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Riemann Integral of Functions from $\mathbb{R}$ into Real Banach Space[1]

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Summary. In this article we deal with the Riemann integral of functions from $\mathbb{R}$ into a real Banach space. The last theorem establishes the integrability of continuous functions on the closed interval of reals. To prove the integrability we defined uniform continuity for functions from $\mathbb{R}$ into a real normed space, and proved related theorems. We also stated some properties of finite sequences of elements of a real normed space and finite sequences of real numbers.

In addition we proved some theorems about the convergence of sequences. We applied definitions introduced in the previous article [21] to the proof of integrability.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [1], [7], [22], [4], [8], [14], [9], [10], [21], [15], [16], [17], [18], [28], [26], [5], [27], [2], [23], [24], [3], [11], [19], [25], [32], [33], [30], [12], [20], [31], and [13].

1. Some Properties of Continuous Functions

In this paper s_1, s_2, q_1 denote sequences of real numbers.

Let X be a real normed space and f be a partial function from $\mathbb{R}$ to the carrier of X. We say that f is uniformly continuous if and only if

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(Def. 1) Let us consider a real number r. Suppose $0 < r$. Then there exists a real number s such that

(i) $0 < s$, and

(ii) for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Now we state the propositions:

(1) Let us consider a set X, a real normed space Y, and a partial function f from $\mathbb{R}$ to the carrier of Y. Then $f{\restriction}X$ is uniformly continuous if and only if for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}(f{\restriction}X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$. PROOF: If $f{\restriction}X$ is uniformly continuous, then for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}(f{\restriction}X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$ by [11, (80)]. Consider s being a real number such that $0 < s$ and for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}(f{\restriction}X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$. $\square$

(2) Let us consider sets X, X_1, a real normed space Y, and a partial function f from $\mathbb{R}$ to the carrier of Y. Suppose

(i) $f{\restriction}X$ is uniformly continuous, and

(ii) $X_1 \subseteq X$.

Then $f{\restriction}X_1$ is uniformly continuous. The theorem is a consequence of (1).

(3) Let us consider a real normed space X, a partial function f from $\mathbb{R}$ to the carrier of X, and a subset Z of $\mathbb{R}$. Suppose

(i) $Z \subseteq \operatorname{dom} f$, and

(ii) Z is compact, and

(iii) $f{\restriction}Z$ is continuous.

Then $f{\restriction}Z$ is uniformly continuous. The theorem is a consequence of (1).

2. Some Properties of Sequences

Now we state the proposition:

(4) Let us consider a real normed space X, natural numbers n, m, a function a from $\operatorname{Seg} n \times \operatorname{Seg} m$ into X, and finite sequences p, q of elements of X. Suppose

(i) $\operatorname{dom} p = \operatorname{Seg} n$, and

(ii) for every natural number i such that $i \in \operatorname{dom} p$ there exists a finite sequence r of elements of X such that $\operatorname{dom} r = \operatorname{Seg} m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \operatorname{dom} r$ holds $r(j) = a(i, j)$, and

(iii) $\operatorname{dom} q = \operatorname{Seg} m$, and

(iv) for every natural number j such that $j \in \operatorname{dom} q$ there exists a finite sequence s of elements of X such that $\operatorname{dom} s = \operatorname{Seg} n$ and $q(j) = \sum s$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) = a(i, j)$.

Then $\sum p = \sum q$. PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every natural number m for every function a from $\operatorname{Seg} \$_1 \times \operatorname{Seg} m$ into X for every finite sequences p, q of elements of X such that $\operatorname{dom} p = \operatorname{Seg} \$_1$ and for every natural number i such that $i \in \operatorname{dom} p$ there exists a finite sequence r of elements of X such that $\operatorname{dom} r = \operatorname{Seg} m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \operatorname{dom} r$ holds $r(j) = a(i, j)$ and $\operatorname{dom} q = \operatorname{Seg} m$ and for every natural number j such that $j \in \operatorname{dom} q$ there exists a finite sequence s of elements of X such that $\operatorname{dom} s = \operatorname{Seg} \$_1$ and $q(j) = \sum s$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) = a(i, j)$ holds $\sum p = \sum q$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [4, (5)], [2, (11)], [13, (95)]. For every natural number n, $\mathcal{P}[n]$ from [2, Sch. 2]. $\square$

Let A be a subset of $\mathbb{R}$. The extension of $\operatorname{vol}(A)$ yielding a real number is defined by the term

(Def. 2) $$\begin{cases} 0, & \textbf{if } A \text{ is empty,} \\ \operatorname{vol}(A), & \textbf{otherwise}. \end{cases}$$

In the sequel n denotes an element of $\mathbb{N}$ and a, b denote real numbers.

Now we state the propositions:

(5) Let us consider a real bounded subset A of $\mathbb{R}$. Then $0 \leqslant$ the extension of $\operatorname{vol}(A)$.

(6) Let us consider a non empty closed interval subset A of $\mathbb{R}$, a Division D of A, and a finite sequence q of elements of $\mathbb{R}$. Suppose

(i) $\operatorname{dom} q = \operatorname{Seg} \operatorname{len} D$, and

(ii) for every natural number i such that $i \in \operatorname{Seg} \operatorname{len} D$ holds $q(i) = \operatorname{vol}(\operatorname{divset}(D, i))$.

Then $\sum q = \operatorname{vol}(A)$. PROOF: Set $p = \text{lower\_volume}(\chi_{A,A}, D)$. For every natural number k such that $k \in \operatorname{dom} q$ holds $q(k) = p(k)$ by [15, (19)]. $\square$

(7) Let us consider a real normed space Y, a point E of Y, a finite sequence q of elements of $\mathbb{R}$, and a finite sequence S of elements of Y. Suppose

(i) $\operatorname{len} S = \operatorname{len} q$, and

(ii) for every natural number i such that $i \in \operatorname{dom} S$ there exists a real number r such that $r = q(i)$ and $S(i) = r \cdot E$.

Then $\sum S = \sum q \cdot E$. PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence q of elements of $\mathbb{R}$ for every finite sequence S of elements of Y such that $\$_1 = \operatorname{len} S$ and $\operatorname{len} S = \operatorname{len} q$ and for every natural number i such that $i \in \operatorname{dom} S$ there exists a real number r such that $r = q(i)$ and $S(i) = r \cdot E$ holds $\sum S = \sum q \cdot E$. $\mathcal{P}[0]$ by [30, (10)], [12, (72)], [30, (43)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. $\square$

(8) Let us consider a non empty closed interval subset A of $\mathbb{R}$, a Division D of A, a non empty closed interval subset B of $\mathbb{R}$, and a finite sequence v of elements of $\mathbb{R}$. Suppose

(i) $B \subseteq A$, and

(ii) $\operatorname{len} D = \operatorname{len} v$, and

(iii) for every natural number i such that $i \in \operatorname{dom} v$ holds $v(i) =$ the extension of $\operatorname{vol}(B \cap \operatorname{divset}(D, i))$.

Then $\sum v = \operatorname{vol}(B)$. The theorem is a consequence of (5). PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non empty closed interval subset A of $\mathbb{R}$ for every Division D of A for every non empty closed interval subset B of $\mathbb{R}$ for every finite sequence v of elements of $\mathbb{R}$ such that $\$_1 = \operatorname{len} D$ and $B \subseteq A$ and $\operatorname{len} D = \operatorname{len} v$ and for every natural number k such that $k \in \operatorname{dom} v$ holds $v(k) =$ the extension of $\operatorname{vol}(B \cap \operatorname{divset}(D, k))$ holds $\sum v = \operatorname{vol}(B)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [29, (29)], [4, (4)], [2, (11)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. $\square$

(9) Let us consider a real normed space Y, a finite sequence x_3 of elements of Y, and a finite sequence y of elements of $\mathbb{R}$. Suppose

(i) $\operatorname{len} x_3 = \operatorname{len} y$, and

(ii) for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x_3$ there exists a point v of Y such that $v = x_{3i}$ and $y(i) = \|v\|$.

Then $\|\sum x_3\| \leqslant \sum y$. PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence x_3 of elements of Y for every finite sequence y of elements of $\mathbb{R}$ such that $\$_1 = \operatorname{len} x_3$ and $\operatorname{len} x_3 = \operatorname{len} y$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x_3$ there exists a point v of Y such that $v = x_{3i}$ and $y(i) = \|v\|$ holds $\|\sum x_3\| \leqslant \sum y$. $\mathcal{P}[0]$ by [30, (43)], [12, (72)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. $\square$

(10) Let us consider a real normed space Y, a finite sequence p of elements of Y, and a finite sequence q of elements of $\mathbb{R}$. Suppose

(i) $\operatorname{len} p = \operatorname{len} q$, and

(ii) for every natural number j such that $j \in \operatorname{dom} p$ holds $\|p_j\| \leqslant q(j)$.

Then $\|\sum p\| \leqslant \sum q$. The theorem is a consequence of (9). Proof: Define $\mathcal{Q}[\text{natural number}, \text{set}] \equiv$ there exists a point v of Y such that $v = p_{\$_1}$ and $\$_2 = \|v\|$. For every natural number i such that $i \in \operatorname{Seg} \operatorname{len} p$ there exists an element x of $\mathbb{R}$ such that $\mathcal{Q}[i, x]$. Consider u being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} u = \operatorname{Seg} \operatorname{len} p$ and for every natural number i such that $i \in \operatorname{Seg} \operatorname{len} p$ holds $\mathcal{Q}[i, u(i)]$ from [4, Sch. 5]. For every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} p$ there exists a point v of Y such that $v = p_i$ and $u(i) = \|v\|$. □

(11) Let us consider an element j of $\mathbb{N}$, a non empty closed interval subset A of $\mathbb{R}$, and a Division D_1 of A. Suppose $j \in \operatorname{dom} D_1$. Then $\operatorname{vol}(\operatorname{divset}(D_1, j)) \leqslant \delta_{D_1}$.

(12) Let us consider a non empty closed interval subset A of $\mathbb{R}$, a Division D of A, and a real number r. Suppose $\delta_D < r$. Let us consider a natural number i and real numbers s, t. If $i \in \operatorname{dom} D$ and $s, t \in \operatorname{divset}(D, i)$, then $|s - t| < r$. The theorem is a consequence of (11).

(13) Let us consider a real Banach space X, a non empty closed interval subset A of $\mathbb{R}$, and a function h from A into the carrier of X. Suppose a real number r. Suppose $0 < r$. Then there exists a real number s such that

(i) $0 < s$, and

(ii) for every real numbers x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} h$ and $|x_1 - x_2| < s$ holds $\|h_{x_1} - h_{x_2}\| < r$.

Let us consider a division sequence T of A and a middle volume sequence S of h and T. Suppose

(iii) δ_T is convergent, and

(iv) $\lim \delta_T = 0$.

Then middle $\operatorname{sum}(h, S)$ is convergent. The theorem is a consequence of (8), (7), (4), (12), (5), (10), and (6). Proof: For every division sequence T of A and for every middle volume sequence S of h and T such that δ_T is convergent and $\lim \delta_T = 0$ holds middle $\operatorname{sum}(h, S)$ is convergent by [32, (57)], [15, (9)], [17, (9)]. □

The scheme *ExRealSeq2X* deals with a non empty set $\mathcal{D}$ and a unary functor $\mathcal{F}$, $\mathcal{G}$ yielding an element of $\mathcal{D}$ and states that

(Sch. 1) There exists a sequence s of $\mathcal{D}$ such that for every natural number n, $s(2 \cdot n) = \mathcal{F}(n)$ and $s(2 \cdot n + 1) = \mathcal{G}(n)$.

Now we state the propositions:

(14) Let us consider a natural number n. Then there exists a natural number k such that $n = 2 \cdot k$ or $n = 2 \cdot k + 1$.

(15) Let us consider a non empty closed interval subset A of $\mathbb{R}$ and division sequences T_2, T of A. Then there exists a division sequence T_1 of A such that for every natural number i, $T_1(2 \cdot i) = T_2(i)$ and $T_1(2 \cdot i + 1) = T(i)$. The theorem is a consequence of (14).

(16) Let us consider a non empty closed interval subset A of $\mathbb{R}$ and division sequences T_2, T, T_1 of A. Suppose

(i) δ_{T_2} is convergent, and

(ii) $\lim \delta_{T_2} = 0$, and

(iii) δ_T is convergent, and

(iv) $\lim \delta_T = 0$, and

(v) for every natural number i, $T_1(2 \cdot i) = T_2(i)$ and $T_1(2 \cdot i + 1) = T(i)$.

Then

(vi) δ_{T_1} is convergent, and

(vii) $\lim \delta_{T_1} = 0$.

The theorem is a consequence of (14).

(17) Let us consider a real normed space X, a non empty closed interval subset A of $\mathbb{R}$, a function h from A into the carrier of X, division sequences T_2, T, T_1 of A, a middle volume sequence S_7 of h and T_2, and a middle volume sequence S of h and T. Suppose a natural number i. Then

(i) $T_1(2 \cdot i) = T_2(i)$, and

(ii) $T_1(2 \cdot i + 1) = T(i)$.

Then there exists a middle volume sequence S_1 of h and T_1 such that for every natural number i, $S_1(2 \cdot i) = S_7(i)$ and $S_1(2 \cdot i + 1) = S(i)$. The theorem is a consequence of (14). PROOF: Reconsider $S_2 = S_7$, $S_3 = S$ as a sequence of (the carrier of X)*. Define $\mathcal{F}$(natural number) $=$ $S_{2\$_1}$. Define $\mathcal{G}$(natural number) $= S_{3\$_1}$. Consider S_1 being a sequence of (the carrier of X)* such that for every natural number n, $S_1(2 \cdot n) = \mathcal{F}(n)$ and $S_1(2 \cdot n + 1) = \mathcal{G}(n)$ from *ExRealSeq2X*. For every element i of $\mathbb{N}$, $S_1(i)$ is a middle volume of h and $T_1(i)$. $\square$

(18) Let us consider a real normed space X and sequences S_4, S_6, S_5 of X. Suppose

(i) S_5 is convergent, and

(ii) for every natural number i, $S_5(2 \cdot i) = S_4(i)$ and $S_5(2 \cdot i + 1) = S_6(i)$.

Then

(iii) S_4 is convergent, and

(iv) $\lim S_4 = \lim S_5$, and

(v) S_6 is convergent, and

(vi) $\lim S_6 = \lim S_5$.

The theorem is a consequence of (14). PROOF: For every real number r such that $0 < r$ there exists a natural number m_1 such that for every natural number i such that $m_1 \leqslant i$ holds $\|S_4(i) - \lim S_5\| < r$ by [2, (11)]. For every real number r such that $0 < r$ there exists a natural number m_1 such that for every natural number i such that $m_1 \leqslant i$ holds $\|S_6(i) - \lim S_5\| < r$ by [2, (11)]. $\square$

(19) Let us consider a real Banach space X and a continuous partial function f from $\mathbb{R}$ to the carrier of X. If $a \leqslant b$ and $[a, b] \subseteq \operatorname{dom} f$, then f is integrable on $[a, b]$. The theorem is a consequence of (3), (13), (15), (17), (16), and (18). PROOF: Set $A = [a, b]$. Reconsider $h = f \restriction A$ as a function from A into the carrier of X. Consider T_2 being a division sequence of A such that δ_{T_2} is convergent and $\lim \delta_{T_2} = 0$. Set $S_7 =$ the middle volume sequence of h and T_2. Set $I = \lim \text{middle sum}(h, S_7)$. For every division sequence T of A and for every middle volume sequence S of h and T such that δ_T is convergent and $\lim \delta_T = 0$ holds middle sum(h, S) is convergent and $\lim \text{middle sum}(h, S) = I$. $\square$

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