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Random Variables and Product of Probability Spaces[1]

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Summary. We have been working on the formalization of the probability and the randomness. In [15] and [16], we formalized some theorems concerning the real-valued random variables and the product of two probability spaces. In this article, we present the generalized formalization of [15] and [16]. First, we formalize the random variables of arbitrary set and prove the equivalence between random variable on Σ, Borel sets and a real-valued random variable on Σ. Next, we formalize the product of countably infinite probability spaces.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [14], [12], [4], [11], [18], [7], [8], [5], [2], [3], [9], [13], [22], [15], [16], [20], [21], [17], [19], [6], and [10].

1. Random Variables

In this paper Ω, Ω_1, Ω_2 denote non empty sets, Σ denotes a σ-field of subsets of Ω, S_1 denotes a σ-field of subsets of Ω_1, and S_2 denotes a σ-field of subsets of Ω_2.

Now we state the proposition:

(1) Let us consider a non empty set B and a function f. Then $f^{-1}(\bigcup B) = \bigcup\{f^{-1}(Y)$ where Y is an element of B : not contradiction$\}$.

Let us consider a function f from Ω_1 into Ω_2, a sequence B of subsets of Ω_2, and a sequence D of subsets of Ω_1. Now we state the propositions:

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(2) If for every element n of $\mathbb{N}$, $D(n) = f^{-1}(B(n))$, then $f^{-1}(\bigcup B) = \bigcup D$.

(3) If for every element n of $\mathbb{N}$, $D(n) = f^{-1}(B(n))$, then $f^{-1}(\text{Intersection}\, B) = \text{Intersection}\, D$.

Now we state the propositions:

(4) Let us consider a function F from Ω into $\mathbb{R}$ and a real number r. Suppose F is a real-valued random variable on Σ. Then $F^{-1}(]-\infty, r[) \in \Sigma$. Proof: Consider X being an element of Σ such that $X = \Omega$ and F is measurable on X. For every element z, $z \in F^{-1}(]-\infty, r[)$ iff $z \in \Omega_\Sigma \cap \text{LE-dom}(F, r)$. □

(5) Let us consider a function F from Ω into $\mathbb{R}$. Suppose F is a real-valued random variable on Σ. Then $\{x$ where x is an element of the Borel sets : $F^{-1}(x)$ is element of $\Sigma\}$ is a σ-field of subsets of $\mathbb{R}$. The theorem is a consequence of (4) and (3). Proof: Set $S = \{x$ where x is an element of the Borel sets : $F^{-1}(x)$ is an element of $\Sigma\}$. For every element x such that $x \in S$ holds $x \in$ the Borel sets. Set $r_0 =$ the element of $\mathbb{R}$. Reconsider $y_0 = \text{halfline}(r_0)$ as an element of the Borel sets. For every subset A of $\mathbb{R}$ such that $A \in S$ holds $A^{\mathrm{c}} \in S$. For every sequence A_1 of subsets of $\mathbb{R}$ such that $\text{rng}\, A_1 \subseteq S$ holds $\text{Intersection}\, A_1 \in S$. □

Let us consider a function f from Ω into $\mathbb{R}$. Now we state the propositions:

(6) Suppose f is a real-valued random variable on Σ. Then $\{x$ where x is an element of the Borel sets : $f^{-1}(x)$ is an element of $\Sigma\} =$ the Borel sets.

(7) f is random variable on Σ and the Borel sets if and only if f is a real-valued random variable on Σ.

(8) The set of random variables on Σ and the Borel sets $=$ the real-valued random variables set on Σ.

Let us consider Ω_1, Ω_2, S_1, and S_2. Let F be a function from Ω_1 into Ω_2. We say that F is (S_1, S_2)-random variable-like if and only if

(Def. 1) F is random variable on S_1 and S_2.

Observe that there exists a function from Ω_1 into Ω_2 which is (S_1, S_2)-random variable-like.

A random variable of S_1 and S_2 is an (S_1, S_2)-random variable-like function from Ω_1 into Ω_2. Now we state the proposition:

(9) Let us consider a function f from Ω into $\mathbb{R}$. Then f is a random variable of Σ and the Borel sets if and only if f is a real-valued random variable on Σ.

Let F be a function. We say that F is random variable family-like if and only if

(Def. 2) Let us consider a set x. Suppose $x \in \text{dom}\, F$. Then there exist non empty sets Ω_1, Ω_2 and there exists a σ-field S_1 of subsets of Ω_1 and there exists a σ-field S_2 of subsets of Ω_2 and there exists a random variable f of S_1 and S_2 such that $F(x) = f$.

One can verify that there exists a function which is random variable family-like.

A random variable family is a random variable family-like function. In this paper F denotes a random variable of S_1 and S_2.

Let Y be a non empty set, S be a σ-field of subsets of Y, and F be a function. We say that F is S-measure valued if and only if

(Def. 3) Let us consider a set x. If $x \in \operatorname{dom} F$, then there exists a σ-measure M on S such that $F(x) = M$.

Note that there exists a function which is S-measure valued.

Let F be a function. We say that F is S-probability valued if and only if

(Def. 4) Let us consider a set x. If $x \in \operatorname{dom} F$, then there exists a probability P on S such that $F(x) = P$.

Let us note that there exists a function which is S-probability valued.

Let X, Y be non empty sets. One can verify that there exists an S-probability valued function which is X-defined.

One can verify that there exists an X-defined S-probability valued function which is total.

Let Y be a non empty set. Let us note that every function which is S-probability valued is also S-measure valued.

Let F be a function. We say that F is S-random variable family if and only if

(Def. 5) Let us consider a set x. Suppose $x \in \operatorname{dom} F$. Then there exists a real-valued random variable Z on S such that $F(x) = Z$.

Observe that there exists a function which is S-random variable family.

Now we state the propositions:

(10) Let us consider an element y of S_2. Suppose $y \neq \emptyset$. Then $\{z$ where z is an element of $\Omega_1 : F(z)$ is an element of $y\} = F^{-1}(y)$. Proof: Set $D = \{z$ where z is an element of $\Omega_1 : F(z)$ is an element of $y\}$. For every element x, $x \in D$ iff $x \in F^{-1}(y)$. $\square$

(11) Let us consider a random variable F of S_1 and S_2. Then

(i) $\{x$ where x is a subset of $\Omega_1 :$ there exists an element y of S_2 such that $x = F^{-1}(y)\} \subseteq S_1$, and

(ii) $\{x$ where x is a subset of $\Omega_1 :$ there exists an element y of S_2 such that $x = F^{-1}(y)\}$ is a σ-field of subsets of Ω_1.

The theorem is a consequence of (3). Proof: Set $S = \{x$ where x is a subset of $\Omega_1 :$ there exists an element y of S_2 such that $x = F^{-1}(y)\}$. For every element x such that $x \in S$ holds $x \in S_1$. For every subset A of

Ω_1 such that $A \in S$ holds $A^c \in S$. For every sequence A_1 of subsets of Ω_1 such that $\operatorname{rng} A_1 \subseteq S$ holds Intersection $A_1 \in S$. $\square$

Let us consider Ω_1, Ω_2, S_1, and S_2. Let M be a measure on S_1 and F be a random variable of S_1 and S_2. The functor the image measure of F and M yielding a measure on S_2 is defined by

(Def. 6) Let us consider an element y of S_2. Then $it(y) = M(F^{-1}(y))$.

Let M be a σ-measure on S_1. Note that the image measure of F and M is σ-additive.

Now we state the proposition:

(12) Let us consider a probability P on S_1 and a random variable F of S_1 and S_2. Then (the image measure of F and P2M P)$(\Omega_2) = 1$.

Let us consider Ω_1, Ω_2, S_1, and S_2. Let P be a probability on S_1 and F be a random variable of S_1 and S_2. The functor probability(F, P) yielding a probability on S_2 is defined by the term

(Def. 7) M2P the image measure of F and P2M P.

Now we state the propositions:

(13) Let us consider a probability P on S_1 and a random variable F of S_1 and S_2. Then probability(F, P) = the image measure of F and P2M P. The theorem is a consequence of (12).

(14) Let us consider a probability P on S_1, a random variable F of S_1 and S_2, and a set y. If $y \in S_2$, then (probability(F, P))$(y) = P(F^{-1}(y))$. The theorem is a consequence of (13).

(15) Every function from Ω_1 into Ω_2 is a random variable of the trivial σ-field of Ω_1 and the trivial σ-field of Ω_2.

(16) Let us consider a non empty set S. Then every non empty finite sequence of elements of S is a random variable of the trivial σ-field of Seg len F and the trivial σ-field of S. The theorem is a consequence of (15).

(17) Let us consider finite non empty sets V, S, a random variable G of the trivial σ-field of V and the trivial σ-field of S, and a set y. Suppose $y \in$ the trivial σ-field of S. Then (probability(G, the trivial probability of V))$(y) = \frac{\overline{\overline{G^{-1}(y)}}}{\overline{\overline{V}}}$. The theorem is a consequence of (14).

(18) Let us consider a finite non empty set S, a non empty finite sequence s of elements of S, and a set x. Suppose $x \in S$. Then there exists a random variable G of the trivial σ-field of Seg len s and the trivial σ-field of S such that

(i) $G = s$, and

(ii) (probability(G, the trivial probability of Seg len s))$(\{x\}) = \mathrm{Prob}_{\mathrm{D}}(x, s)$.

The theorem is a consequence of (16) and (17).

2. Product of Probability Spaces

Let D be a non-empty many sorted set indexed by $\mathbb{N}$ and n be a natural number. One can check that $D(n)$ is non empty.

Let S, F be many sorted sets indexed by $\mathbb{N}$. We say that F is σ-field S-sequence-like if and only if

(Def. 8) Let us consider a natural number n. Then $F(n)$ is a σ-field of subsets of $S(n)$.

Let S be a many sorted set indexed by $\mathbb{N}$. Let us observe that there exists a many sorted set indexed by $\mathbb{N}$ which is σ-field S-sequence-like.

Let D be a many sorted set indexed by $\mathbb{N}$. A σ-field sequence of D is a σ-field D-sequence-like many sorted set indexed by $\mathbb{N}$. Let S be a σ-field sequence of D and n be a natural number. Note that the functor $S(n)$ yields a σ-field of subsets of $D(n)$. Let D be a non-empty many sorted set indexed by $\mathbb{N}$. Let M be a many sorted set indexed by $\mathbb{N}$. We say that M is S-probability sequence-like if and only if

(Def. 9) Let us consider a natural number n. Then $M(n)$ is a probability on $S(n)$.

Observe that there exists a many sorted set indexed by $\mathbb{N}$ which is S-probability sequence-like.

A probability sequence of S is an S-probability sequence-like many sorted set indexed by $\mathbb{N}$. Let P be a probability sequence of S and n be a natural number. One can verify that the functor $P(n)$ yields a probability on $S(n)$. Let D be a many sorted set indexed by $\mathbb{N}$. The functor the product domain D yielding a many sorted set indexed by $\mathbb{N}$ is defined by

(Def. 10) (i) $it(0) = D(0)$, and

(ii) for every natural number i, $it(i+1) = it(i) \times D(i+1)$.

Now we state the proposition:

(19) Let us consider a many sorted set D indexed by $\mathbb{N}$. Then

(i) (the product domain $D)(0) = D(0)$, and

(ii) (the product domain $D)(1) = D(0) \times D(1)$, and

(iii) (the product domain $D)(2) = D(0) \times D(1) \times D(2)$, and

(iv) (the product domain $D)(3) = D(0) \times D(1) \times D(2) \times D(3)$.

Let D be a non-empty many sorted set indexed by $\mathbb{N}$. Let us note that the product domain D is non-empty.

Let D be a finite-yielding many sorted set indexed by $\mathbb{N}$. One can check that the product domain D is finite-yielding.

Let us consider Ω and Σ. Let P be a set. Assume P is a probability on Σ. The functor modetrans(P, Σ) yielding a probability on Σ is defined by the term

(Def. 11) P.

Let D be a finite-yielding non-empty many sorted set indexed by $\mathbb{N}$. The functor the trivial σ-field sequence D yielding a σ-field sequence of D is defined by

(Def. 12) Let us consider a natural number n. Then $it(n) =$ the trivial σ-field of $D(n)$.

Let P be a probability sequence of the trivial σ-field sequence D and n be a natural number. One can check that the functor $P(n)$ yields a probability on the trivial σ-field of $D(n)$. The functor ProductProbability(P, D) yielding a many sorted set indexed by $\mathbb{N}$ is defined by

(Def. 13) (i) $it(0) = P(0)$, and

(ii) for every natural number i, $it(i+1) =$ Product-Probability((the product domain $D)(i), D(i+1),$ modetrans $(it(i),$ the trivial σ-field of (the product domain $D)(i)), P(i+1))$.

Let us consider a finite-yielding non-empty many sorted set D indexed by $\mathbb{N}$, a probability sequence P of the trivial σ-field sequence D, and a natural number n. Now we state the propositions:

(20) (ProductProbability$(P, D))(n)$ is a probability on the trivial σ-field of (the product domain $D)(n)$.

(21) There exists a probability P_4 on the trivial σ-field of (the product domain $D)(n)$ such that

(i) $P_4 =$ (ProductProbability$(P, D))(n)$, and

(ii) (ProductProbability$(P, D))(n+1) =$ Product-Probability((the product domain $D)(n), D(n+1), P_4, P(n+1))$.

Now we state the proposition:

(22) Let us consider a finite-yielding non-empty many sorted set D indexed by $\mathbb{N}$ and a probability sequence P of the trivial σ-field sequence D. Then

(i) (ProductProbability$(P, D))(0) = P(0)$, and

(ii) (ProductProbability$(P, D))(1) =$ Product-Probability$(D(0), D(1), P(0), P(1))$, and

(iii) there exists a probability P_1 on the trivial σ-field of $D(0) \times D(1)$ such that $P_1 =$ (ProductProbability$(P, D))(1)$ and (ProductProbability$(P, D))(2) =$ Product-Probability$(D(0) \times D(1), D(2), P_1, P(2))$, and

(iv) there exists a probability P_2 on the trivial σ-field of $D(0) \times D(1) \times D(2)$ such that $P_2 =$ (ProductProbability$(P, D))(2)$ and (ProductProbability$(P, D))(3) =$ Product-Probability$(D(0) \times D(1) \times D(2), D(3), P_2, P(3))$, and

(v) there exists a probability P_3 on the trivial σ-field of $D(0) \times D(1) \times D(2) \times D(3)$ such that $P_3 =$ (ProductProbability$(P, D))(3)$ and

(ProductProbability(P, D))(4) = Product-Probability$(D(0) \times D(1) \times D(2) \times D(3), D(4), P_3, P(4))$.

The theorem is a consequence of (19) and (21).

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.

[6] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[12] Peter Jaeger. Elementary introduction to stochastic finance in discrete time. *Formalized Mathematics*, 20(**1**):1–5, 2012. doi:10.2478/v10037-012-0001-5.

[13] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.

[14] Hiroyuki Okazaki. Probability on finite and discrete set and uniform distribution. *Formalized Mathematics*, 17(**2**):173–178, 2009. doi:10.2478/v10037-009-0020-z.

[15] Hiroyuki Okazaki and Yasunari Shidama. Probability on finite set and real-valued random variables. *Formalized Mathematics*, 17(**2**):129–136, 2009. doi:10.2478/v10037-009-0014-x.

[16] Hiroyuki Okazaki and Yasunari Shidama. Probability measure on discrete spaces and algebra of real-valued random variables. *Formalized Mathematics*, 18(**4**):213–217, 2010. doi:10.2478/v10037-010-0026-6.

[17] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[18] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1 (**1**):187–190, 1990.

[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

[21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[22] Bo Zhang, Hiroshi Yamazaki, and Yatsuka Nakamura. The relevance of measure and probability, and definition of completeness of probability. *Formalized Mathematics*, 14 (**4**):225–229, 2006. doi:10.2478/v10037-006-0026-8.

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