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Banach's Continuous Inverse Theorem and Closed Graph Theorem[1]

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Summary. In this article we formalize one of the most important theorems of linear operator theory – the Closed Graph Theorem commonly used in a standard text book such as [10] in Chapter 24.3. It states that a surjective closed linear operator between Banach spaces is bounded.

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The terminology and notation used here have been introduced in the following articles: [3], [4], [2], [15], [11], [14], [1], [5], [13], [12], [19], [20], [16], [7], [17], [8], [18], [9], and [6].

Let X, Y be non empty normed structures, let x be a point of X, and let y be a point of Y. Then $\langle x, y\rangle$ is a point of $X \times Y$.

Let X, Y be non empty normed structures, let s_1 be a sequence of X, and let s_2 be a sequence of Y. Then $\langle s_1, s_2\rangle$ is a sequence of $X \times Y$.

We now state several propositions:

(1) Let X, Y be real linear spaces and T be a linear operator from X into Y. Suppose T is bijective. Then T^{-1} is a linear operator from Y into X and $\operatorname{rng}(T^{-1}) =$ the carrier of X.

(2) Let X, Y be non empty linear topological spaces, T be a linear operator from X into Y, and S be a function from Y into X. Suppose T is bijective

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and open and $S = T^{-1}$. Then S is a linear operator from Y into X, onto, and continuous.

(3) For all real normed spaces X, Y and for every linear operator f from X into Y holds $0_Y = f(0_X)$.

(4) Let X, Y be real normed spaces, f be a linear operator from X into Y, and x be a point of X. Then f is continuous in x if and only if f is continuous in 0_X.

(5) Let X, Y be real normed spaces and f be a linear operator from X into Y. Then f is continuous on the carrier of X if and only if f is continuous in 0_X.

(6) Let X, Y be real normed spaces and f be a linear operator from X into Y. Then f is Lipschitzian if and only if f is continuous on the carrier of X.

(7) Let X, Y be real Banach spaces and T be a Lipschitzian linear operator from X into Y. Suppose T is bijective. Then T^{-1} is a Lipschitzian linear operator from Y into X.

(8) Let X, Y be real normed spaces, s_1 be a sequence of X, s_2 be a sequence of Y, x be a point of X, and y be a point of Y. Then s_1 is convergent and $\lim s_1 = x$ and s_2 is convergent and $\lim s_2 = y$ if and only if $\langle s_1, s_2 \rangle$ is convergent and $\lim \langle s_1, s_2 \rangle = \langle\!\langle x, y \rangle\!\rangle$.

Let X, Y be real normed spaces and let T be a partial function from X to Y. The functor graph(T) yields a subset of $X \times Y$ and is defined as follows:

(Def. 1) graph(T) $= T$.

Let X, Y be real normed spaces and let T be a non empty partial function from X to Y. Observe that graph(T) is non empty.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. Note that graph(T) is linearly closed.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. The functor graphNrm(T) yielding a function from graph(T) into $\mathbb{R}$ is defined as follows:

(Def. 2) graphNrm(T) = (the norm of $X \times Y$)$\restriction$graph(T).

Let X, Y be real normed spaces and let T be a partial function from X to Y. We say that T is closed if and only if:

(Def. 3) graph(T) is closed.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. The functor graphNSP(T) yields a non empty normed structure and is defined by:

(Def. 4) $\mathrm{graphNSP}(T) = \langle \mathrm{graph}(T), \mathrm{Zero}(\mathrm{graph}(T), X \times Y), \mathrm{Add}(\mathrm{graph}(T), X \times Y), \mathrm{Mult}(\mathrm{graph}(T), X \times Y), \mathrm{graphNrm}(T) \rangle$.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. One can check that graphNSP(T) is Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, and scalar unital.

One can prove the following proposition

(9) For all real normed spaces X, Y and for every linear operator T from X into Y holds graphNSP(T) is a subspace of $X \times Y$.

Let X, Y be real normed spaces and let T be a linear operator from X into Y. Note that graphNSP(T) is reflexive, discernible, and real normed space-like.

We now state several propositions:

(10) Let X be a real normed space, Y be a real Banach space, and X_0 be a subset of Y. Suppose that

(i) X is a subspace of Y,

(ii) the carrier of $X = X_0$,

(iii) the norm of $X =$ (the norm of Y)$\restriction$(the carrier of X), and

(iv) X_0 is closed.

Then X is complete.

(11) Let X, Y be real Banach spaces and T be a linear operator from X into Y. If T is closed, then graphNSP(T) is complete.

(12) Let X, Y be real normed spaces and T be a non empty partial function from X to Y. Then T is closed if and only if for every sequence s_3 of X such that $\operatorname{rng} s_3 \subseteq \operatorname{dom} T$ and s_3 is convergent and $T_* s_3$ is convergent holds $\lim s_3 \in \operatorname{dom} T$ and $\lim(T_* s_3) = T(\lim s_3)$.

(13) Let X, Y be real normed spaces, T be a non empty partial function from X to Y, and T_0 be a linear operator from X into Y. If T_0 is Lipschitzian and $\operatorname{dom} T$ is closed and $T = T_0$, then T is closed.

(14) Let X, Y be real normed spaces, T be a non empty partial function from X to Y, and S be a non empty partial function from Y to X. If T is closed and one-to-one and $S = T^{-1}$, then S is closed.

(15) For all real normed spaces X, Y and for every point x of X and for every point y of Y holds $\|x\| \leq \|\langle x, y\rangle\|$ and $\|y\| \leq \|\langle x, y\rangle\|$.

Let X, Y be real Banach spaces. Note that every linear operator from X into Y which is closed is also Lipschitzian.

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