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# Double Sequences and Limits[1]

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**Summary.** Double sequences are important extension of the ordinary notion of a sequence. In this article we formalized three types of limits of double sequences and the theory of these limits.

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The notation and terminology used in this paper have been introduced in the following articles: [3], [4], [13], [5], [15], [6], [7], [16], [10], [1], [2], [8], [11], [18], [12], [17], and [9].

In this paper $R$, $R_1$, $R_2$ denote functions from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, $r_1$, $r_2$ denote convergent sequences of real numbers, $n$, $m$, $N$, $M$ denote natural numbers, and $e$, $r$ denote real numbers.

Let us consider $R$. We say that $R$ is p-convergent if and only if

(Def. 1) There exists a real number $p$ such that for every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n,m) - p| < e$.

Assume $R$ is p-convergent. The functor P-lim $R$ yielding a real number is defined by

(Def. 2) Let us consider a real number $e$. Suppose $0 < e$. Then there exists a natural number $N$ such that for every natural numbers $n$, $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n,m) - it| < e$.

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We say that $R$ is convergent in the first coordinate if and only if

(Def. 3) Let us consider an element $m$ of $\mathbb{N}$. Then $\mathrm{curry}'(R, m)$ is convergent.

We say that $R$ is convergent in the second coordinate if and only if

(Def. 4) Let us consider an element $n$ of $\mathbb{N}$. Then $\mathrm{curry}(R, n)$ is convergent.

The lim in the first coordinate of $R$ yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 5) Let us consider an element $m$ of $\mathbb{N}$. Then $it(m) = \lim \mathrm{curry}'(R, m)$.

The lim in the second coordinate of $R$ yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by

(Def. 6) Let us consider an element $n$ of $\mathbb{N}$. Then $it(n) = \lim \mathrm{curry}(R, n)$.

Assume the lim in the first coordinate of $R$ is convergent. The first coordinate major iterated lim of $R$ yielding a real number is defined by

(Def. 7) Let us consider a real number $e$. Suppose $0 < e$. Then there exists a natural number $M$ such that for every natural number $m$ such that $m \geqslant M$ holds $|(\text{the lim in the first coordinate of } R)(m) - it| < e$.

Assume the lim in the second coordinate of $R$ is convergent. The second coordinate major iterated lim of $R$ yielding a real number is defined by

(Def. 8) Let us consider a real number $e$. Suppose $0 < e$. Then there exists a natural number $N$ such that for every natural number $n$ such that $n \geqslant N$ holds $|(\text{the lim in the second coordinate of } R)(n) - it| < e$.

Let $R$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. We say that $R$ is uniformly convergent in the first coordinate if and only if

(Def. 9) (i) $R$ is convergent in the first coordinate, and

(ii) for every real number $e$ such that $e > 0$ there exists a natural number $M$ such that for every natural number $m$ such that $m \geqslant M$ for every natural number $n$, $|R(n, m) - (\text{the lim in the first coordinate of } R)(n)| < e$.

We say that $R$ is uniformly convergent in the second coordinate if and only if

(Def. 10) (i) $R$ is convergent in the second coordinate, and

(ii) for every real number $e$ such that $e > 0$ there exists a natural number $N$ such that for every natural number $n$ such that $n \geqslant N$ for every natural number $m$, $|R(n, m) - (\text{the lim in the second coordinate of } R)(m)| < e$.

Let us consider $R$. We say that $R$ is non-decreasing if and only if

(Def. 11) Let us consider natural numbers $n_1, m_1, n_2, m_2$. If $n_1 \geqslant n_2$ and $m_1 \geqslant m_2$, then $R(n_1, m_1) \geqslant R(n_2, m_2)$.

We say that $R$ is non-increasing if and only if

(Def. 12) Let us consider natural numbers $n_1, m_1, n_2, m_2$. If $n_1 \geqslant n_2$ and $m_1 \geqslant m_2$, then $R(n_1, m_1) \leqslant R(n_2, m_2)$.

Now we state the proposition:

(1) Let us consider real numbers $a$, $b$, $c$. If $a \leqslant b \leqslant c$, then $|b| \leqslant |a|$ or $|b| \leqslant |c|$.

Note that every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-decreasing and p-convergent is also lower bounded and upper bounded and every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-increasing and p-convergent is also lower bounded and upper bounded.

Let $r$ be an element of $\mathbb{R}$. Let us note that $\mathbb{N} \times \mathbb{N} \longmapsto r$ is p-convergent convergent in the first coordinate and convergent in the second coordinate as a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$.

Now we state the proposition:

(2) Let us consider an element $r$ of $\mathbb{R}$. Then $\text{P-lim}(\mathbb{N} \times \mathbb{N} \longmapsto r) = r$. Proof: Set $R = \mathbb{N} \times \mathbb{N} \longmapsto r$. For every natural numbers $n$, $m$, $R(n, m) = r$ by [15, (70)]. □

Note that there exists a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is p-convergent, convergent in the first coordinate, and convergent in the second coordinate.

In this paper $P_1$ denotes a p-convergent function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$.

Let $P_4$ be a p-convergent convergent in the second coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Note that the lim in the second coordinate of $P_4$ is convergent.

Now we state the proposition:

(3) Suppose $R$ is p-convergent and convergent in the second coordinate. Then $\text{P-lim}\, R =$ the second coordinate major iterated lim of $R$. Proof: Consider $z$ being a real number such that for every $e$ such that $0 < e$ there exists a natural number $N_1$ such that for every $n$ and $m$ such that $n \geqslant N_1$ and $m \geqslant N_1$ holds $|R(n, m) - z| < e$. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|(\text{the lim in the second coordinate of } R)(n) - z| < e$ by [4, (63), (60)]. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|(\text{the lim in the second coordinate of } R)(n) - \text{P-lim}\, R| < e$ by [4, (60), (63)]. □

Let $P_3$ be a p-convergent convergent in the first coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Let us note that the lim in the first coordinate of $P_3$ is convergent.

Now we state the proposition:

(4) Suppose $R$ is p-convergent and convergent in the first coordinate. Then $\text{P-lim}\, R =$ the first coordinate major iterated lim of $R$. Proof: Consider $z$ being a real number such that for every $e$ such that $0 < e$ there exists a natural number $N_1$ such that for every $n$ and $m$ such that $n \geqslant N_1$ and $m \geqslant N_1$ holds $|R(n, m) - z| < e$. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|(\text{the lim in the first coordinate of } R)(n) - z| < e$ by [4, (63), (60)]. For every $e$ such that $0 < e$

there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|$(the lim in the first coordinate of $R)(n) - \text{P-lim}\, R| < e$ by [4, (60), (63)]. $\square$

One can verify that every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-decreasing and upper bounded is also p-convergent convergent in the first coordinate and convergent in the second coordinate and every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-increasing and lower bounded is also p-convergent convergent in the first coordinate and convergent in the second coordinate.

Now we state the propositions:

(5) Suppose $R$ is uniformly convergent in the first coordinate and the lim in the first coordinate of $R$ is convergent. Then

(i) $R$ is p-convergent, and

(ii) $\text{P-lim}\, R =$ the first coordinate major iterated lim of $R$.

(6) Suppose $R$ is uniformly convergent in the second coordinate and the lim in the second coordinate of $R$ is convergent. Then

(i) $R$ is p-convergent, and

(ii) $\text{P-lim}\, R =$ the second coordinate major iterated lim of $R$.

Let us consider $R$. We say that $R$ is Cauchy if and only if

(Def. 13) Let us consider a real number $e$. Suppose $e > 0$. Then there exists a natural number $N$ such that for every natural numbers $n_1, n_2, m_1, m_2$ such that $N \leqslant n_1 \leqslant n_2$ and $N \leqslant m_1 \leqslant m_2$ holds $|R(n_2, m_2) - R(n_1, m_1)| < e$.

Now we state the propositions:

(7) $R$ is p-convergent if and only if $R$ is Cauchy. Proof: Define $\mathcal{R}$(element of $\mathbb{N}$) $= R(\$_1, \$_1)$. Consider $s_1$ being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element $n$ of $\mathbb{N}$, $s_1(n) = \mathcal{R}(n)$ from [7, Sch. 4]. Reconsider $z = \lim s_1$ as a complex number. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ and $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n, m) - z| < e$ by [4, (63)]. $\square$

(8) Let us consider a function $R$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Suppose

(i) $R$ is non-decreasing, or

(ii) $R$ is non-increasing.

Then $R$ is p-convergent if and only if $R$ is lower bounded and upper bounded.

Let $X$, $Y$ be non empty sets, $H$ be a binary operation on $Y$, and $f$, $g$ be functions from $X$ into $Y$. Observe that the functor $H_{f,g}$ yields a function from $X \times X$ into $Y$. Now we state the propositions:

(9) (i) $\cdot_{\mathbb{R}_{r_1, r_2}}$ is convergent in the first coordinate and convergent in the second coordinate, and

(ii) the lim in the first coordinate of $\cdot_{\mathbb{R}_{r_1, r_2}}$ is convergent, and

(iii) the first coordinate major iterated lim of $\cdot_{\mathbb{R}\, r_1,r_2} = \lim r_1 \cdot \lim r_2$, and

(iv) the lim in the second coordinate of $\cdot_{\mathbb{R} r_1,r_2}$ is convergent, and

(v) the second coordinate major iterated lim of $\cdot_{\mathbb{R}\, r_1,r_2} = \lim r_1 \cdot \lim r_2$, and

(vi) $\cdot_{\mathbb{R} r_1,r_2}$ is p-convergent, and

(vii) P-lim $\cdot_{\mathbb{R} r_1,r_2} = \lim r_1 \cdot \lim r_2$.

PROOF: Set $R = \cdot_{\mathbb{R} r_1,r_2}$. For every $n$ and $m$, $R(n,m) = r_1(n) \cdot r_2(m)$ by [5, (77)]. For every element $m$ of $\mathbb{N}$ and for every real number $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|(\text{curry}'(R,m))(n) - \lim r_1 \cdot r_2(m)| < e$ by [4, (47), (65), (44)]. For every element $m$ of $\mathbb{N}$, $\text{curry}'(R,m)$ is convergent. For every element $m$ of $\mathbb{N}$ and for every real number $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|(\text{curry}(R,m))(n) - r_1(m) \cdot \lim r_2| < e$ by [4, (47), (65), (44)]. For every element $m$ of $\mathbb{N}$, $\text{curry}(R,m)$ is convergent. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds |(the lim in the first coordinate of $R)(n) - \lim r_1 \cdot \lim r_2| < e$ by [4, (46), (65)]. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds |(the lim in the second coordinate of $R)(n) - \lim r_1 \cdot \lim r_2| < e$ by [4, (46), (65)]. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ and $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n,m) - \lim r_1 \cdot \lim r_2| < e$ by [12, (3)], [4, (63), (46), (65)]. □

(10) (i) $+_{\mathbb{R} r_1,r_2}$ is convergent in the first coordinate and convergent in the second coordinate, and

(ii) the lim in the first coordinate of $+_{\mathbb{R} r_1,r_2}$ is convergent, and

(iii) the first coordinate major iterated lim of $+_{\mathbb{R}\, r_1,r_2} = \lim r_1 + \lim r_2$, and

(iv) the lim in the second coordinate of $+_{\mathbb{R} r_1,r_2}$ is convergent, and

(v) the second coordinate major iterated lim of $+_{\mathbb{R}\, r_1,r_2} = \lim r_1 + \lim r_2$, and

(vi) $+_{\mathbb{R} r_1,r_2}$ is p-convergent, and

(vii) P-lim $+_{\mathbb{R} r_1,r_2} = \lim r_1 + \lim r_2$.

PROOF: Set $R = +_{\mathbb{R} r_1,r_2}$. For every $n$ and $m$, $R(n,m) = r_1(n) + r_2(m)$ by [5, (77)]. For every element $m$ of $\mathbb{N}$ and for every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural number $n$ such that $n \geqslant N$ holds $|(\text{curry}'(R,m))(n) - (\lim r_1 + r_2(m))| < e$. For every element $m$ of $\mathbb{N}$, $\text{curry}'(R,m)$ is convergent. For every element $m$ of $\mathbb{N}$ and for every real number $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|(\text{curry}(R,m))(n) - (r_1(m) + \lim r_2)| < e$. For every element $m$ of $\mathbb{N}$, $\text{curry}(R,m)$ is convergent. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|(\text{the lim in the first coordinate of } R)(n) - (\lim r_1 + \lim r_2)| < e$. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ such that $n \geqslant N$ holds $|(\text{the lim in the second coordinate of } R)(n) - (\lim r_1 + \lim r_2)| < e$. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ and $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n, m) - (\lim r_1 + \lim r_2)| < e$ by [4, (56)]. □

(11) Suppose $R_1$ is p-convergent and $R_2$ is p-convergent. Then

(i) $R_1 + R_2$ is p-convergent, and

(ii) $\text{P-lim}(R_1 + R_2) = \text{P-lim}\, R_1 + \text{P-lim}\, R_2$.

(12) Suppose $R_1$ is p-convergent and $R_2$ is p-convergent. Then

(i) $R_1 - R_2$ is p-convergent, and

(ii) $\text{P-lim}(R_1 - R_2) = \text{P-lim}\, R_1 - \text{P-lim}\, R_2$.

(13) Let us consider a function $R$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ and a real number $r$. Suppose $R$ is p-convergent. Then

(i) $r \cdot R$ is p-convergent, and

(ii) $\text{P-lim}(r \cdot R) = r \cdot \text{P-lim}\, R$.

(14) If $R$ is p-convergent and for every natural numbers $n$, $m$, $R(n, m) \geqslant r$, then $\text{P-lim}\, R \geqslant r$.

(15) Suppose $R_1$ is p-convergent and $R_2$ is p-convergent and for every natural numbers $n$, $m$, $R_1(n, m) \leqslant R_2(n, m)$. Then $\text{P-lim}\, R_1 \leqslant \text{P-lim}\, R_2$. The theorem is a consequence of (12) and (14).

(16) Suppose $R_1$ is p-convergent and $R_2$ is p-convergent and $\text{P-lim}\, R_1 = \text{P-lim}\, R_2$ and for every natural numbers $n$, $m$, $R_1(n, m) \leqslant R(n, m) \leqslant R_2(n, m)$. Then

(i) $R$ is p-convergent, and

(ii) $\text{P-lim}\, R = \text{P-lim}\, R_1$.

PROOF: For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ and $m$ such that $n \geqslant N$ and $m \geqslant N$ holds $|R(n, m) - \text{P-lim}\, R_1| < e$ by [14, (4), (5), (1)]. □

Let $X$ be a non empty set and $s_1$ be a function from $\mathbb{N} \times \mathbb{N}$ into $X$. A subsequence of $s_1$ is a function from $\mathbb{N} \times \mathbb{N}$ into $X$ and is defined by

(Def. 14) There exist increasing sequences $N$, $M$ of $\mathbb{N}$ such that for every natural numbers $n$, $m$, $it(n, m) = s_1(N(n), M(m))$.

Let us consider $P_1$. Observe that every subsequence of $P_1$ is p-convergent.

Now we state the proposition:

(17) Let us consider a subsequence $P_2$ of $P_1$. Then $\text{P-lim}\, P_2 = \text{P-lim}\, P_1$.

Let $R$ be a convergent in the first coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Note that every subsequence of $R$ is convergent in the first coordinate.

Now we state the proposition:

(18) Let us consider a subsequence $R_1$ of $R$. Suppose

(i) $R$ is convergent in the first coordinate, and

(ii) the lim in the first coordinate of $R$ is convergent.

Then

(iii) the lim in the first coordinate of $R_1$ is convergent, and

(iv) the first coordinate major iterated lim of $R_1$ = the first coordinate major iterated lim of $R$.

Proof: Consider $I_1$, $I_2$ being increasing sequences of $\mathbb{N}$ such that for every natural numbers $n$, $m$, $R_1(n, m) = R(I_1(n), I_2(m))$. For every $e$ such that $0 < e$ there exists $N$ such that for every $m$ such that $m \geqslant N$ holds |(the lim in the first coordinate of $R_1)(m)$ − the first coordinate major iterated lim of $R| < e$. □

Let $R$ be a convergent in the second coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. One can check that every subsequence of $R$ is convergent in the second coordinate.

Now we state the proposition:

(19) Let us consider a subsequence $R_1$ of $R$. Suppose

(i) $R$ is convergent in the second coordinate, and

(ii) the lim in the second coordinate of $R$ is convergent.

Then

(iii) the lim in the second coordinate of $R_1$ is convergent, and

(iv) the second coordinate major iterated lim of $R_1$ = the second coordinate major iterated lim of $R$.

Proof: Consider $I_1$, $I_2$ being increasing sequences of $\mathbb{N}$ such that for every $n$ and $m$, $R_1(n, m) = R(I_1(n), I_2(m))$. For every $e$ such that $0 < e$ there exists $N$ such that for every $m$ such that $m \geqslant N$ holds |(the lim in the second coordinate of $R_1)(m)$ − the second coordinate major iterated lim of $R| < e$. □

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