Riemann Integral of Functions from \( \mathbb{R} \) into Real Banach Space

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Summary. In this article we deal with the Riemann integral of functions from \( \mathbb{R} \) into a real Banach space. The last theorem establishes the integrability of continuous functions on the closed interval of reals. To prove the integrability we defined uniform continuity for functions from \( \mathbb{R} \) into a real normed space, and proved related theorems. We also stated some properties of finite sequences of elements of a real normed space and finite sequences of real numbers.

In addition we proved some theorems about the convergence of sequences. We applied definitions introduced in the previous article \cite{21} to the proof of integrability.

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The notation and terminology used in this paper have been introduced in the following articles: \cite{6, 11, 7, 22, 4, 8, 14, 9, 10, 21, 15, 16, 17, 18, 28, 26, 5, 27, 2, 23, 24, 3, 11, 19, 25, 32, 33, 30, 12, 20, 31, and 13}.

1. Some Properties of Continuous Functions

In this paper \( s_1, s_2, q_1 \) denote sequences of real numbers.

Let \( X \) be a real normed space and \( f \) be a partial function from \( \mathbb{R} \) to the carrier of \( X \). We say that \( f \) is uniformly continuous if and only if

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(Def. 1) Let us consider a real number $r$. Suppose $0 < r$. Then there exists a real number $s$ such that

(i) $0 < s$, and

(ii) for every real numbers $x_1, x_2$ such that $x_1, x_2 \in \text{dom } f$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$.

Now we state the propositions:

(1) Let us consider a set $X$, a real normed space $Y$, and a partial function $f$ from $\mathbb{R}$ to the carrier of $Y$. Then $f|X$ is uniformly continuous if and only if for every real number $r$ such that $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real numbers $x_1, x_2$ such that $x_1, x_2 \in \text{dom}(f|X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$. 

Proof: If $f|X$ is uniformly continuous, then for every real number $r$ such that $0 < r$ there exists a real number $s$ such that $0 < s$ and for every real numbers $x_1, x_2$ such that $x_1, x_2 \in \text{dom}(f|X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$ by (80)]. Consider $s$ being a real number such that $0 < s$ and for every real numbers $x_1, x_2$ such that $x_1, x_2 \in \text{dom}(f|X)$ and $|x_1 - x_2| < s$ holds $\|f_{x_1} - f_{x_2}\| < r$. □

(2) Let us consider sets $X, X_1$, a real normed space $Y$, and a partial function $f$ from $\mathbb{R}$ to the carrier of $Y$. Suppose

(i) $f|X$ is uniformly continuous, and

(ii) $X_1 \subseteq X$.

Then $f|X_1$ is uniformly continuous. The theorem is a consequence of (1).

(3) Let us consider a real normed space $X$, a partial function $f$ from $\mathbb{R}$ to the carrier of $X$, and a subset $Z$ of $\mathbb{R}$. Suppose

(i) $Z \subseteq \text{dom } f$, and

(ii) $Z$ is compact, and

(iii) $f|Z$ is continuous.

Then $f|Z$ is uniformly continuous. The theorem is a consequence of (1).

2. Some Properties of Sequences

Now we state the proposition:

(4) Let us consider a real normed space $X$, natural numbers $n, m$, a function $a$ from $\text{Seg } n \times \text{Seg } m$ into $X$, and finite sequences $p, q$ of elements of $X$. Suppose

(i) $\text{dom } p = \text{Seg } n$, and
(ii) for every natural number \( i \) such that \( i \in \text{dom} p \) there exists a finite sequence \( r \) of elements of \( X \) such that \( \text{dom} r = \text{Seg} m \) and \( p(i) = \sum r \) and for every natural number \( j \) such that \( j \in \text{dom} r \) holds \( r(j) = a(i, j) \), and

(iii) \( \text{dom} q = \text{Seg} m \), and

(iv) for every natural number \( j \) such that \( j \in \text{dom} q \) there exists a finite sequence \( s \) of elements of \( X \) such that \( \text{dom} s = \text{Seg} n \) and \( q(j) = \sum s \) and for every natural number \( i \) such that \( i \in \text{dom} s \) holds \( s(i) = a(i, j) \).

Then \( \sum p = \sum q \).

**Proof:** Define \( \mathcal{P}[\text{natural number}] \equiv \) for every natural number \( m \) for every function \( a \) from \( \text{Seg} S_1 \times \text{Seg} m \) into \( X \) for every finite sequences \( p, q \) of elements of \( X \) such that \( \text{dom} p = \text{Seg} S_1 \) and for every natural number \( i \) such that \( i \in \text{dom} p \) there exists a finite sequence \( r \) of elements of \( X \) such that \( \text{dom} r = \text{Seg} m \) and \( p(i) = \sum r \) and for every natural number \( j \) such that \( j \in \text{dom} r \) holds \( r(j) = a(i, j) \) and \( \text{dom} q = \text{Seg} m \) and for every natural number \( j \) such that \( j \in \text{dom} q \) there exists a finite sequence \( s \) of elements of \( X \) such that \( \text{dom} s = \text{Seg} S_1 \) and \( q(j) = \sum s \) and for every natural number \( i \) such that \( i \in \text{dom} s \) holds \( s(i) = a(i, j) \).

For every natural number \( n \) such that \( \mathcal{P}[n] \) holds \( \mathcal{P}[n + 1] \) by [4, (5)], [2, (11)], [13, (95)]. For every natural number \( n, \mathcal{P}[n] \) from [2, Sch. 2]. □

Let \( A \) be a subset of \( \mathbb{R} \). The extension of \( \text{vol}(A) \) yielding a real number is defined by the term

\[
(\text{Def. 2}) \quad \left\{ \begin{array}{ll}
0, & \text{if } A \text{ is empty}, \\
\text{vol}(A), & \text{otherwise}.
\end{array} \right.
\]

In the sequel \( n \) denotes an element of \( \mathbb{N} \) and \( a, b \) denote real numbers.

Now we state the propositions:

(5) Let us consider a real bounded subset \( A \) of \( \mathbb{R} \). Then \( 0 \leq \text{the extension of } \text{vol}(A) \).

(6) Let us consider a non empty closed interval subset \( A \) of \( \mathbb{R} \), a Division \( D \) of \( A \), and a finite sequence \( q \) of elements of \( \mathbb{R} \). Suppose

(i) \( \text{dom} q = \text{Seg} \text{len} D \), and

(ii) for every natural number \( i \) such that \( i \in \text{Seg} \text{len} D \) holds \( q(i) = \text{vol}(\text{divset}(D, i)) \).

Then \( \sum q = \text{vol}(A) \).

**Proof:** Set \( p = \text{lower}_\text{volume}(\chi_{A,A}, D) \). For every natural number \( k \) such that \( k \in \text{dom} q \) holds \( q(k) = p(k) \) by [15, (19)]. □

(7) Let us consider a real normed space \( Y \), a point \( E \) of \( Y \), a finite sequence \( q \) of elements of \( \mathbb{R} \), and a finite sequence \( S \) of elements of \( Y \). Suppose

(i) \( \text{len} S = \text{len} q \), and
(ii) for every natural number $i$ such that $i \in \text{dom } S$ there exists a real number $r$ such that $r = q(i)$ and $S(i) = r \cdot E$.

Then $\sum S = \sum q \cdot E$. Proof: Define $P[\text{natural number}] \equiv \text{for every finite sequence } q \text{ of elements of } \mathbb{R} \text{ for every finite sequence } S \text{ of elements of } Y$ such that $S_1 = \text{len } S$ and $\text{len } S = \text{len } q$ and for every natural number $i$ such that $i \in \text{dom } S$ there exists a real number $r$ such that $r = q(i)$ and $S(i) = r \cdot E$ holds $\sum S = \sum q \cdot E$. $P[0]$ by [30] (10), [12] (72), [30] (43)]. For every natural number $i$, $P[i]$ from [2] Sch. 2. □

(8) Let us consider a non empty closed interval subset $A$ of $\mathbb{R}$, a Division $D$ of $A$, a non empty closed interval subset $B$ of $\mathbb{R}$, and a finite sequence $v$ of elements of $\mathbb{R}$. Suppose

(i) $B \subseteq A$, and

(ii) len $D = \text{len } v$, and

(iii) for every natural number $i$ such that $i \in \text{dom } v$ holds $v(i) = \text{the extension of } \text{vol}(B \cap \text{divset}(D, i))$.

Then $\sum v = \text{vol}(B)$. The theorem is a consequence of (5). Proof: Define $P[\text{natural number}] \equiv \text{for every non empty closed interval subset } A \text{ of } \mathbb{R} \text{ for every Division } D \text{ of } A \text{ for every non empty closed interval subset } B \text{ of } \mathbb{R} \text{ for every finite sequence } v \text{ of elements of } \mathbb{R} \text{ such that } S_1 = \text{len } D$ and $B \subseteq A$ and $\text{len } D = \text{len } v$ and for every natural number $k$ such that $k \in \text{dom } v$ holds $v(k) = \text{the extension of } \text{vol}(B \cap \text{divset}(D, k))$ holds $\sum v = \text{vol}(B)$. For every natural number $i$ such that $P[i]$ holds $P[i + 1]$ by [29] (29), [11] (4), [2] (11)]. For every natural number $i$, $P[i]$ from [2] Sch. 2. □

(9) Let us consider a real normed space $Y$, a finite sequence $x_3$ of elements of $Y$, and a finite sequence $y$ of elements of $\mathbb{R}$. Suppose

(i) len $x_3 = \text{len } y$, and

(ii) for every element $i$ of $\mathbb{N}$ such that $i \in \text{dom } x_3$ there exists a point $v$ of $Y$ such that $v = x_{3i}$ and $y(i) = \|v\|$.

Then $\|\sum x_3\| \leq \sum y$. Proof: Define $P[\text{natural number}] \equiv \text{for every finite sequence } x_3 \text{ of elements of } Y \text{ for every finite sequence } y \text{ of elements of } \mathbb{R} \text{ such that } S_1 = \text{len } x_3$ and $\text{len } x_3 = \text{len } y$ and for every element $i$ of $\mathbb{N}$ such that $i \in \text{dom } x_3$ there exists a point $v$ of $Y$ such that $v = x_{3i}$ and $y(i) = \|v\|$ holds $\|\sum x_3\| \leq \sum y$. $P[0]$ by [30] (43), [12] (72)]. For every natural number $i$, $P[i]$ from [2] Sch. 2. □

(10) Let us consider a real normed space $Y$, a finite sequence $p$ of elements of $Y$, and a finite sequence $q$ of elements of $\mathbb{R}$. Suppose

(i) len $p = \text{len } q$, and

(ii) for every natural number $j$ such that $j \in \text{dom } p$ holds $\|p_j\| \leq q(j)$. 


Then $\|\sum p\| \leq \sum q$. The theorem is a consequence of (9). Proof: Define $Q[\text{natural number}, \text{set}] \equiv$ there exists a point $v$ of $Y$ such that $v = p_i$, and $\|v\|$. For every natural number $i$ such that $i \in \text{Seg len } p$ there exists an element $x$ of $\mathbb{R}$ such that $Q[i, x]$. Consider $u$ being a finite sequence of elements of $\mathbb{R}$ such that $\text{dom } u = \text{Seg len } p$ and for every natural number $i$ such that $i \in \text{Seg len } p$ holds $Q[i, u(i)]$ from [4 Sch. 5]. For every element $i$ of $\mathbb{N}$ such that $i \in \text{dom } p$ there exists a point $v$ of $Y$ such that $v = p_i$ and $u(i) = \|v\|$.

(11) Let us consider an element $j$ of $\mathbb{N}$, a non empty closed interval subset $A$ of $\mathbb{R}$, and a Division $D_1$ of $A$. Suppose $j \in \text{dom } D_1$. Then $\text{vol}(\text{divset}(D_1, j)) \leq \delta_{D_1}$.

(12) Let us consider a non empty closed interval subset $A$ of $\mathbb{R}$, a Division $D$ of $A$, and a real number $r$. Suppose $\delta_D < r$. Let us consider a natural number $i$ and real numbers $s, t$. If $i \in \text{dom } D$ and $s, t \in \text{divset}(D, i)$, then $|s - t| < r$. The theorem is a consequence of (11).

(13) Let us consider a real Banach space $X$, a non empty closed interval subset $A$ of $\mathbb{R}$, and a function $h$ from $A$ into the carrier of $X$. Suppose a real number $r$. Suppose $0 < r$. Then there exists a real number $s$ such that

(i) $0 < s$, and

(ii) for every real numbers $x_1, x_2$ such that $x_1, x_2 \in \text{dom } h$ and $|x_1 - x_2| < s$ holds $\|h_{x_1} - h_{x_2}\| < r$.

Let us consider a division sequence $T$ of $A$ and a middle volume sequence $S$ of $h$ and $T$. Suppose

(iii) $\delta_T$ is convergent, and

(iv) $\lim \delta_T = 0$.

Then middle sum($h, S$) is convergent. The theorem is a consequence of (8), (7), (4), (12), (5), (10), and (6). Proof: For every division sequence $T$ of $A$ and for every middle volume sequence $S$ of $h$ and $T$ such that $\delta_T$ is convergent and $\lim \delta_T = 0$ holds middle sum($h, S$) is convergent by [32 (57)], [15 (9)], [17 (9)].

The scheme $\text{ExRealSeq2X}$ deals with a non empty set $D$ and a unary functor $\mathcal{F}, \mathcal{G}$ yielding an element of $D$ and states that

(Sch. 1) There exists a sequence $s$ of $D$ such that for every natural number $n$, $s(2 \cdot n) = \mathcal{F}(n)$ and $s(2 \cdot n + 1) = \mathcal{G}(n)$.

Now we state the propositions:

(14) Let us consider a natural number $n$. Then there exists a natural number $k$ such that $n = 2 \cdot k$ or $n = 2 \cdot k + 1$. 
(15) Let us consider a non empty closed interval subset $A$ of $\mathbb{R}$ and division sequences $T_2$, $T$ of $A$. Then there exists a division sequence $T_1$ of $A$ such that for every natural number $i$, $T_1(2 \cdot i) = T_2(i)$ and $T_1(2 \cdot i + 1) = T(i)$. The theorem is a consequence of (14).

(16) Let us consider a non empty closed interval subset $A$ of $\mathbb{R}$ and division sequences $T_2$, $T$, $T_1$ of $A$. Suppose

(i) $\delta_{T_2}$ is convergent, and
(ii) $\lim \delta_{T_2} = 0$, and
(iii) $\delta_T$ is convergent, and
(iv) $\lim \delta_T = 0$, and
(v) for every natural number $i$, $T_1(2 \cdot i) = T_2(i)$ and $T_1(2 \cdot i + 1) = T(i)$.

Then

(vi) $\delta_{T_1}$ is convergent, and
(vii) $\lim \delta_{T_1} = 0$.

The theorem is a consequence of (14).

(17) Let us consider a real normed space $X$, a non empty closed interval subset $A$ of $\mathbb{R}$, a function $h$ from $A$ into the carrier of $X$, division sequences $T_2$, $T$, $T_1$ of $A$, a middle volume sequence $S_7$ of $h$ and $T_2$, and a middle volume sequence $S$ of $h$ and $T$. Suppose a natural number $i$. Then

(i) $T_1(2 \cdot i) = T_2(i)$, and
(ii) $T_1(2 \cdot i + 1) = T(i)$.

Then there exists a middle volume sequence $S_1$ of $h$ and $T_1$ such that for every natural number $i$, $S_1(2 \cdot i) = S_7(i)$ and $S_1(2 \cdot i + 1) = S(i)$.

The theorem is a consequence of (14). PROOF: Reconsider $S_2 = S_7$, $S_3 = S$ as a sequence of (the carrier of $X)^*$. Define $F$(natural number) = $S_2S_1$. Define $G$(natural number) = $S_3$. Consider $S_1$ being a sequence of (the carrier of $X)^*$ such that for every natural number $n$, $S_1(2 \cdot n) = F(n)$ and $S_1(2 \cdot n + 1) = G(n)$ from $ExRealSeq2X$. For every element $i$ of $\mathbb{N}$, $S_1(i)$ is a middle volume of $h$ and $T_1(i)$. □

(18) Let us consider a real normed space $X$ and sequences $S_4$, $S_6$, $S_5$ of $X$. Suppose

(i) $S_5$ is convergent, and
(ii) for every natural number $i$, $S_5(2 \cdot i) = S_4(i)$ and $S_5(2 \cdot i + 1) = S_6(i)$.

Then

(iii) $S_4$ is convergent, and
(iv) $\lim S_4 = \lim S_5$, and
(v) $S_6$ is convergent, and
(vi) \( \lim S_6 = \lim S_5 \).

The theorem is a consequence of (14). **Proof:** For every real number \( r \) such that \( 0 < r \) there exists a natural number \( m_1 \) such that for every natural number \( i \) such that \( m_1 \leq i \) holds \( \| S_4(i) - \lim S_5 \| < r \) by [2] (11). For every real number \( r \) such that \( 0 < r \) there exists a natural number \( m_1 \) such that for every natural number \( i \) such that \( m_1 \leq i \) holds \( \| S_6(i) - \lim S_5 \| < r \) by [2] (11). □

(19) Let us consider a real Banach space \( X \) and a continuous partial function \( f \) from \( R \) to the carrier of \( X \). If \( a \leq b \) and \( [a, b] \subseteq \text{dom} f \), then \( f \) is integrable on \( [a, b] \). The theorem is a consequence of (3), (13), (15), (17), (16), and (18). **Proof:** Set \( A = [a, b] \). Reconsider \( h = f \upharpoonright A \) as a function from \( A \) into the carrier of \( X \). Consider \( T_2 \) being a division sequence of \( A \) such that \( \delta T_2 \) is convergent and \( \lim \delta T_2 = 0 \). Set \( S_7 = \text{the middle volume sequence of } h \text{ and } T_2 \). Set \( I = \lim \text{middle sum}(h, S_7) \).

For every division sequence \( T \) of \( A \) and for every middle volume sequence \( S \) of \( h \) and \( T \) such that \( \delta T \) is convergent and \( \lim \delta T = 0 \) holds middle sum \( (h, S) \) is convergent and \( \lim \text{middle sum}(h, S) = I \). □

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