z$. The theorem is a consequence of (2) and (3).

(15) If $x \rightarrow y \rightarrow_+ z$, then $x \rightarrow_* z$. The theorem is a consequence of (2), (10), and (3).

(16) If $x \rightarrow_{01} y \rightarrow z$, then $x \rightarrow_* z$. The theorem is a consequence of (9), (2), and (3).

(17) If $x \rightarrow_{01} y \rightarrow_{01} z$, then $x \rightarrow_* z$. The theorem is a consequence of (9) and (3).

(18) If $x \rightarrow_{01} y \rightarrow_* z$, then $x \rightarrow_* z$. The theorem is a consequence of (9) and (3).

(19) If $x \rightarrow_{01} y \rightarrow_+ z$, then $x \rightarrow_* z$. The theorem is a consequence of (9), (10), and (3).

(20) If $x \rightarrow_* y \rightarrow z$, then $x \rightarrow_* z$. The theorem is a consequence of (2) and (3).

(21) If $x \rightarrow_* y \rightarrow_{01} z$, then $x \rightarrow_* z$. The theorem is a consequence of (9) and (3).

(22) If $x \rightarrow_* y \rightarrow_+ z$, then $x \rightarrow_* z$. The theorem is a consequence of (10) and (3).

(23) If $x \rightarrow_+ y \rightarrow z$, then $x \rightarrow_* z$. The theorem is a consequence of (10), (2), and (3).

(24) If $x \to_+ y \to_{01} z$, then $x \to_* z$. The theorem is a consequence of (10), (9), and (3).

(25) If $x \to_+ y \to_+ z$, then $x \to_* z$. The theorem is a consequence of (10) and (3).

(26) If $x \to y \to z$, then $x \to_+ z$.

(27) If $x \to y \to_{01} z$, then $x \to_+ z$.

(28) If $x \to y \to_+ z$, then $x \to_+ z$.

(29) If $x \to_{01} y \to z$, then $x \to_+ z$.

(30) If $x \to_{01} y \to_+ z$, then $x \to_+ z$. The theorem is a consequence of (4) and (18).

(31) If $x \to_* y \to_+ z$, then $x \to_+ z$. The theorem is a consequence of (4) and (3).

(32) If $x \to_+ y \to z$, then $x \to_+ z$.

(33) If $x \to_+ y \to_{01} z$, then $x \to_+ z$.

(34) If $x \to_+ y \to_* z$, then $x \to_+ z$.

(35) If $x \to_+ y \to_+ z$, then $x \to_+ z$.

(36) If $x \leftrightarrow_{01} y$, then $x \leftrightarrow_* y$.

(37) If $x \leftrightarrow_+ y$, then $x \leftrightarrow_* y$. The theorem is a consequence of (6) and (7).

(38) If $x \leftrightarrow y$, then $x \leftrightarrow_+ y$.

(39) If $x \leftrightarrow y \leftrightarrow z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (6) and (7).

(40) If $x \leftrightarrow y \leftrightarrow_{01} z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (6), (36), and (7).

(41) If $x \leftrightarrow_{01} y \leftrightarrow z$, then $x \leftrightarrow_* z$.

(42) If $x \leftrightarrow y \leftrightarrow_* z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (6) and (7).

(43) If $x \leftrightarrow_* y \leftrightarrow z$, then $x \leftrightarrow_* z$.

(44) If $x \leftrightarrow y \leftrightarrow_+ z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (6), (37), and (7).

(45) If $x \leftrightarrow_+ y \leftrightarrow z$, then $x \leftrightarrow_* z$.

(46) If $x \leftrightarrow_{01} y \leftrightarrow_{01} z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (36) and (7).

(47) If $x \leftrightarrow_{01} y \leftrightarrow_* z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (36) and (7).

(48) If $x \leftrightarrow_* y \leftrightarrow_{01} z$, then $x \leftrightarrow_* z$.

(49) If $x \leftrightarrow_{01} y \leftrightarrow_+ z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (36), (37), and (7).

(50) If $x \leftrightarrow_+ y \leftrightarrow_{01} z$, then $x \leftrightarrow_* z$.

(51) If $x \leftrightarrow_* y \leftrightarrow_+ z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (37) and (7).

(52) If $x \leftrightarrow_+ y \leftrightarrow_+ z$, then $x \leftrightarrow_* z$. The theorem is a consequence of (37) and (7).

(53) If $x \leftrightarrow y \leftrightarrow z$, then $x \leftrightarrow_+ z$.

(54) If $x \leftrightarrow y \leftrightarrow_{01} z$, then $x \leftrightarrow_+ z$.

(55) If $x \leftrightarrow y \leftrightarrow_+ z$, then $x \leftrightarrow_+ z$.

(56) If $x \leftrightarrow_{01} y \leftrightarrow_+ z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (8) and (47).

(57) If $x \leftrightarrow_* y \leftrightarrow_+ z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (8) and (7).

(58) If $x \leftrightarrow_+ y \leftrightarrow_+ z$, then $x \leftrightarrow_+ z$.

(59) If $x \leftrightarrow_{01} y$, then $x \leftarrow y$ or $x = y$ or $x \rightarrow y$.

(60) If $x \leftarrow y$ or $x = y$ or $x \rightarrow y$, then $x \leftrightarrow_{01} y$.

(61) If $x \leftrightarrow_{01} y$, then $x \leftarrow_{01} y$ or $x \rightarrow y$.

(62) If $x \leftarrow_{01} y$ or $x \rightarrow y$, then $x \leftrightarrow_{01} y$.

Let us assume that $x \leftrightarrow_{01} y$. Now we state the propositions:

(63) (i) $x \leftarrow_{01} y$, or

(ii) $x \rightarrow_+ y$.

(64) (i) $x \leftarrow_{01} y$, or

(ii) $x \leftrightarrow y$.

Now we state the propositions:

(65) If $x \leftarrow_{01} y$ or $x \leftrightarrow y$, then $x \leftrightarrow_{01} y$.

(66) If $x \leftrightarrow_* y \rightarrow z$, then $x \leftrightarrow_+ z$.

(67) If $x \leftrightarrow_+ y \rightarrow z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (37).

Let us assume that $x \leftrightarrow_{01} y$. Now we state the propositions:

(68) (i) $x \leftarrow_{01} y$, or

(ii) $x \rightarrow y$.

(69) (i) $x \leftarrow_{01} y$, or

(ii) $x \rightarrow_+ y$.

Now we state the propositions:

(70) If $x \leftarrow_{01} y$ or $x \rightarrow y$, then $x \leftrightarrow_{01} y$.

(71) If $x \leftarrow_{01} y$ or $x \leftrightarrow y$, then $x \leftrightarrow_{01} y$.

(72) If $x \leftrightarrow_{01} y$, then $x \leftarrow_{01} y$ or $x \leftrightarrow y$.

(73) If $x \leftrightarrow_+ y \rightarrow z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (37).

(74) If $x \leftrightarrow_* y \rightarrow z$, then $x \leftrightarrow_+ z$.

(75) If $x \leftrightarrow_{01} y \rightarrow z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (36).

(76) If $x \leftrightarrow_+ y \to_{01} z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (70) and (56).

(77) If $x \leftrightarrow y \to_{01} z$, then $x \leftrightarrow_+ z$. The theorem is a consequence of (70), (38), and (56).

(78) If $x \to y \to z \to u$, then $x \to_+ u$.

(79) If $x \to y \to_{01} z \to u$, then $x \to_+ u$.

(80) If $x \to y \to_* z \to u$, then $x \to_+ u$.

(81) If $x \to y \to_+ z \to u$, then $x \to_+ u$. The theorem is a consequence of (15) and (4).

(82) If $x \to_* y$, then $x \leftrightarrow_* y$. PROOF: Define $\mathcal{P}[\text{element of } X] \equiv x \leftrightarrow_* \$_1$. For every y and z such that $y \to z$ and $\mathcal{P}[y]$ holds $\mathcal{P}[z]$. $\mathcal{P}[y]$ from *Star1*. $\square$

(83) Suppose for every x and y such that $x \to z$ and $x \to y$ holds $y \to z$. If $x \to z$ and $x \to_* y$, then $y \to z$. PROOF: Define $\mathcal{P}[\text{element of } X] \equiv \$_1 \to z$. For every u and v such that $u \to_* v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$ from *Star*. $\square$

(84) If for every x and y such that $x \to y$ holds $y \to x$, then for every x and y such that $x \leftrightarrow_* y$ holds $x \to_* y$. PROOF: Define $\mathcal{P}[\text{element of } X] \equiv x \to_* \$_1$. For every u and v such that $u \leftrightarrow v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$. For every u and v such that $u \leftrightarrow_* v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$ from *Star2*. $\square$

(85) If $x \to_* y$, then $x = y$ or $x \to_+ y$. PROOF: Define $\mathcal{P}[\text{element of } X] \equiv x = \$_1$ or $x \to_+ \$_1$. For every y and z such that $y \to z$ and $\mathcal{P}[y]$ holds $\mathcal{P}[z]$. $\mathcal{P}[y]$ from *Star1*. $\square$

(86) If for every x, y, and z such that $x \to y \to z$ holds $x \to z$, then for every x and y such that $x \to_+ y$ holds $x \to y$. PROOF: Consider z such that $x \to z$ and $z \to_* y$. Define $\mathcal{P}[\text{element of } X] \equiv x \to \$_1$. $\mathcal{P}[y]$ from *Star1*. $\square$

2. EXAMPLES OF AN ABSTRACT REDUCTION SYSTEM

The scheme *ARSex* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{R}$ and states that

(Sch. 7) There exists a strict non empty ARS X such that the carrier of $X = \mathcal{A}$ and for every elements x, y of X, $x \to y$ iff $\mathcal{R}[x, y]$.

The functors: ARS_{01} and ARS_{02} yielding strict ARS's are defined by conditions,

(Def. 9) (i) the carrier of $\mathrm{ARS}_{01} = \{0, 1\}$, and

(ii) the reduction of $\mathrm{ARS}_{01} = \{0\} \times \{0, 1\}$,

(Def. 10) (i) the carrier of $\mathrm{ARS}_{02} = \{0, 1, 2\}$, and

(ii) the reduction of $\mathrm{ARS}_{02} = \{0\} \times \{0, 1, 2\}$,

respectively. One can check that ARS_{01} is non empty and ARS_{02} is non empty.

From now on i, j, k denote elements of ARS_{01}.

Now we state the propositions:

(87) Let us consider a set s. Then s is an element of ARS_{01} if and only if $s = 0$ or $s = 1$.

(88) $i \to j$ if and only if $i = 0$. The theorem is a consequence of (87).

In the sequel l, m, n denote elements of ARS_{02}.

Now we state the propositions:

(89) Let us consider a set s. Then s is an element of ARS_{02} if and only if $s = 0$ or $s = 1$ or $s = 2$.

(90) $m \to n$ if and only if $m = 0$. The theorem is a consequence of (89).

3. Normal Forms

Let us consider X and x. We say that x is a normal form if and only if

(Def. 11) There exists no y such that $x \to y$.

Now we state the proposition:

(91) x is a normal form if and only if x is a normal form w.r.t. the reduction of X. Proof: If x is a normal form, then x is a normal form w.r.t. the reduction of X by [13, (87)]. □

Let us consider X, x, and y. We say that x is a normal form of y if and only if

(Def. 12) (i) x is a normal form, and

(ii) $y \to_* x$.

Now we state the proposition:

(92) x is a normal form of y if and only if x is a normal form of y w.r.t. the reduction of X. The theorem is a consequence of (91).

Let us consider X and x. We say that x is normalizable if and only if

(Def. 13) There exists y such that y is a normal form of x.

Now we state the proposition:

(93) x is normalizable if and only if x has a normal form w.r.t. the reduction of X. The theorem is a consequence of (92).

Let us consider X. We say that X is WN if and only if

(Def. 14) x is normalizable.

We say that X is SN if and only if

(Def. 15) Let us consider a function f from $\mathbb{N}$ into the carrier of X. Then there exists a natural number i such that $f(i) \not\to f(i+1)$.

We say that X is UN* if and only if

(Def. 16) If y is a normal form of x and z is a normal form of x, then $y = z$.

We say that X is UN if and only if

(Def. 17) If x is a normal form and y is a normal form and $x \leftrightarrow_* y$, then $x = y$.

We say that X is NF if and only if

(Def. 18) If x is a normal form and $x \leftrightarrow_* y$, then $y \rightarrow_* x$.

Now we state the propositions:

(94) X is WN if and only if the reduction of X is weakly-normalizing. The theorem is a consequence of (93).

(95) If X is SN, then the reduction of X is strongly-normalizing.

(96) If X is not empty and the reduction of X is strongly-normalizing, then X is SN.

From now on A denotes a set.

Now we state the proposition:

(97) X is SN if and only if there exists no A and there exists z such that $z \in A$ and for every x such that $x \in A$ there exists y such that $y \in A$ and $x \rightarrow y$.

The scheme *notSN* deals with an ARS $\mathcal{X}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 8) $\mathcal{X}$ is not SN

provided

- there exists an element x of $\mathcal{X}$ such that $\mathcal{P}[x]$ and
- for every element x of $\mathcal{X}$ such that $\mathcal{P}[x]$ there exists an element y of $\mathcal{X}$ such that $\mathcal{P}[y]$ and $x \rightarrow y$.

Now we state the propositions:

(98) X is UN if and only if the reduction of X has unique normal form property. PROOF: Set $R =$ the reduction of X. If X is UN, then R has unique normal form property by (91), [6, (28), (31)]. x is a normal form w.r.t. R and y is a normal form w.r.t. R and x and y are convertible w.r.t. R. $\square$

(99) X is NF if and only if the reduction of X has normal form property. PROOF: Set $R =$ the reduction of X. If X is NF, then R has normal form property by (91), [6, (28), (31), (12)]. $\square$

Let us consider X and x. Assume there exists y such that y is a normal form of x and for every y and z such that y is a normal form of x and z is a normal form of x holds $y = z$. The functor nf x yielding an element of X is defined by

(Def. 19) *it* is a normal form of x.

Now we state the propositions:

(100) Suppose there exists y such that y is a normal form of x and for every y and z such that y is a normal form of x and z is a normal form of x holds

$y = z$. Then $\mathrm{nf}\, x = \mathrm{nf}_\alpha(x)$, where α is the reduction of X. The theorem is a consequence of (92).

(101) If x is a normal form and $x \to_* y$, then $x = y$. The theorem is a consequence of (85).

(102) If x is a normal form, then x is a normal form of x.

(103) If x is a normal form and $y \to x$, then x is a normal form of y.

(104) If x is a normal form and $y \to_{01} x$, then x is a normal form of y.

(105) If x is a normal form and $y \to_+ x$, then x is a normal form of y.

(106) If x is a normal form of y and y is a normal form of x, then $x = y$.

(107) If x is a normal form of y and $z \to y$, then x is a normal form of z.

(108) If x is a normal form of y and $z \to_* y$, then x is a normal form of z.

(109) If x is a normal form of y and $z \to_* x$, then x is a normal form of z.

Let us consider X. One can check that every element of X which is a normal form is also normalizable.

Now we state the propositions:

(110) If x is normalizable and $y \to x$, then y is normalizable.

(111) X is WN if and only if for every x, there exists y such that y is a normal form of x.

(112) If for every x, x is a normal form, then X is WN. The theorem is a consequence of (102).

One can verify that every ARS which is SN is also WN.

Now we state the propositions:

(113) If $x \neq y$ and for every a and b, $a \to b$ iff $a = x$, then y is a normal form and x is normalizable. The theorem is a consequence of (2).

(114) There exists X such that

(i) X is WN, and

(ii) X is not SN.

PROOF: Define $\mathcal{R}[\mathrm{set}, \mathrm{set}] \equiv \$_1 = 0$. Consider X being a strict non empty ARS such that the carrier of $X = \{0, 1\}$ and for every elements x, y of X, $x \to y$ iff $\mathcal{R}[x, y]$ from *ARSex*. X is WN. $\square$

One can verify that every ARS which is NF is also UN* and every ARS which is NF is also UN and every ARS which is UN is also UN*.

Now we state the proposition:

(115) If X is WN and UN* and x is a normal form and $x \leftrightarrow_* y$, then $y \to_* x$.
PROOF: Define $\mathcal{P}[\text{element of } X] \equiv \$_1 \to_* x$. For every y and z such that $y \leftrightarrow z$ and $\mathcal{P}[y]$ holds $\mathcal{P}[z]$. For every y and z such that $y \leftrightarrow_* z$ and $\mathcal{P}[y]$ holds $\mathcal{P}[z]$ from *Star2*. $\square$

Observe that every ARS which is WN and UN* is also NF and every ARS which is WN and UN* is also UN.

Now we state the propositions:

(116) If y is a normal form of x and z is a normal form of x and $y \neq z$, then $x \to_+ y$. The theorem is a consequence of (85) and (101).

(117) If X is WN and UN*, then $\mathrm{nf}\, x$ is a normal form of x.

(118) If X is WN and UN* and y is a normal form of x, then $y = \mathrm{nf}\, x$.

Let us assume that X is WN and UN*. Now we state the propositions:

(119) $\mathrm{nf}\, x$ is a normal form. The theorem is a consequence of (117).

(120) $\mathrm{nf}\, \mathrm{nf}\, x = \mathrm{nf}\, x$. The theorem is a consequence of (119), (102), and (118).

Now we state the propositions:

(121) If X is WN and UN* and $x \to_* y$, then $\mathrm{nf}\, x = \mathrm{nf}\, y$. The theorem is a consequence of (117), (108), and (118).

(122) If X is WN and UN* and $x \leftrightarrow_* y$, then $\mathrm{nf}\, x = \mathrm{nf}\, y$. Proof: Define $\mathcal{P}[\text{element of } X] \equiv \mathrm{nf}\, x = \mathrm{nf}\, \$_1$. For every z and u such that $z \leftrightarrow u$ and $\mathcal{P}[z]$ holds $\mathcal{P}[u]$. $\mathcal{P}[y]$ from *Star2A*. $\square$

(123) If X is WN and UN* and $\mathrm{nf}\, x = \mathrm{nf}\, y$, then $x \leftrightarrow_* y$. The theorem is a consequence of (117), (82), and (7).

4. Divergence and Convergence

Let us consider X, x, and y. We say that $x \swarrow^{*}\!\searrow y$ if and only if

(Def. 20) There exists z such that $x \leftarrow_* z \to_* y$.

Observe that the predicate is symmetric and reflexive. We say that $x \searrow_{*}\!\swarrow y$ if and only if

(Def. 21) There exists z such that $x \to_* z \leftarrow_* y$.

One can check that the predicate is symmetric and reflexive. We say that $x \swarrow^{01}\!\searrow y$ if and only if

(Def. 22) There exists z such that $x \leftarrow_{01} z \to_{01} y$.

Observe that the predicate is symmetric and reflexive. We say that $x \searrow_{01}\!\swarrow y$ if and only if

(Def. 23) There exists z such that $x \to_{01} z \leftarrow_{01} y$.

One can check that the predicate is symmetric and reflexive.

Now we state the propositions:

(124) $x \swarrow^{*}\!\searrow y$ if and only if x and y are divergent w.r.t. the reduction of X.

(125) $x \searrow_{*}\!\swarrow y$ if and only if x and y are convergent w.r.t. the reduction of X.

(126) $x \swarrow^{01}\!\searrow y$ if and only if x and y are divergent at most in 1 step w.r.t. the reduction of X.

(127) $x \searrow_{01}\swarrow y$ if and only if x and y are convergent at most in 1 step w.r.t. the reduction of X.

Let us consider X. We say that X is DIAMOND if and only if

(Def. 24) If $x \swarrow^{01}\searrow y$, then $x \searrow_{01}\swarrow y$.

We say that X is CONF if and only if

(Def. 25) If $x \swarrow^{*}\searrow y$, then $x \searrow_{*}\swarrow y$.

We say that X is CR if and only if

(Def. 26) If $x \leftrightarrow_{*} y$, then $x \searrow_{*}\swarrow y$.

We say that X is WCR if and only if

(Def. 27) If $x \swarrow^{01}\searrow y$, then $x \searrow_{*}\swarrow y$.

We say that X is COMP if and only if

(Def. 28) X is SN and CONF.

The scheme *isCR* deals with a non empty ARS $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and states that

(Sch. 9) $\mathcal{X}$ is CR

provided

- for every element x of $\mathcal{X}$, $x \to_{*} \mathcal{F}(x)$ and
- for every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow_{*} y$ holds $\mathcal{F}(x) = \mathcal{F}(y)$.

The scheme *isCOMP* deals with a non empty ARS $\mathcal{X}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{X}$ and states that

(Sch. 10) $\mathcal{X}$ is COMP

provided

- $\mathcal{X}$ is SN and
- for every element x of $\mathcal{X}$, $x \to_{*} \mathcal{F}(x)$ and
- for every elements x, y of $\mathcal{X}$ such that $x \leftrightarrow_{*} y$ holds $\mathcal{F}(x) = \mathcal{F}(y)$.

Now we state the propositions:

(128) If $x \swarrow^{01}\searrow y$, then $x \swarrow^{*}\searrow y$. The theorem is a consequence of (9).

(129) If $x \searrow_{01}\swarrow y$, then $x \searrow_{*}\swarrow y$. The theorem is a consequence of (9).

Let us assume that $x \to y$. Now we state the propositions:

(130) $x \swarrow^{01}\searrow y$.

(131) $x \searrow_{01}\swarrow y$.

Let us assume that $x \to_{01} y$. Now we state the propositions:

(132) $x \swarrow^{01}\searrow y$.

(133) $x \searrow_{01}\swarrow y$.

Let us assume that $x \leftrightarrow y$. Now we state the propositions:

(134) $x \swarrow^{01}\searrow y$.

(135) $x \searrow_{01}\swarrow y$.

Let us assume that $x \leftrightarrow_{01} y$. Now we state the propositions:

(136) $x \swarrow^{01}\searrow y$. The theorem is a consequence of (59).

(137) $x \searrow_{01}\swarrow y$. The theorem is a consequence of (59).

Now we state the proposition:

(138) If $x \to y$, then $x \searrow_{*}\swarrow y$.

Let us assume that $x \to_{*} y$. Now we state the propositions:

(139) $x \searrow_{*}\swarrow y$.

(140) $x \swarrow^{*}\searrow y$.

Let us assume that $x \to_{+} y$. Now we state the propositions:

(141) $x \searrow_{*}\swarrow y$. The theorem is a consequence of (10).

(142) $x \swarrow^{*}\searrow y$. The theorem is a consequence of (10).

Now we state the propositions:

(143) If $x \to y$ and $x \to z$, then $y \swarrow^{01}\searrow z$.

(144) If $x \to y$ and $z \to y$, then $x \searrow_{01}\swarrow z$.

(145) If $x \searrow_{*}\swarrow z \leftarrow y$, then $x \searrow_{*}\swarrow y$. The theorem is a consequence of (14).

(146) If $x \searrow_{*}\swarrow z \leftarrow_{01} y$, then $x \searrow_{*}\swarrow y$. The theorem is a consequence of (18).

(147) If $x \searrow_{*}\swarrow z \leftarrow_{*} y$, then $x \searrow_{*}\swarrow y$. The theorem is a consequence of (3).

(148) If $x \swarrow^{*}\searrow y$, then $x \leftrightarrow_{*} y$. The theorem is a consequence of (82) and (7).

(149) If $x \searrow_{*}\swarrow y$, then $x \leftrightarrow_{*} y$. The theorem is a consequence of (82) and (7).

5. Church-Rosser Property

Now we state the propositions:

(150) X is DIAMOND if and only if the reduction of X is subcommutative. PROOF: Set $R =$ the reduction of X. If X is DIAMOND, then R is subcommutative by [23, (15)], (127). □

(151) X is CONF if and only if the reduction of X is confluent. PROOF: Set $R =$ the reduction of X. If X is CONF, then R is confluent by [6, (37), (32)], (124), (125). x and y are divergent w.r.t. R. □

(152) X is CR if and only if the reduction of X has Church-Rosser property. PROOF: Set $R =$ the reduction of X. If X is CR, then R has Church-Rosser property by [6, (32)], (125), [6, (38)]. □

(153) X is WCR if and only if the reduction of X is locally-confluent. PROOF: Set $R =$ the reduction of X. If X is WCR, then R is locally-confluent by [23, (15)], (125). □

(154) Let us consider a non empty ARS X. Then X is COMP if and only if the reduction of X is complete. The theorem is a consequence of (151), (95), and (96).

(155) If X is DIAMOND and $x \leftarrow_* z \rightarrow_{01} y$, then there exists u such that $x \rightarrow_{01} u \leftarrow_* y$. PROOF: Define $\mathcal{P}[\text{element of } X] \equiv$ there exists u such that $\$_1 \rightarrow_{01} u \leftarrow_* y$. For every u and v such that $u \rightarrow v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$. For every u and v such that $u \rightarrow_* v$ and $\mathcal{P}[u]$ holds $\mathcal{P}[v]$ from *Star*. $\square$

(156) If X is DIAMOND and $x \leftarrow_{01} y \rightarrow_* z$, then there exists u such that $x \rightarrow_* u \leftarrow_{01} z$. The theorem is a consequence of (155).

One can verify that every ARS which is DIAMOND is also CONF and every ARS which is DIAMOND is also CR and every ARS which is CR is also WCR and every ARS which is CR is also CONF and every ARS which is CONF is also CR.

Now we state the proposition:

(157) If X is non CONF and WN, then there exists x and there exists y and there exists z such that y is a normal form of x and z is a normal form of x and $y \neq z$. The theorem is a consequence of (108).

NEWMAN LEMMA: Every ARS which is SN and WCR is also CR and every ARS which is CR is also NF and every ARS which is WN and UN is also CR and every ARS which is SN and CR is also COMP and every ARS which is COMP is also CR WCR NF UN UN* and WN.

Now we state the proposition:

(158) If X is COMP, then for every x and y such that $x \leftrightarrow_* y$ holds $\mathrm{nf}\, x = \mathrm{nf}\, y$.

Observe that every ARS which is WN and UN* is also CR and every ARS which is SN and UN* is also COMP.

6. Term Rewriting Systems

We consider TRS structures which extend ARS's and universal algebra structures and are systems

$$\langle \text{a carrier}, \text{a characteristic}, \text{a reduction} \rangle$$

where the carrier is a set, the characteristic is a finite sequence of operational functions of the carrier, the reduction is a binary relation on the carrier.

One can verify that there exists a TRS structure which is non empty, non-empty, and strict.

Let S be a non empty universal algebra structure. We say that S is group-like if and only if

(Def. 29) (i) $\mathrm{Seg}\, 3 \subseteq \mathrm{dom}(\text{the characteristic of } S)$, and

(ii) for every non empty homogeneous partial function f from (the carrier of S)* to the carrier of S, if $f =$ (the characteristic of S)(1), then arity $f = 0$ and if $f =$ (the characteristic of S)(2), then arity $f = 1$ and if $f =$ (the characteristic of S)(3), then arity $f = 2$.

Now we state the propositions:

(159) Let us consider a non empty set X and non empty homogeneous partial functions f_1, f_2, f_3 from X^* to X. Suppose

(i) arity $f_1 = 0$, and

(ii) arity $f_2 = 1$, and

(iii) arity $f_3 = 2$.

Let us consider a non empty universal algebra structure S. Suppose

(iv) the carrier of $S = X$, and

(v) $\langle f_1, f_2, f_3 \rangle \subseteq$ the characteristic of S.

Then S is group-like.

(160) Let us consider a non empty set X, non empty quasi total homogeneous partial functions f_1, f_2, f_3 from X^* to X, and a non empty universal algebra structure S. Suppose

(i) the carrier of $S = X$, and

(ii) $\langle f_1, f_2, f_3 \rangle =$ the characteristic of S.

Then S is quasi total and partial. Proof: S is quasi total by [7, (89)], [19, (1)], [7, (45)]. $\square$

Let S be a non empty non-empty universal algebra structure, o be an operation of S, and a be an element of $\operatorname{dom} o$. Let us note that the functor $o(a)$ yields an element of S. One can check that every operation of S is non empty.

Note that every element of $\operatorname{dom} o$ is relation-like and function-like.

Let S be a partial non empty non-empty universal algebra structure. Let us observe that every operation of S is homogeneous.

Let S be a quasi total non empty non-empty universal algebra structure. One can check that every operation of S is quasi total.

Now we state the propositions:

(161) Let us consider a non empty non-empty universal algebra structure S. Suppose S is group-like. Then

(i) 1 is an operation symbol of S, and

(ii) 2 is an operation symbol of S, and

(iii) 3 is an operation symbol of S.

(162) Let us consider a partial non empty non-empty universal algebra structure S. Suppose S is group-like. Then

(i) arity $\mathrm{Den}(1(\in \mathrm{dom}(\text{the characteristic of } S)), S) = 0$, and

(ii) arity $\mathrm{Den}(2(\in \mathrm{dom}(\text{the characteristic of } S)), S) = 1$, and

(iii) arity $\mathrm{Den}(3(\in \mathrm{dom}(\text{the characteristic of } S)), S) = 2$.

The theorem is a consequence of (161).

Let S be a non empty non-empty TRS structure. We say that S is invariant if and only if

(Def. 30) Let us consider an operation symbol o of S, elements a, b of $\mathrm{dom}\,\mathrm{Den}(o, S)$, and a natural number i. Suppose $i \in \mathrm{dom}\, a$. Let us consider elements x, y of S. Suppose

(i) $x = a(i)$, and

(ii) $b = a +\cdot (i, y)$, and

(iii) $x \to y$.

Then $(\mathrm{Den}(o, S))(a) \to (\mathrm{Den}(o, S))(b)$.

We say that S is compatible if and only if

(Def. 31) Let us consider an operation symbol o of S and elements a, b of $\mathrm{dom}\,\mathrm{Den}(o, S)$. Suppose a natural number i. Suppose $i \in \mathrm{dom}\, a$. Let us consider elements x, y of S. If $x = a(i)$ and $y = b(i)$, then $x \to y$. Then $(\mathrm{Den}(o, S))(a) \to_* (\mathrm{Den}(o, S))(b)$.

Now we state the proposition:

(163) Let us consider a natural number n, a non empty set X, and an element x of X. Then there exists a non empty homogeneous quasi total partial function f from X^* to X such that

(i) arity $f = n$, and

(ii) $f = X^n \longmapsto x$.

Proof: Set $f = X^n \longmapsto x$. f is quasi total by [9, (132), (133)]. f is homogeneous by [9, (132)]. $\square$

Let X be a non empty set, O be a finite sequence of operational functions of X, and r be a binary relation on X. Observe that $\langle X, O, r\rangle$ is non empty.

Let O be a non empty non-empty finite sequence of operational functions of X. Let us note that $\langle X, O, r\rangle$ is non-empty.

Let x be an element of X. The functor $\mathrm{TotalTRS}(X, x)$ yielding a non empty non-empty strict TRS structure is defined by

(Def. 32) (i) the carrier of $it = X$, and

(ii) the characteristic of $it = \langle X^0 \longmapsto x, X^1 \longmapsto x, X^2 \longmapsto x\rangle$, and

(iii) the reduction of $it = \nabla_X$.

One can verify that $\mathrm{TotalTRS}(X, x)$ is quasi total partial group-like and invariant and there exists a non empty non-empty TRS structure which is strict, quasi total, partial, group-like, and invariant.

Let S be a group-like quasi total partial non empty non-empty TRS structure. The functor 1_S yielding an element of S is defined by the term

(Def. 33) $(\mathrm{Den}(1(\in \mathrm{dom}(\text{the characteristic of } S)), S))(\emptyset)$.

Let a be an element of S. The functor a^{-1} yielding an element of S is defined by the term

(Def. 34) $(\mathrm{Den}(2(\in \mathrm{dom}(\text{the characteristic of } S)), S))(\langle a \rangle)$.

Let b be an element of S. The functor $a \cdot b$ yielding an element of S is defined by the term

(Def. 35) $(\mathrm{Den}(3(\in \mathrm{dom}(\text{the characteristic of } S)), S))(\langle a, b \rangle)$.

In the sequel S denotes a group-like quasi total partial invariant non empty non-empty TRS structure and a, b, c denote elements of S.

Let us assume that $a \to b$. Now we state the propositions:

(164) $a^{-1} \to b^{-1}$. The theorem is a consequence of (162).

(165) $a \cdot c \to b \cdot c$. The theorem is a consequence of (162).

(166) $c \cdot a \to c \cdot b$. The theorem is a consequence of (162).

7. Idea of Knuth-Bendix Algorithm

In the sequel S denotes a group-like quasi total partial non empty non-empty TRS structure and a, b, c denote elements of S.

Let us consider S. We say that S is (R1) if and only if

(Def. 36) $1_S \cdot a \to a$.

We say that S is (R2) if and only if

(Def. 37) $a^{-1} \cdot a \to 1_S$.

We say that S is (R3) if and only if

(Def. 38) $(a \cdot b) \cdot c \to a \cdot (b \cdot c)$.

We say that S is (R4) if and only if

(Def. 39) $a^{-1} \cdot (a \cdot b) \to b$.

We say that S is (R5) if and only if

(Def. 40) $(1_S)^{-1} \cdot a \to a$.

We say that S is (R6) if and only if

(Def. 41) $(a^{-1})^{-1} \cdot 1_S \to a$.

We say that S is (R7) if and only if

(Def. 42) $(a^{-1})^{-1} \cdot b \to a \cdot b$.

We say that S is (R8) if and only if

(Def. 43) $a \cdot 1_S \to a$.

We say that S is (R9) if and only if

(Def. 44) $(a^{-1})^{-1} \to a$.

We say that S is (R10) if and only if

(Def. 45) $(1_S)^{-1} \to 1_S$.

We say that S is (R11) if and only if

(Def. 46) $a \cdot a^{-1} \to 1_S$.

We say that S is (R12) if and only if

(Def. 47) $a \cdot (a^{-1} \cdot b) \to b$.

We say that S is (R13) if and only if

(Def. 48) $a \cdot (b \cdot (a \cdot b)^{-1}) \to 1_S$.

We say that S is (R14) if and only if

(Def. 49) $a \cdot (b \cdot a)^{-1} \to b^{-1}$.

We say that S is (R15) if and only if

(Def. 50) $(a \cdot b)^{-1} \to b^{-1} \cdot a^{-1}$.

In the sequel S denotes a group-like quasi total partial invariant non empty non-empty TRS structure and a, b, c denote elements of S.

Now we state the propositions:

(167) If S is (R1), (R2), and (R3), then $a^{-1} \cdot (a \cdot b) \swarrow^{*}\searrow b$. The theorem is a consequence of (2), (165), and (3).

(168) If S is (R1) and (R4), then $(1_S)^{-1} \cdot a \swarrow^{*}\searrow a$. The theorem is a consequence of (2) and (166).

(169) If S is (R2) and (R4), then $(a^{-1})^{-1} \cdot 1_S \swarrow^{*}\searrow a$. The theorem is a consequence of (2) and (166).

(170) If S is (R1), (R3), and (R6), then $(a^{-1})^{-1} \cdot b \swarrow^{*}\searrow a \cdot b$. The theorem is a consequence of (2), (166), (3), and (165).

(171) If S is (R6) and (R7), then $a \cdot 1_S \swarrow^{*}\searrow a$. The theorem is a consequence of (2).

(172) If S is (R6) and (R8), then $(a^{-1})^{-1} \swarrow^{*}\searrow a$. The theorem is a consequence of (2).

(173) If S is (R5) and (R8), then $(1_S)^{-1} \swarrow^{*}\searrow 1_S$. The theorem is a consequence of (2).

(174) If S is (R2) and (R9), then $a \cdot a^{-1} \swarrow^{*}\searrow 1_S$. The theorem is a consequence of (2) and (165).

(175) If S is (R1), (R3), and (R11), then $a \cdot (a^{-1} \cdot b) \swarrow^{*}\searrow b$. The theorem is a consequence of (2), (165), and (12).

(176) If S is (R3) and (R11), then $a \cdot (b \cdot (a \cdot b)^{-1}) \swarrow^{*}\searrow 1_S$. The theorem is a consequence of (2).

(177) If S is (R4), (R8), and (R13), then $a \cdot (b \cdot a)^{-1} \swarrow^{*}\searrow b^{-1}$. The theorem is a consequence of (2), (166), and (12).

(178) If S is (R4) and (R14), then $(a \cdot b)^{-1} \swarrow^{*}\searrow b^{-1} \cdot a^{-1}$. The theorem is a consequence of (2) and (166).

(179) If S is (R1) and (R10), then $(1_S)^{-1} \cdot a \rightarrow_* a$. The theorem is a consequence of (165) and (12).

(180) If S is (R8) and (R9), then $(a^{-1})^{-1} \cdot 1_S \rightarrow_* a$. The theorem is a consequence of (12).

(181) If S is (R9), then $(a^{-1})^{-1} \cdot b \rightarrow_* a \cdot b$. The theorem is a consequence of (2) and (165).

(182) If S is (R11) and (R14), then $a \cdot (b \cdot (a \cdot b)^{-1}) \rightarrow_* 1_S$. The theorem is a consequence of (166) and (12).

(183) If S is (R12) and (R15), then $a \cdot (b \cdot a)^{-1} \rightarrow_* b^{-1}$. The theorem is a consequence of (166) and (12).

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