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The Friendship Theorem[1]

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Summary. In this article we prove the friendship theorem according to the article [1], which states that if a group of people has the property that any pair of persons have exactly one common friend, then there is a universal friend, i.e. a person who is a friend of every other person in the group.

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The papers [3], [2], [6], [7], [11], [8], [9], [15], [14], [4], [13], [5], [17], [18], [12], [16], and [10] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: x, y, z are sets, i, k, n are natural numbers, R is a binary relation, P is a finite binary relation, and p, q are finite sequences.

Let us consider P, x. Observe that $P^{\circ}x$ is finite.

We now state several propositions:

(1) $\overline{\overline{R}} = \overline{\overline{R^{\smile}}}$.

(2) If R is symmetric, then $R^{\circ}x = R^{-1}(x)$.

(3) If $(p_{\lfloor k}) \frown (p \restriction k) = (q_{\lfloor n}) \frown (q \restriction n)$ and $k \leq n \leq \operatorname{len} p$, then $p = (q_{\lfloor n -' k}) \frown (q \restriction (n -' k))$.

(4) If $n \in \operatorname{dom} q$ and $p = (q_{\lfloor n}) \frown (q \restriction n)$, then $q = (p_{\lfloor \operatorname{len} p -' n}) \frown (p \restriction (\operatorname{len} p -' n))$.

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(5) If $(p_{\lfloor k}) \frown (p{\restriction}k) = (q_{\lfloor n}) \frown (q{\restriction}n)$, then there exists i such that $p = (q_{\lfloor i}) \frown (q{\restriction}i)$.

The scheme *Sch* deals with a non empty set $\mathcal{A}$, a non zero natural number $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a cardinal number C such that $\mathcal{B} \cdot C = \overline{\overline{\{F \in \mathcal{A}^{\mathcal{B}} \colon \mathcal{P}[F]\}}}$

provided the following requirements are met:

- For all finite sequences p, q of elements of $\mathcal{A}$ such that $p \frown q$ is $\mathcal{B}$-element and $\mathcal{P}[p \frown q]$ holds $\mathcal{P}[q \frown p]$, and
- For every element p of $\mathcal{A}^{\mathcal{B}}$ such that $\mathcal{P}[p]$ and for every natural number i such that $i < \mathcal{B}$ and $p = (p_{\lfloor i}) \frown (p{\restriction}i)$ holds $i = 0$.

One can prove the following propositions:

(6) Let X be a non empty set, A be a non empty finite subset of X, and P be a function from X into 2^X. Suppose that for every x such that $x \in X$ holds $\overline{\overline{P(x)}} = n$. Then $\overline{\overline{\{F \in X^{k+1} \colon F(1) \in A \ \wedge\ \bigwedge_i (i \in \operatorname{Seg} k \ \Rightarrow\ F(i+1) \in P(F(i)))\}}} = \overline{\overline{A}} \cdot n^k$.

(7) If $\operatorname{len} p$ is prime and there exists i such that $0 < i < \operatorname{len} p$ and $p = (p_{\lfloor i}) \frown (p{\restriction}i)$, then $\operatorname{rng} p \subseteq \{p(1)\}$.

2. The Friendship Graph

Let us consider R and let x be an element of field R. We say that x is universal friend if and only if:

(Def. 1) For every y such that $y \in \operatorname{field} R \setminus \{x\}$ holds $\langle x, y\rangle \in R$.

Let R be a binary relation. We say that R has universal friend if and only if:

(Def. 2) There exists an element of field R which is universal friend.

Let R be a binary relation. We introduce R is without universal friend as an antonym of R has universal friend.

Let R be a binary relation. We say that R is friendship graph like if and only if:

(Def. 3) For all x, y such that $x, y \in \operatorname{field} R$ and $x \neq y$ there exists z such that $R^{\circ}x \cap \operatorname{Coim}(R, y) = \{z\}$.

Let us observe that there exists a binary relation which is finite, symmetric, irreflexive, and friendship graph like.

A friendship graph is a finite symmetric irreflexive friendship graph like binary relation.

In the sequel F_1 is a friendship graph.

The following propositions are true:

(8) $2 \mid \overline{\overline{F_1{}^\circ x}}$.

(9) If x, $y \in \operatorname{field} F_1$ and $\langle x, y\rangle \notin F_1$, then $\overline{\overline{F_1{}^\circ x}} = \overline{\overline{F_1{}^\circ y}}$.

(10) If F_1 is without universal friend and $x \in \operatorname{field} F_1$, then $\overline{\overline{F_1{}^\circ x}} > 2$.

(11) If F_1 is without universal friend and x, $y \in \operatorname{field} F_1$, then $\overline{\overline{F_1{}^\circ x}} = \overline{\overline{F_1{}^\circ y}}$.

(12) If F_1 is without universal friend and $x \in \operatorname{field} F_1$, then $\overline{\overline{\operatorname{field} F_1}} = 1 + \overline{\overline{F_1{}^\circ x}} \cdot (\overline{\overline{F_1{}^\circ x}} - 1)$.

(13) For all elements x, y of $\operatorname{field} F_1$ such that x is universal friend and $x \neq y$ there exists z such that $F_1{}^\circ y = \{x, z\}$ and $F_1{}^\circ z = \{x, y\}$.

3. The Friendship Theorem

Next we state the proposition

(14) If F_1 is non empty, then F_1 has universal friend.

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