Differential Equations on Functions from $\mathbb{R}$ into Real Banach Space

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Summary. In this article, we describe the differential equations on functions from $\mathbb{R}$ into real Banach space. The descriptions are based on the article [20]. As preliminary to the proof of these theorems, we proved some properties of differentiable functions on real normed space. For the proof we referred to descriptions and theorems in the article [21] and the article [32]. And applying the theorems of Riemann integral introduced in the article [22], we proved the ordinary differential equations on real Banach space. We referred to the methods of proof in [30].

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The notation and terminology used in this paper have been introduced in the following articles: [29], [5], [11], [3], [6], [7], [19], [13], [34], [31], [33], [1], [15], [25], [32], [18], [24], [23], [26], [27], [20], [2], [8], [14], [16], [28], [12], [37], [38], [9], [35], [36], [17], and [10].

1. SOME PROPERTIES OF DIFFERENTIABLE FUNCTIONS ON REAL NORMED SPACE

From now on $Y$ denotes a real normed space.

Now we state the propositions:

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Let us consider a real normed space $Y$, a function $J$ from $\langle E^1, \| \cdot \| \rangle$ into $\mathbb{R}$, a point $x_0$ of $\langle E^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle E^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $J = \text{proj}(1, 1)$, and 
(ii) $x_0 \in \text{dom } f$, and 
(iii) $y_0 \in \text{dom } g$, and 
(iv) $x_0 = \langle y_0 \rangle$, and 
(v) $f = g \cdot J$.

Then $f$ is continuous in $x_0$ if and only if $g$ is continuous in $y_0$. Proof: If $f$ is continuous in $x_0$, then $g$ is continuous in $y_0$ by [14, (2)], [6, (39)], [37, (36)]. □

Let us consider a real normed space $Y$, a function $I$ from $\mathbb{R}$ into $\langle E^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle E^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle E^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and 
(ii) $x_0 \in \text{dom } f$, and 
(iii) $y_0 \in \text{dom } g$, and 
(iv) $x_0 = \langle y_0 \rangle$, and 
(v) $f \cdot I = g$.

Then $f$ is continuous in $x_0$ if and only if $g$ is continuous in $y_0$. Proof: If $f$ is continuous in $x_0$, then $g$ is continuous in $y_0$ by [14, (1)], [21, (33)], [26, (15)]. □

Let us consider a function $I$ from $\mathbb{R}$ into $\langle E^1, \| \cdot \| \rangle$. Suppose $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$. Then

(i) for every rest $R$ of $\langle E^1, \| \cdot \| \rangle$, $Y, R \cdot I$ is a rest of $Y$, and 
(ii) for every linear operator $L$ from $\langle E^1, \| \cdot \| \rangle$ into $Y, L \cdot I$ is a linear of $Y$.

Proof: For every rest $R$ of $\langle E^1, \| \cdot \| \rangle$, $Y, R \cdot I$ is a rest of $Y$ by [15 (23)], [5, (47)], [14, (3)]. Reconsider $L_0 = L$ as a function from $\mathbb{R}$ into $Y$. Reconsider $L_1 = L_0 \cdot I$ as a partial function from $\mathbb{R}$ to $Y$. Reconsider $r = L_1(jj)$ as a point of $Y$. For every real number $p, L_{1p} = p \cdot r$ by [6, (13)], [14, (3)], [6, (12)]. □

Let us consider a function $J$ from $\langle E^1, \| \cdot \| \rangle$ into $\mathbb{R}$. Suppose $J = \text{proj}(1, 1)$. Then

(i) for every rest $R$ of $Y, R \cdot J$ is a rest of $\langle E^1, \| \cdot \| \rangle$, and 
(ii) for every linear $L$ of $Y, L \cdot J$ is a Lipschitzian linear operator from $\langle E^1, \| \cdot \| \rangle$ into $Y$. 


5. Let us consider a function $I$ from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1,1)$ qua function)$^{-1}$, and

(ii) $x_0 \in \text{dom} \, f$, and

(iii) $y_0 \in \text{dom} \, g$, and

(iv) $x_0 = \langle y_0 \rangle$, and

(v) $f \cdot I = g$, and

(vi) $f$ is differentiable in $x_0$.

Then

(vii) $g$ is differentiable in $y_0$, and

(viii) $g'(y_0) = f'(x_0)(\langle 1 \rangle)$, and

(ix) for every element $r$ of $\mathbb{R}$, $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$.

The theorem is a consequence of (3). Proof: Consider $N_1$ being a neighbourhood of $x_0$ such that $N_1 \subseteq \text{dom} \, f$ and there exists a point $L$ of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $Y$ and there exists a rest $R$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $Y$ such that for every point $x$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x \in N_1$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}$. Consider $e$ being a real number such that $0 < e$ and $\{z, \text{where } z \text{ is a point of } \langle \mathcal{E}^1, \| \cdot \| \rangle : \|z - x_0\| < e \} \subseteq N_1$. Consider $L$ being a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $Y, R$ being a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $Y$ such that for every point $x_3$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $x_3 \in N_1$ holds $f_{x_3} - f_{x_0} = L(x_3 - x_0) + R_{x_3-x_0}$. Reconsider $R_0 = R \cdot I$ as a rest of $Y$. Reconsider $L_0 = L \cdot I$ as a linear of $Y$. Set $N = \{z, \text{where } z \text{ is a point of } \langle \mathcal{E}^1, \| \cdot \| \rangle : \|z - x_0\| < e \}. N \subseteq \text{the carrier of } \langle \mathcal{E}^1, \| \cdot \| \rangle$. Set $N_0 = \{z, \text{where } z \text{ is an element of } \mathbb{R} : |z - y_0| < e \}. \{y_0 - e, y_0 + e \} \subseteq N_0$ by [28] (1)]. $N_0 \subseteq [y_0 - e, y_0 + e]$ by [28] (1)]. For every real number $y_1$ such that $y_1 \in N_0$ holds $(f \cdot I)y_1 - (f \cdot I)y_0 = L_{y_1-y_0} + R_{y_1-y_0}$ by [6] (12)], [17] (35)], [14] (3)].

6. Let us consider a function $I$ from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a real number $y_0$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $Y$. Suppose

(i) $I = (\text{proj}(1,1)$ qua function)$^{-1}$, and

(ii) $x_0 \in \text{dom} \, f$, and

(iii) $y_0 \in \text{dom} \, g$, and
(iv) $x_0 = \langle y_0 \rangle$, and
(v) $f \cdot I = g$.

Then $f$ is differentiable in $x_0$ if and only if $g$ is differentiable in $y_0$. The theorem is a consequence of (5) and (4). **Proof:** Reconsider $J = \text{proj}(1, 1)$ as a function from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $\mathbb{R}$. Consider $N_0$ being a neighbourhood of $y_0$ such that $N_0 \subseteq \text{dom}(f \cdot I)$ and there exists a linear $L$ of $Y$ and there exists a rest $R$ of $Y$ such that for every real number $y$ such that $y \in N_0$ holds $(f \cdot I)_y - (f \cdot I)_{y_0} = L_{y-y_0} + R_{y-y_0}$. Consider $e_0$ being a real number such that $0 < e_0$ and $N_0 = \| y_0 - e_0, y_0 + e_0 \|$. Reconsider $e = e_0$ as an element of $\mathbb{R}$. Set $N = \{ z, \text{ where } z \text{ is a point of } \langle \mathcal{E}^1, \| \cdot \| \rangle : \| z - x_0 \| < e \}$. Consider $L$ being a linear of $Y$, $R$ being a rest of $Y$ such that for every real number $y_1$ such that $y_1 \in N_0$ holds $(f \cdot I)_{y_1} - (f \cdot I)_{y_0} = L_{y_1-y_0} + R_{y_1-y_0}$. Reconsider $R_0 = R \cdot J$ as a rest of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, $Y$. Reconsider $L_0 = L \cdot J$ as a Lipschitzian linear operator from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $Y$. $N \subseteq$ the carrier of $\langle \mathcal{E}^1, \| \cdot \| \rangle$. For every point $y$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $y \in N$ holds $f_y - f_{x_0} = L_0(y-y_0) + R_{0y-x_0}$ by [6, (13)], [7, (35)], [14, (4)]. □

(7) Let us consider a function $J$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $\mathbb{R}$, a point $x_0$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $Y$. Suppose
(i) $J = \text{proj}(1, 1)$, and
(ii) $x_0 \in \text{dom } f$, and
(iii) $y_0 \in \text{dom } g$, and
(iv) $x_0 = \langle y_0 \rangle$, and
(v) $f = g \cdot J$.

Then $f$ is differentiable in $x_0$ if and only if $g$ is differentiable in $y_0$. The theorem is a consequence of (6).

(8) Let us consider a function $I$ from $\mathbb{R}$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a point $x_0$ of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, an element $y_0$ of $\mathbb{R}$, a partial function $g$ from $\mathbb{R}$ to $Y$, and a partial function $f$ from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ to $Y$. Suppose
(i) $I = (\text{proj}(1, 1) \text{ qua function})^{-1}$, and
(ii) $x_0 \in \text{dom } f$, and
(iii) $y_0 \in \text{dom } g$, and
(iv) $x_0 = \langle y_0 \rangle$, and
(v) $f \cdot I = g$, and
(vi) $f$ is differentiable in $x_0$.

Then $\| g' (y_0) \| = \| f' (x_0) \|$. The theorem is a consequence of (5). **Proof:** Reconsider $d_1 = f' (x_0)$ as a Lipschitzian linear operator from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $Y$. Set $A = \text{PreNorms} (d_1)$. For every real number $r$ such that $r \in A$ holds $r \leq \| g' (y_0) \|$ by [14, (1), (4)]. □
Let us consider real numbers $a, b, z$ and points $p, q, x$ of $\langle E^1, \| \cdot \| \rangle$. Now we state the propositions:

(9) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then

(i) if $z \in ]a, b[$, then $x \in ]p, q[$, and

(ii) if $x \in ]p, q[$, then $a \neq b$ and if $a < b$, then $z \in ]a, b[$ and if $a > b$, then $z \in ]b, a[$.

(10) Suppose $p = \langle a \rangle$ and $q = \langle b \rangle$ and $x = \langle z \rangle$. Then

(i) if $z \in [a, b]$, then $x \in [p, q]$, and

(ii) if $x \in [p, q]$, then if $a \leq b$, then $z \in [a, b]$ and if $a > b$, then $z \in [b, a]$.

Now we state the propositions:

(11) Let us consider real numbers $a, b$, points $p, q$ of $\langle E^1, \| \cdot \| \rangle$, and a function $I$ from $\mathbb{R}$ into $\langle E^1, \| \cdot \| \rangle$. Suppose

(i) $p = \langle a \rangle$, and

(ii) $q = \langle b \rangle$, and

(iii) $I = (\text{proj}(1, 1) \quad \text{qua function})^{-1}$.

Then

(iv) if $a \leq b$, then $I^0[a, b] = [p, q]$, and

(v) if $a < b$, then $I^0[a, b] = ]p, q[$.

The theorem is a consequence of (10) and (9).

(12) Let us consider a real normed space $Y$, a partial function $g$ from $\mathbb{R}$ to the carrier of $Y$, and real numbers $a, b, M$. Suppose

(i) $a \leq b$, and

(ii) $[a, b] \subseteq \text{dom } g$, and

(iii) for every real number $x$ such that $x \in [a, b]$ holds $g$ is continuous in $x$, and

(iv) for every real number $x$ such that $x \in ]a, b[$ holds $g$ is differentiable in $x$, and

(v) for every real number $x$ such that $x \in ]a, b[$ holds $\|g'(x)\| \leq M$.

Then $\|g_b - g_a\| \leq M \cdot |b - a|$. The theorem is a consequence of (11), (10), (1), (9), (7), and (8).
2. Differential Equations

In the sequel $X, Y$ denote real Banach spaces, $Z$ denotes an open subset of $\mathbb{R}$, $a, b, c, d, e, r, x_0$ denote real numbers, $y_0$ denotes a vector of $X$, and $G$ denotes a function from $X$ into $X$.

Now we state the propositions:

(13) Let us consider a real Banach space $X$, a partial function $F$ from $\mathbb{R}$ to the carrier of $X$, and a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $[a, b] \subseteq \text{dom } f$, and
(ii) $]a, b[ \subseteq \text{dom } F$, and

(iii) for every real number $x$ such that $x \in ]a, b[ \text{ holds } F_x = \int_a^x f(x)dx$, and

(iv) $x_0 \in ]a, b[$, and

(v) $f$ is continuous in $x_0$.

Then

(vi) $F$ is differentiable in $x_0$, and

(vii) $F'(x_0) = f_{x_0}$.

(14) Let us consider a partial function $F$ from $\mathbb{R}$ to the carrier of $X$ and a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $\text{dom } f = [a, b]$, and
(ii) $\text{dom } F = [a, b]$, and

(iii) for every real number $t$ such that $t \in [a, b]$ holds $F_t = \int_a^t f(x)dx$.

Let us consider a real number $x$. If $x \in [a, b]$, then $F$ is continuous in $x$.

(15) Let us consider a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$. If $a \in \text{dom } f$, then $\int_a^a f(x)dx = 0_X$.

Let us consider a continuous partial function $f$ from $\mathbb{R}$ to the carrier of $X$ and a partial function $g$ from $\mathbb{R}$ to the carrier of $X$. Now we state the propositions:

(16) Suppose $a \leq b$ and $\text{dom } f = [a, b]$ and for every real number $t$ such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t f(x)dx$. Then $g_a = y_0$. 
(17) Suppose \( \text{dom } f = [a, b] \) and \( \text{dom } g = [a, b] \) and \( Z = ]a, b[ \) and for every real number \( t \) such that \( t \in ]a, b[ \) holds \( g_t = y_0 + \int_a^t f(x)dx \). Then

(i) \( g \) is continuous and differentiable on \( Z \), and
(ii) for every real number \( t \) such that \( t \in Z \) holds \( g'(t) = f_t \).

Let us consider a partial function \( f \) from \( \mathbb{R} \) to the carrier of \( X \). Now we state the propositions:

(18) Suppose \( a \leq b \) and \( [a, b] \subseteq \text{dom } f \) and for every real number \( x \) such that \( x \in [a, b] \) holds \( f \) is continuous in \( x \) and \( f \) is differentiable on \( ]a, b[ \) and for every real number \( x \) such that \( x \in ]a, b[ \) holds \( f'(x) = 0_X \). Then \( f_b = f_a \).

(19) Suppose \( [a, b] \subseteq \text{dom } f \) and for every real number \( x \) such that \( x \in [a, b] \) holds \( f \) is continuous in \( x \) and \( f \) is differentiable on \( ]a, b[ \) and for every real number \( x \) such that \( x \in ]a, b[ \) holds \( f'(x) = 0_X \). Then \( f|]a, b[ \) is constant.

Now we state the propositions:

(20) Let us consider a continuous partial function \( f \) from \( \mathbb{R} \) to the carrier of \( X \). Suppose

(i) \( [a, b] = \text{dom } f \), and
(ii) \( f|]a, b[ \) is constant.

Let us consider a real number \( x \). If \( x \in [a, b] \), then \( f_x = f_a \).

(21) Let us consider continuous partial functions \( y, G_1 \) from \( \mathbb{R} \) to the carrier of \( X \) and a partial function \( g \) from \( \mathbb{R} \) to the carrier of \( X \). Suppose

(i) \( a \leq b \), and
(ii) \( Z = ]a, b[ \), and
(iii) \( \text{dom } y = [a, b] \), and
(iv) \( \text{dom } g = [a, b] \), and
(v) \( \text{dom } G_1 = [a, b] \), and
(vi) \( y \) is differentiable on \( Z \), and
(vii) \( y_a = y_0 \), and
(viii) for every real number \( t \) such that \( t \in Z \) holds \( y'(t) = G_1t \), and
(ix) for every real number \( t \) such that \( t \in [a, b] \) holds \( g_t = y_0 + \int_a^t G_1(x)dx \).

Then \( y = g \). The theorem is a consequence of (17), (16), (19), and (20).

**Proof:** Reconsider \( h = y - g \) as a continuous partial function from \( \mathbb{R} \) to the carrier of \( X \). For every real number \( x \) such that \( x \in \text{dom } h \) holds \( h_x = 0_X \) by \([35](15)\). For every element \( x \) of \( \mathbb{R} \) such that \( x \in \text{dom } y \) holds \( y(x) = g(x) \) by \([35](21)\). □
Let $X$ be a real Banach space, $y_0$ be a vector of $X$, $G$ be a function from $X$ into $X$, and $a, b$ be real numbers. Assume $a \leq b$ and $G$ is continuous on $\text{dom } G$. The functor $\text{Fredholm}(G, a, b, y_0)$ yielding a function from the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ into the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ is defined by

(Def. 1) Let us consider a vector $x$ of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$. Then there exist continuous partial functions $f, g, G_1$ from $\mathbb{R}$ to the carrier of $X$ such that

(i) $x = f$, and
(ii) $it(x) = g$, and
(iii) $\text{dom } f = [a, b]$, and
(iv) $\text{dom } g = [a, b]$, and
(v) $G_1 = G \cdot f$, and
(vi) for every real number $t$ such that $t \in [a, b]$ holds $g_t = y_0 + \int_a^t G_1(x)dx$.

Now we state the propositions:

(22) Suppose $a \leq b$ and $0 < r$ and for every vectors $y_1, y_2$ of $X$, $\parallel G_{y_1} - G_{y_2} \parallel \leq r \cdot \parallel y_1 - y_2 \parallel$. Let us consider vectors $u, v$ of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$ and continuous partial functions $g, h$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $g = (\text{Fredholm}(G, a, b, y_0))(u)$, and
(ii) $h = (\text{Fredholm}(G, a, b, y_0))(v)$.

Let us consider a real number $t$. Suppose $t \in [a, b]$. Then $\parallel g_t - h_t \parallel \leq (r \cdot (t - a)) \cdot \parallel u - v \parallel$. PROOF: Set $F = \text{Fredholm}(G, a, b, y_0)$. Consider $f_1, g_1, G_3$ being continuous partial functions from $\mathbb{R}$ to the carrier of $X$ such that $u = f_1$ and $F(u) = g_1$ and $\text{dom } f_1 = [a, b]$ and $\text{dom } g_1 = [a, b]$ and $G_3 = G \cdot f_1$ and for every real number $t$ such that $t \in [a, b]$ holds $g_{1t} = y_0 + \int_a^t G_3(x)dx$. Consider $f_2, g_2, G_5$ being continuous partial functions from $\mathbb{R}$ to the carrier of $X$ such that $v = f_2$ and $F(v) = g_2$ and $\text{dom } f_2 = [a, b]$ and $\text{dom } g_2 = [a, b]$ and $G_5 = G \cdot f_2$ and for every real number $t$ such that $t \in [a, b]$ holds $g_{2t} = y_0 + \int_a^t G_5(x)dx$. Set $G_4 = G_3 - G_5$.

For every real number $x$ such that $x \in [a, t]$ holds $\parallel G_{4x} \parallel \leq r \cdot \parallel u - v \parallel$ by [20] (26), [21] (12). □

(23) Suppose $a \leq b$ and $0 < r$ and for every vectors $y_1, y_2$ of $X$, $\parallel G_{y_1} - G_{y_2} \parallel \leq r \cdot \parallel y_1 - y_2 \parallel$. Let us consider vectors $u, v$ of the $\mathbb{R}$-norm space of
Differential equations on functions from $\mathbb{R}$.

continuous functions of $[a, b]$ and $X$, an element $m$ of $\mathbb{N}$, and continuous partial functions $g$, $h$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $g = (\text{Fredholm}(G, a, b, y_0))^{m+1}(u)$, and
(ii) $h = (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)$.

Let us consider a real number $t$. Suppose $t \in [a, b]$. Then

$$\|g_t - h_t\| \leq \frac{(r \cdot (t-a))^{m+1}}{(m+1)!} \cdot \|u - v\|.$$ The theorem is a consequence of (22).

**Proof:** Set $F = \text{Fredholm}(G, a, b, y_0)$. Define $P[n]$ for every continuous partial functions $g$, $h$ from $\mathbb{R}$ to the carrier of $X$ such that $g = F^{m+1}(u_1)$ and $h = F^{m+1}(v_1)$ for every real number $t$ such that $t \in [a, b]$ holds

$$\|g_t - h_t\| \leq \frac{(r \cdot (t-a))^{m+1}}{(m+1)!} \cdot \|u_1 - v_1\|.$$ The theorem is a consequence of (23).

(24) Let us consider a natural number $m$. Suppose

(i) $a \leq b$, and
(ii) $0 < r$, and
(iii) for every vectors $y_1$, $y_2$ of $X$, $\|Gy_1 - Gy_2\| \leq r \cdot \|y_1 - y_2\|$.

Let us consider vectors $u$, $v$ of the $\mathbb{R}$-norm space of continuous functions of $[a, b]$ and $X$.

Then

$$\|(\text{Fredholm}(G, a, b, y_0))^{m+1}(u) - (\text{Fredholm}(G, a, b, y_0))^{m+1}(v)\| \leq \frac{(r \cdot (b-a))^{m+1}}{(m+1)!} \cdot \|u - v\|.$$ The theorem is a consequence of (23).

(25) If $a < b$ and $G$ is Lipschitzian on the carrier of $X$, then there exists a natural number $m$ such that $(\text{Fredholm}(G, a, b, y_0))^{m+1}$ is contraction. The theorem is a consequence of (24).

(26) If $a < b$ and $G$ is Lipschitzian on the carrier of $X$, then Fredholm($G, a, b, y_0$) has unique fixpoint. The theorem is a consequence of (25).

(27) Let us consider continuous partial functions $f$, $g$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $\text{dom } f = [a, b]$, and
(ii) $\text{dom } g = [a, b]$, and
(iii) $Z = [a, b]$, and
(iv) $a < b$, and
(v) $G$ is Lipschitzian on the carrier of $X$, and
(vi) $g = (\text{Fredholm}(G, a, b, y_0))(f)$.

Then

(vii) $g_a = y_0$, and
(viii) $g$ is differentiable on $Z$, and
(ix) for every real number $t$ such that $t \in Z$ holds $g'(t) = (G \cdot f)_t$.

The theorem is a consequence of (17) and (16).

(28) Let us consider a continuous partial function $y$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $a < b$, and

(ii) $Z = ]a, b[$, and

(iii) $G$ is Lipschitzian on the carrier of $X$, and

(iv) $\text{dom } y = [a, b]$, and

(v) $y$ is differentiable on $Z$, and

(vi) $y_a = y_0$, and

(vii) for every real number $t$ such that $t \in Z$ holds $y'(t) = G(y_t)$.

Then $y$ is a fixpoint of $\text{Fredholm}(G, a, b, y_0)$. The theorem is a consequence of (21). Proof: Consider $f$, $g$, $G_1$ being continuous partial functions from $\mathbb{R}$ to the carrier of $X$ such that $y = f$ and $(\text{Fredholm}(G, a, b, y_0))(y) = g$ and dom $f = [a, b]$ and dom $g = [a, b]$ and $G_1 = G \cdot f$ and for every real number $t$ such that $t \in [a, b]$ holds $g_t = y_0 + \frac{a}{q} \int G_1(x) dx$. For every real number $t$ such that $t \in Z$ holds $y'(t) = G_1 t$ by [6, (13)]. □

(29) Let us consider continuous partial functions $y_1, y_2$ from $\mathbb{R}$ to the carrier of $X$. Suppose

(i) $a < b$, and

(ii) $Z = ]a, b[$, and

(iii) $G$ is Lipschitzian on the carrier of $X$, and

(iv) $\text{dom } y_1 = [a, b]$, and

(v) $y_1$ is differentiable on $Z$, and

(vi) $y_{1a} = y_0$, and

(vii) for every real number $t$ such that $t \in Z$ holds $y_1'(t) = G(y_{1t})$, and

(viii) $\text{dom } y_2 = [a, b]$, and

(ix) $y_2$ is differentiable on $Z$, and

(x) $y_{2a} = y_0$, and

(xi) for every real number $t$ such that $t \in Z$ holds $y_2'(t) = G(y_{2t})$.

Then $y_1 = y_2$. The theorem is a consequence of (26) and (28).

(30) Suppose $a < b$ and $Z = ]a, b[$ and $G$ is Lipschitzian on the carrier of $X$. Then there exists a continuous partial function $y$ from $\mathbb{R}$ to the carrier of $X$ such that
Differential equations on functions from \( \mathbb{R} \) ... 271

(i) \( \text{dom } y = [a, b] \), and

(ii) \( y \) is differentiable on \( Z \), and

(iii) \( y_a = y_0 \), and

(iv) for every real number \( t \) such that \( t \in Z \) holds \( y'(t) = G(y_t) \).

The theorem is a consequence of (26) and (27).

References

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