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The Properties of Sets of Temporal Logic Subformulas[1]

Mariusz Giero[2]
Department of Logic, Informatics and Philosophy of Science
University of Białystok
Plac Uniwersytecki 1, 15-420 Białystok, Poland

Summary. This is a second preliminary article to prove the completeness theorem of an extension of basic propositional temporal logic. We base it on the proof of completeness for basic propositional temporal logic given in [17]. We introduce two modified definitions of a subformula. In the former one we treat until-formula as indivisible. In the latter one, we extend the set of subformulas of until-formulas by a special disjunctive formula. This is needed to construct a temporal model. We also define an ordered positive-negative pair of finite sequences of formulas (PNP). PNPs represent states of a temporal model.

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The notation and terminology used here have been introduced in the following papers: [21], [11], [24], [18], [4], [1], [26], [8], [22], [27], [10], [20], [2], [3], [5], [9], [12], [19], [6], [7], [16], [15], [23], [25], [13], and [14].

1. Preliminaries

For simplicity, we adopt the following convention: A, B, p, q, r are elements of the LTLB-WFF, n is an element of $\mathbb{N}$, X is a subset of the LTLB-WFF, g is a function from the LTLB-WFF into *Boolean*, and x is a set.

Next we state two propositions:

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(1) Let X be a non empty set, t be a finite sequence of elements of X, and k be a natural number. If $k+1 \leq \operatorname{len} t$, then $t_{\lfloor k} = \langle t(k+1) \rangle \frown (t_{\lfloor k+1})$.

(2) $\mathbb{N} \longmapsto \emptyset$ is a LTL Model.

Let us consider X. We say that X is without implication if and only if:

(Def. 1) For every p such that $p \in X$ holds p is not conditional.

Let D be a set. The functor D^*_{1-1} yielding a set is defined by:

(Def. 2) For every x holds $x \in D^*_{1-1}$ iff x is a one-to-one finite sequence of elements of D.

Let D be a set. One can verify that D^*_{1-1} is non empty.

Let D be a finite set. Observe that D^*_{1-1} is finite.

We now state the proposition

(3) For all sets D_1, D_2 such that $D_1 \subseteq D_2$ holds ${D_1}^*_{1-1} \subseteq {D_2}^*_{1-1}$.

Let a_1 be a set and let a_2 be a subset of a_1. Then ${a_2}^*_{1-1}$ is a non empty subset of ${a_1}^*_{1-1}$.

Next we state the proposition

(4) For all one-to-one finite sequences F, G such that $\operatorname{rng} F$ misses $\operatorname{rng} G$ holds $F \frown G$ is one-to-one.

Let X be a set and let f, g be one-to-one finite sequences of elements of X. Let us assume that $\operatorname{rng} f$ misses $\operatorname{rng} g$. The functor $f \frown g$ yielding a one-to-one finite sequence of elements of X is defined as follows:

(Def. 3) $f \frown g = f \frown g$.

2. Set of Subformulas where an Until-formula is treated as Indivisible and its Properties

The function $\dot{\tau}$ from the LTLB-WFF into $2^{\text{the LTLB-WFF}}$ is defined as follows:

(Def. 4) $\dot{\tau}(\bot_t) = \{\bot_t\}$ and $\dot{\tau}(\operatorname{prop} n) = \{\operatorname{prop} n\}$ and $\dot{\tau}(A \Rightarrow B) = \{A \Rightarrow B\} \cup \dot{\tau}(A) \cup \dot{\tau}(B)$ and $\dot{\tau}(A\,\mathcal{U}\,B) = \{A\,\mathcal{U}\,B\}$.

One can prove the following propositions:

(5) If A is not conditional, then $\dot{\tau}(A) = \{A\}$.

(6) $p \in \dot{\tau}(p)$.

Let us consider p. Observe that $\dot{\tau}(p)$ is non empty and finite.

One can prove the following propositions:

(7) If $p \Rightarrow q \in \dot{\tau}(r)$, then $p, q \in \dot{\tau}(r)$.

(8) If $p \in \dot{\tau}(q)$, then $\dot{\tau}(p) \subseteq \dot{\tau}(q)$.

(9) If $p\,\mathcal{U}\,q \in \dot{\tau}(\neg A)$, then $p\,\mathcal{U}\,q \in \dot{\tau}(A)$.

(10) If $p\,\mathcal{U}\,q \in \dot{\tau}(A \mathbin{\&\&} B)$, then $p\,\mathcal{U}\,q \in \dot{\tau}(A)$ or $p\,\mathcal{U}\,q \in \dot{\tau}(B)$.

(11) If $p \in \dot{\tau}(q)$ and $p \neq q$, then $\operatorname{len} p < \operatorname{len} q$.

(12) $\dot{\tau}(p) \subseteq \dot{\tau}(\neg p)$.

(13) $\dot{\tau}(q) \subseteq \dot{\tau}(p \,\&\&\, q)$.

(14) $\dot{\tau}(q) \subseteq \dot{\tau}(p \vee q)$.

Let us consider X. The functor $\tau(X)$ yields a subset of the LTLB-WFF and is defined as follows:

(Def. 5) $x \in \tau(X)$ iff there exists A such that $A \in X$ and $x \in \dot{\tau}(A)$.

We now state two propositions:

(15) $\tau(X) = \bigcup\{\dot{\tau}(p); p$ ranges over elements of the LTLB-WFF: $p \in X\}$.

(16) $X \subseteq \tau(X)$.

Let X be an empty subset of the LTLB-WFF. One can check that $\tau(X)$ is empty.

Let X be a finite subset of the LTLB-WFF. Note that $\tau(X)$ is finite.

Let X be a non empty subset of the LTLB-WFF. One can verify that $\tau(X)$ is non empty.

The following propositions are true:

(17) $\tau(\tau(X)) = \tau(X)$.

(18) If X is without implication, then $\tau(X) = X$.

(19) If $p \Rightarrow q \in \tau(X)$, then p, $q \in \tau(X)$.

(20) If $p \,\&\&\, q \in \tau(X)$, then p, $q \in \tau(X)$.

(21) If $p \vee q \in \tau(X)$, then p, $q \in \tau(X)$.

(22) If $\mathrm{UN}(p, q) \in \tau(X)$, then p, q, $p \,\mathcal{U}\, q \in \tau(X)$.

(23) If $p \in \tau(X)$, then $\dot{\tau}(p) \subseteq \tau(X)$.

3. Extended Set of Subformulas and its Properties

The function $\dot{\sigma}$ from the LTLB-WFF into $2^{\text{the LTLB-WFF}}$ is defined by:

(Def. 6) $\dot{\sigma}(\perp_t) = \{\perp_t\}$ and $\dot{\sigma}(\mathrm{prop}\, n) = \{\mathrm{prop}\, n\}$ and $\dot{\sigma}(A \Rightarrow B) = \{A \Rightarrow B\} \cup \dot{\sigma}(A) \cup \dot{\sigma}(B)$ and $\dot{\sigma}(A \,\mathcal{U}\, B) = \dot{\tau}(\mathrm{UN}(A, B)) \cup \dot{\sigma}(A) \cup \dot{\sigma}(B)$.

One can prove the following propositions:

(24) $p \,\mathcal{U}\, q \in \dot{\sigma}(p \,\mathcal{U}\, q)$.

(25) $\dot{\tau}(p) \subseteq \dot{\sigma}(p)$.

Let us consider p. Note that $\dot{\sigma}(p)$ is non empty and finite.

The following proposition is true

(26) If $p \in \dot{\sigma}(A \,\mathcal{U}\, B)$, then if $A \,\mathcal{U}\, B \in \dot{\sigma}(q)$, then $p \in \dot{\sigma}(q)$.

Let us consider X. The functor $\sigma(X)$ yielding a subset of $2^{\text{the LTLB-WFF}}$ is defined as follows:

(Def. 7) $\sigma(X) = \{\dot{\sigma}(A); A$ ranges over elements of the LTLB-WFF: $A \in X\}$.

Let X be a finite subset of the LTLB-WFF. Note that $\sigma(X)$ is finite and finite-membered.

4. An Ordered Pair of Finite Sequences of Formulas. PNP-formula, Consistent PNP and Complete PNP

A positive-negative pair is an element of
(the LTLB-WFF)$^{*}_{1-1}$ $\times$ (the LTLB-WFF)$^{*}_{1-1}$.

In the sequel P, Q, P_1, R are positive-negative pairs.

Let us consider P. Then $P_{\mathbf{1}}$ is a one-to-one finite sequence of elements of the LTLB-WFF. Then $P_{\mathbf{2}}$ is a one-to-one finite sequence of elements of the LTLB-WFF.

Let us consider P. The functor $\operatorname{rng} P$ yielding a finite subset of the LTLB-WFF is defined by:

(Def. 8) $\operatorname{rng} P = \operatorname{rng}(P_{\mathbf{1}}) \cup \operatorname{rng}(P_{\mathbf{2}})$.

Let f_1, f_2 be one-to-one finite sequences of elements of the LTLB-WFF. Then $\langle f_1, f_2 \rangle$ is a positive-negative pair.

Let us consider P. The functor $\widehat{P}$ yielding an element of the LTLB-WFF is defined by:

(Def. 9) $\widehat{P} = (\mathrm{conjunction}(P_{\mathbf{1}}))_{\mathrm{len}\,\mathrm{conjunction}(P_{\mathbf{1}})}$
$\&\&(\mathrm{conjunction}\,\mathrm{negation}(P_{\mathbf{2}}))_{\mathrm{len}\,\mathrm{conjunction}\,\mathrm{negation}(P_{\mathbf{2}})}$.

We now state three propositions:

(27) $\widehat{F} = \top_t \,\&\&\, \top_t$, where $F = \langle \varepsilon_{(\text{the LTLB-WFF})}, \varepsilon_{(\text{the LTLB-WFF})} \rangle$.

(28) If $A \in \operatorname{rng}(P_{\mathbf{1}})$, then $\emptyset_{\text{the LTLB-WFF}} \vdash \widehat{P} \Rightarrow A$.

(29) If $A \in \operatorname{rng}(P_{\mathbf{2}})$, then $\emptyset_{\text{the LTLB-WFF}} \vdash \widehat{P} \Rightarrow \neg A$.

Let us consider P. We say that P is inconsistent if and only if:

(Def. 10) $\emptyset_{\text{the LTLB-WFF}} \vdash \neg\widehat{P}$.

Let us consider P. We introduce P is consistent as an antonym of P is inconsistent.

We say that P is complete if and only if:

(Def. 11) $\tau(\operatorname{rng} P) = \operatorname{rng} P$.

One can check that $\langle \varepsilon_{(\text{the LTLB-WFF})}, \varepsilon_{(\text{the LTLB-WFF})} \rangle$ is consistent as a positive-negative pair.

Let us observe that $\langle \varepsilon_{(\text{the LTLB-WFF})}, \varepsilon_{(\text{the LTLB-WFF})} \rangle$ is complete as a positive-negative pair.

One can check that there exists a positive-negative pair which is consistent and complete.

Let P be a consistent positive-negative pair. Observe that $\langle P_{\mathbf{1}}, P_{\mathbf{2}} \rangle$ is consistent as a positive-negative pair.

5. The Properties of Consistent PNPs

One can prove the following propositions:

(30) For every consistent positive-negative pair P holds $\mathrm{rng}(P_{\mathbf{1}})$ misses $\mathrm{rng}(P_{\mathbf{2}})$.

(31) Let P be a consistent positive-negative pair. If $A \notin \mathrm{rng}\, P$, then $\langle\!\langle (P_{\mathbf{1}}) \frown \langle A \rangle, P_{\mathbf{2}} \rangle\!\rangle$ is consistent or $\langle\!\langle P_{\mathbf{1}}, (P_{\mathbf{2}}) \frown \langle A \rangle \rangle\!\rangle$ is consistent.

(32) For every consistent positive-negative pair P holds $\bot_t \notin \mathrm{rng}(P_{\mathbf{1}})$.

(33) Let P be a consistent positive-negative pair. Suppose A, B, $A \Rightarrow B \in \mathrm{rng}\, P$. Then $A \Rightarrow B \in \mathrm{rng}(P_{\mathbf{1}})$ if and only if $A \in \mathrm{rng}(P_{\mathbf{2}})$ or $B \in \mathrm{rng}(P_{\mathbf{1}})$.

(34) Let P be a consistent positive-negative pair. Then there exists a consistent positive-negative pair P_1 such that $\mathrm{rng}(P_{\mathbf{1}}) \subseteq \mathrm{rng}((P_1)_{\mathbf{1}})$ and $\mathrm{rng}(P_{\mathbf{2}}) \subseteq \mathrm{rng}((P_1)_{\mathbf{2}})$ and $\tau(\mathrm{rng}\, P) = \mathrm{rng}\, P_1$.

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