Double Sequences and Limits

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Summary. Double sequences are important extension of the ordinary notion of a sequence. In this article we formalized three types of limits of double sequences and the theory of these limits.

MSC: 54A20 03B35

Keywords: formalization of basic metric space; limits of double sequences

MML identifier: DBLSEQ_1 version: 8.1.02 5.19.1189

The notation and terminology used in this paper have been introduced in the following articles: [3], [4], [13], [5], [15], [6], [7], [16], [10], [11], [2], [8], [11], [18], [12], [17], and [9].

In this paper $R, R_1, R_2$ denote functions from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$, $r_1, r_2$ denote convergent sequences of real numbers, $n, m, N, M$ denote natural numbers, and $e, r$ denote real numbers.

Let us consider $R$. We say that $R$ is $p$-convergent if and only if

(Def. 1) There exists a real number $p$ such that for every real number $e$ such that $0 < e$ there exists a natural number $N$ such that for every natural numbers $n, m$ such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - p| < e$.

Assume $R$ is $p$-convergent. The functor $\text{P-lim} R$ yielding a real number is defined by

(Def. 2) Let us consider a real number $e$. Suppose $0 < e$. Then there exists a natural number $N$ such that for every natural numbers $n, m$ such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - it| < e$.

1This work was supported by JSPS KAKENHI 23500029.
We say that $R$ is convergent in the first coordinate if and only if
(Def. 3) Let us consider an element $m$ of $\mathbb{N}$. Then $\text{curry}'(R, m)$ is convergent.

We say that $R$ is convergent in the second coordinate if and only if
(Def. 4) Let us consider an element $n$ of $\mathbb{N}$. Then $\text{curry}(R, n)$ is convergent.

The lim in the first coordinate of $R$ yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by
(Def. 5) Let us consider an element $m$ of $\mathbb{N}$. Then $\text{it}(m) = \lim \text{curry}'(R, m)$.

The lim in the second coordinate of $R$ yielding a function from $\mathbb{N}$ into $\mathbb{R}$ is defined by
(Def. 6) Let us consider an element $n$ of $\mathbb{N}$. Then $\text{it}(n) = \lim \text{curry}(R, n)$.

Assume the lim in the first coordinate of $R$ is convergent. The first coordinate major iterated lim of $R$ yielding a real number is defined by
(Def. 7) Let us consider a real number $e$. Suppose $0 < e$. Then there exists a natural number $M$ such that for every natural number $m$ such that $m \geq M$ holds $|(\text{the lim in the first coordinate of } R)(m) - \text{it}| < e$.

Assume the lim in the second coordinate of $R$ is convergent. The second coordinate major iterated lim of $R$ yielding a real number is defined by
(Def. 8) Let us consider a real number $e$. Suppose $0 < e$. Then there exists a natural number $N$ such that for every natural number $n$ such that $n \geq N$ holds $|(\text{the lim in the second coordinate of } R)(n) - \text{it}| < e$.

Let $R$ be a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. We say that $R$ is uniformly convergent in the first coordinate if and only if
(Def. 9) (i) $R$ is convergent in the first coordinate, and
(ii) for every real number $e$ such that $e > 0$ there exists a natural number $M$ such that for every natural number $m$ such that $m \geq M$ for every natural number $n$, $|R(n, m) - (\text{the lim in the first coordinate of } R)(n)| < e$.

We say that $R$ is uniformly convergent in the second coordinate if and only if
(Def. 10) (i) $R$ is convergent in the second coordinate, and
(ii) for every real number $e$ such that $e > 0$ there exists a natural number $N$ such that for every natural number $n$ such that $n \geq N$ for every natural number $m$, $|R(n, m) - (\text{the lim in the second coordinate of } R)(m)| < e$.

Let us consider $R$. We say that $R$ is non-decreasing if and only if
(Def. 11) Let us consider natural numbers $n_1, m_1, n_2, m_2$. If $n_1 \geq n_2$ and $m_1 \geq m_2$, then $R(n_1, m_1) \geq R(n_2, m_2)$.

We say that $R$ is non-increasing if and only if
(Def. 12) Let us consider natural numbers $n_1, m_1, n_2, m_2$. If $n_1 \geq n_2$ and $m_1 \geq m_2$, then $R(n_1, m_1) \leq R(n_2, m_2)$. 
Now we state the proposition:

(1) Let us consider real numbers \( a, b, c \). If \( a \leq b \leq c \), then \( |b| \leq |a| \) or \( |b| \leq |c| \).

Note that every function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \) which is non-decreasing and \( p \)-convergent is also lower bounded and upper bounded and every function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \) which is non-increasing and \( p \)-convergent is also lower bounded and upper bounded.

Let \( r \) be an element of \( \mathbb{R} \). Let us note that \( \mathbb{N} \times \mathbb{N} \mapsto r \) is \( p \)-convergent convergent in the first coordinate and convergent in the second coordinate as a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \).

Now we state the proposition:

(2) Let us consider an element \( r \) of \( \mathbb{R} \). Then \( \text{P-lim}(\mathbb{N} \times \mathbb{N} \mapsto r) = r \). Proof:

Set \( R = \mathbb{N} \times \mathbb{N} \mapsto r \). For every natural numbers \( n, m \), \( R(n, m) = r \) by [15 (70)]. □

Note that there exists a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \) which is \( p \)-convergent, convergent in the first coordinate, and convergent in the second coordinate.

In this paper \( P_1 \) denotes a \( p \)-convergent function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \).

Let \( P_4 \) be a \( p \)-convergent convergent in the second coordinate function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). Note that the \( \text{lim} \) in the second coordinate of \( P_4 \) is convergent.

Now we state the proposition:

(3) Suppose \( R \) is \( p \)-convergent and convergent in the second coordinate.

Then \( \text{P-lim} R = \) the second coordinate major iterated lim of \( R \). Proof:

Consider \( z \) being a real number such that for every \( e \) such that \( 0 < e \) there exists a natural number \( N_1 \) such that for every \( n \) and \( m \) such that \( n \geq N_1 \) and \( m \geq N_1 \) holds \( |R(n, m) - z| < e \). For every \( e \) such that \( 0 < e \) there exists \( N \) such that for every \( n \) such that \( n \geq N \) holds \( |(the \ \text{lim \ in \ the \ second \ coordinate \ of} \ R(n)) - z| < e \) by [4 (63), (60)]. For every \( e \) such that \( 0 < e \) there exists \( N \) such that for every \( n \) such that \( n \geq N \) holds \( |(the \ \text{lim \ in \ the \ second \ coordinate \ of} \ R(n)) - \text{P-lim} R| < e \) by [4 (63), (60)]. □

Let \( P_3 \) be a \( p \)-convergent convergent in the first coordinate function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{R} \). Let us note that the \( \text{lim} \) in the first coordinate of \( P_3 \) is convergent.

Now we state the proposition:

(4) Suppose \( R \) is \( p \)-convergent and convergent in the first coordinate. Then \( \text{P-lim} R = \) the first coordinate major iterated lim of \( R \). Proof:

Consider \( z \) being a real number such that for every \( e \) such that \( 0 < e \) there exists a natural number \( N_1 \) such that for every \( n \) and \( m \) such that \( n \geq N_1 \) and \( m \geq N_1 \) holds \( |R(n, m) - z| < e \). For every \( e \) such that \( 0 < e \) there exists \( N \) such that for every \( n \) such that \( n \geq N \) holds \( |(the \ \text{lim \ in \ the \ first \ coordinate \ of} \ R(n)) - z| < e \) by [4 (63), (60)]. For every \( e \) such that \( 0 < e \)
there exists $N$ such that for every $n$ such that $n \geq N$ holds $|\text{the lim in the first coordinate of } R(n) - \text{P-lim } R| < e$ by [4] (60), (63). □

One can verify that every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-decreasing and upper bounded is also $p$-convergent convergent in the first coordinate and convergent in the second coordinate and every function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$ which is non-increasing and lower bounded is also $p$-convergent convergent in the first coordinate and convergent in the second coordinate.

Now we state the propositions:

(5) Suppose $R$ is uniformly convergent in the first coordinate and the lim in the first coordinate of $R$ is convergent. Then

(i) $R$ is $p$-convergent, and

(ii) $\text{P-lim } R = \text{the first coordinate major iterated lim of } R$.

(6) Suppose $R$ is uniformly convergent in the second coordinate and the lim in the second coordinate of $R$ is convergent. Then

(i) $R$ is $p$-convergent, and

(ii) $\text{P-lim } R = \text{the second coordinate major iterated lim of } R$.

Let us consider $R$. We say that $R$ is Cauchy if and only if

(Def. 13) Let us consider a real number $e$. Suppose $e > 0$. Then there exists a natural number $N$ such that for every natural numbers $n_1, n_2, m_1, m_2$ such that $N \leq n_1 \leq n_2$ and $N \leq m_1 \leq m_2$ holds $|R(n_2, m_2) - R(n_1, m_1)| < e$.

Now we state the propositions:

(7) $R$ is $p$-convergent if and only if $R$ is Cauchy. **Proof:** Define $\mathcal{R}(\text{element of } \mathbb{N}) = R(s_1, s_1)$. Consider $s_1$ being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element $n$ of $\mathbb{N}$, $s_1(n) = \mathcal{R}(n)$ from [7] Sch. 4. Reconsider $z = \lim s_1$ as a complex number. For every $e$ such that $0 < e$ there exists $N$ such that for every $n$ and $m$ such that $n \geq N$ and $m \geq N$ holds $|R(n, m) - z| < e$ by [4] (63). □

(8) Let us consider a function $R$ from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Suppose

(i) $R$ is non-decreasing, or

(ii) $R$ is non-increasing.

Then $R$ is $p$-convergent if and only if $R$ is lower bounded and upper bounded.

Let $X, Y$ be non empty sets, $H$ be a binary operation on $Y$, and $f, g$ be functions from $X$ into $Y$. Observe that the functor $H_{f,g}$ yields a function from $X \times X$ into $Y$. Now we state the propositions:

(9) (i) $\otimes_{r_1, r_2} R$ is convergent in the first coordinate and convergent in the second coordinate, and

(ii) the lim in the first coordinate of $\otimes_{r_1, r_2} R$ is convergent, and
(iii) the first coordinate major iterated lim of \( \lim_{\mathbb{R}} r_1, r_2 = \lim r_1 \cdot \lim r_2 \), and

(iv) the \( \lim \) in the second coordinate of \( \lim_{\mathbb{R}} r_1, r_2 \) is convergent, and

(v) the second coordinate major iterated lim of \( \lim_{\mathbb{R}} r_1, r_2 = \lim r_1 \cdot \lim r_2 \), and

(vi) \( \lim_{\mathbb{R}} r_1, r_2 \) is \( p \)-convergent, and

\[ \text{Proof:} \quad \text{Set } R = \lim_{\mathbb{R}} r_1, r_2. \text{ For every } n \text{ and } m, \text{ } R(n, m) = r_1(n) \cdot r_2(m) \text{ by } \[5 \] (77). \text{ For every element } m \text{ of } \mathbb{N} \text{ and for every real number } e \text{ such that } 0 < e, \text{ there exists } N \text{ such that for every } n \text{ such that } n \geq N \text{ holds } |(\text{curry}'(R, m))(n) - \lim r_1 \cdot r_2(m)| < e \text{ by } \[4 \] (47), (65), (44)]. \text{ For every element } m \text{ of } \mathbb{N}, \text{ curry}'(R, m) \text{ is convergent. For every element } m \text{ of } \mathbb{N} \text{ and for every real number } e \text{ such that } 0 < e \text{ there exists } N \text{ such that for every } n \text{ such that } n \geq N \text{ holds } |(\text{curry}(R, m))(n) - r_1(m) \cdot \lim r_2| < e \text{ by } \[4 \] (47), (65), (44)]. \text{ For every element } m \text{ of } \mathbb{N}, \text{ curry}(R, m) \text{ is convergent. For every } e \text{ such that } 0 < e \text{ there exists } N \text{ such that for every } n \text{ such that } n \geq N \text{ holds } |(\text{the lim in the first coordinate of } R(n)) - \lim r_1 \cdot \lim r_2| < e \text{ by } \[4 \] (46), (65)]. \text{ For every } e \text{ such that } 0 < e \text{ there exists } N \text{ such that for every } n \text{ and } m \text{ such that } n \geq N \text{ and } m \geq N \text{ holds } |R(n, m) - \lim r_1 \cdot \lim r_2| < e \text{ by } \[2 \] (3), \[4 \] (63), (46), (65)]. \]

\[ 10 \]

(i) \( +\lim_{\mathbb{R}} r_1, r_2 \) is convergent in the first coordinate and convergent in the second coordinate, and

(ii) the \( \lim \) in the first coordinate of \( +\lim_{\mathbb{R}} r_1, r_2 \) is convergent, and

(iii) the first coordinate major iterated lim of \( +\lim_{\mathbb{R}} r_1, r_2 = \lim r_1 + \lim r_2 \), and

(iv) the \( \lim \) in the second coordinate of \( +\lim_{\mathbb{R}} r_1, r_2 \) is convergent, and

(v) the second coordinate major iterated lim of \( +\lim_{\mathbb{R}} r_1, r_2 = \lim r_1 + \lim r_2 \), and

(vi) \( +\lim_{\mathbb{R}} r_1, r_2 \) is \( p \)-convergent, and

\[ \text{Proof:} \quad \text{Set } R = +\lim_{\mathbb{R}} r_1, r_2. \text{ For every } n \text{ and } m, \text{ } R(n, m) = r_1(n) + r_2(m) \text{ by } \[5 \] (77)]. \text{ For every element } m \text{ of } \mathbb{N} \text{ and for every real number } e \text{ such that } 0 < e \text{ there exists a natural number } N \text{ such that for every natural number } n \text{ such that } n \geq N \text{ holds } |(\text{curry}'(R, m))(n) - (\lim r_1 + r_2(m))| < e. \text{ For every element } m \text{ of } \mathbb{N}, \text{ curry}'(R, m) \text{ is convergent. For every element } m \text{ of } \mathbb{N} \text{ and for every real number } e \text{ such that } 0 < e \text{ there exists } N \text{ such that for every } n \text{ such that } n \geq N \text{ holds } |(\text{curry}(R, m))(n) - (r_1(m) + \lim r_2)| < e. \text{ For every element } m \text{ of } \mathbb{N}, \text{ curry}(R, m) \text{ is convergent. For every } e \text{ such} \]
that \(0 < e\) there exists \(N\) such that for every \(n\) such that \(n \geq N\) holds
\(|(\text{the lim in the first coordinate of } R(n) - (\text{lim } r_1 + \text{lim } r_2))| < e\). For every \(e\) such that \(0 < e\) there exists \(N\) such that for every \(n\) such that \(n \geq N\) holds
\(|(\text{the lim in the second coordinate of } R(n) - (\text{lim } r_1 + \text{lim } r_2))| < e\).
For every \(e\) such that \(0 < e\) there exists \(N\) such that for every \(n\) and \(m\) such that \(n \geq N\) and \(m \geq N\) holds
\(|R(n, m) - (\text{lim } r_1 + \text{lim } r_2)| < e\) by \([14]\) (56). □

(11) Suppose \(R_1\) is p-convergent and \(R_2\) is p-convergent. Then
(i) \(R_1 + R_2\) is p-convergent, and
(ii) \(\text{P-lim}(R_1 + R_2) = \text{P-lim } R_1 + \text{P-lim } R_2\).

(12) Suppose \(R_1\) is p-convergent and \(R_2\) is p-convergent. Then
(i) \(R_1 - R_2\) is p-convergent, and
(ii) \(\text{P-lim}(R_1 - R_2) = \text{P-lim } R_1 - \text{P-lim } R_2\).

(13) Let us consider a function \(R\) from \(\mathbb{N} \times \mathbb{N}\) into \(\mathbb{R}\) and a real number \(r\).
Suppose \(R\) is p-convergent. Then
(i) \(r \cdot R\) is p-convergent, and
(ii) \(\text{P-lim}(r \cdot R) = r \cdot \text{P-lim } R\).

(14) If \(R\) is p-convergent and for every natural numbers \(n, m\), \(R(n, m) \geq r\),
then \(\text{P-lim } R \geq r\).

(15) Suppose \(R_1\) is p-convergent and \(R_2\) is p-convergent and for every natural
numbers \(n, m\), \(R_1(n, m) \leq R_2(n, m)\). Then \(\text{P-lim } R_1 \leq \text{P-lim } R_2\). The
theorem is a consequence of (12) and (14).

(16) Suppose \(R_1\) is p-convergent and \(R_2\) is p-convergent and \(\text{P-lim } R_1 = \text{P-lim } R_2\) and for every natural numbers \(n, m\), \(R_1(n, m) \leq R(n, m) \leq R_2(n, m)\). Then
(i) \(R\) is p-convergent, and
(ii) \(\text{P-lim } R = \text{P-lim } R_1\).

**Proof:** For every \(e\) such that \(0 < e\) there exists \(N\) such that for every \(n\) and \(m\) such that \(n \geq N\) and \(m \geq N\) holds
\(|R(n, m) - \text{P-lim } R_1| < e\) by \([14]\) (4), (5), (1). □

Let \(X\) be a non empty set and \(s_1\) be a function from \(\mathbb{N} \times \mathbb{N}\) into \(X\). A
subsequence of \(s_1\) is a function from \(\mathbb{N} \times \mathbb{N}\) into \(X\) and is defined by
(Def. 14) There exist increasing sequences \(N, M\) of \(\mathbb{N}\) such that for every natural
numbers \(n, m, it(n, m) = s_1(N(n), M(m))\).

Let us consider \(P_1\). Observe that every subsequence of \(P_1\) is p-convergent.
Now we state the proposition:

(17) Let us consider a subsequence \(P_2\) of \(P_1\). Then \(\text{P-lim } P_2 = \text{P-lim } P_1\).
Let $R$ be a convergent in the first coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Note that every subsequence of $R$ is convergent in the first coordinate.

Now we state the proposition:

(18) Let us consider a subsequence $R_1$ of $R$. Suppose

(i) $R$ is convergent in the first coordinate, and

(ii) the lim in the first coordinate of $R$ is convergent.

Then

(iii) the lim in the first coordinate of $R_1$ is convergent, and

(iv) the first coordinate major iterated lim of $R_1 = \text{the first coordinate major iterated lim of } R$.

Proof: Consider $I_1, I_2$ being increasing sequences of $\mathbb{N}$ such that for every natural numbers $n, m$, $R_1(n, m) = R(I_1(n), I_2(m))$. For every $e$ such that $0 < e$ there exists $N$ such that for every $m$ such that $m \geq N$ holds $|(\text{the lim in the first coordinate of } R_1)(m) - \text{the first coordinate major iterated lim of } R| < e$. □

Let $R$ be a convergent in the second coordinate function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. One can check that every subsequence of $R$ is convergent in the second coordinate.

Now we state the proposition:

(19) Let us consider a subsequence $R_1$ of $R$. Suppose

(i) $R$ is convergent in the second coordinate, and

(ii) the lim in the second coordinate of $R$ is convergent.

Then

(iii) the lim in the second coordinate of $R_1$ is convergent, and

(iv) the second coordinate major iterated lim of $R_1 = \text{the second coordinate major iterated lim of } R$.

Proof: Consider $I_1, I_2$ being increasing sequences of $\mathbb{N}$ such that for every $n$ and $m$, $R_1(n, m) = R(I_1(n), I_2(m))$. For every $e$ such that $0 < e$ there exists $N$ such that for every $m$ such that $m \geq N$ holds $|(\text{the lim in the second coordinate of } R_1)(m) - \text{the second coordinate major iterated lim of } R| < e$. □

References


Received August 31, 2013