The Derivations of Temporal Logic Formulas¹

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Summary. This is a preliminary article to prove the completeness theorem of an extension of basic propositional temporal logic. We base it on the proof of completeness for basic propositional temporal logic given in [12]. We introduce n -ary connectives and prove their properties. We derive temporal logic formulas.

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The papers [14], [3], [1], [16], [6], [17], [8], [2], [7], [13], [4], [5], [11], [10], [15], and [9] provide the terminology and notation for this paper.

1. PRELIMINARIES

For simplicity, we adopt the following rules: A, B, p, q, r, s are elements of the LTLB-WFF, i, k, n are elements of $\mathbb{N}$, X is a subset of the LTLB-WFF, f, f_1 are finite sequences of elements of the LTLB-WFF, and g is a function from the LTLB-WFF into *Boolean*.

Let f be a finite sequence and let x be an empty set. One can check that $f(x)$ is empty.

We now state three propositions:

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- (1) For every finite sequence f such that $\text{len } f > 0$ and $n > 0$ holds $\text{len}(f \upharpoonright n) > 0$.
- (2) For every finite sequence f such that $\text{len } f = 0$ holds $f \upharpoonright n = f$.
- (3) For all finite sequences f, g such that $\text{rng } f = \text{rng } g$ holds $\text{len } f = 0$ iff $\text{len } g = 0$.

Let us consider A, B . The functor $\text{UN}(A, B)$ yields an element of the LTLB-WFF and is defined by:

(Def. 1) $\text{UN}(A, B) = B \vee (A \&\&(A \mathcal{U} B))$.

One can prove the following proposition

- (4) $\text{VAL}_g(\top_t) = 1$.

Next we state the proposition

- (5) $\text{VAL}_g(p \vee q) = \text{VAL}_g(p) \vee \text{VAL}_g(q)$.

2. n -ARGUMENT CONNECTIVES AND THEIR PROPERTIES

Let us consider f . The functor conjunction f yielding a finite sequence of elements of the LTLB-WFF is defined as follows:

- (Def. 2)(i) $\text{len conjunction } f = \text{len } f$ and $(\text{conjunction } f)(1) = f(1)$ and for every i such that $1 \leq i < \text{len } f$ holds $(\text{conjunction } f)(i + 1) = (\text{conjunction } f)_i \&\& f_{i+1}$ if $\text{len } f > 0$,
- (ii) $\text{conjunction } f = \langle \top_t \rangle$, otherwise.

Let us consider f, A . The functor implication (f, A) yielding a finite sequence of elements of the LTLB-WFF is defined as follows:

- (Def. 3)(i) $\text{len implication}(f, A) = \text{len } f$ and $(\text{implication}(f, A))(1) = \mathcal{G}(f_1) \Rightarrow A$ and for every i such that $1 \leq i < \text{len } f$ holds $(\text{implication}(f, A))(i + 1) = \mathcal{G}(f_{i+1}) \Rightarrow (\text{implication}(f, A))_i$ if $\text{len } f > 0$,
- (ii) $\text{implication}(f, A) = \varepsilon_{(\text{the LTLB-WFF})}$, otherwise.

Let us consider f . The functor negation f yields a finite sequence of elements of the LTLB-WFF and is defined by:

- (Def. 4) $\text{len negation } f = \text{len } f$ and for every i such that $1 \leq i \leq \text{len } f$ holds $(\text{negation } f)(i) = \neg(f_i)$.

Let us consider f . The functor next f yields a finite sequence of elements of the LTLB-WFF and is defined by:

- (Def. 5) $\text{len next } f = \text{len } f$ and for every i such that $1 \leq i \leq \text{len } f$ holds $(\text{next } f)(i) = \mathcal{X}(f_i)$.

We now state a number of propositions:

- (6) If $\text{len } f > 0$, then $(\text{conjunction } f)_1 = f_1$.
- (7) For every natural number i such that $1 \leq i < \text{len } f$ holds $(\text{conjunction } f)_{i+1} = (\text{conjunction } f)_i \&\& f_{i+1}$.

- (8) For every natural number i such that $i \in \text{dom } f$ holds $(\text{negation } f)_i = \neg(f_i)$.
- (9) For every natural number i such that $i \in \text{dom } f$ holds $(\text{next } f)_i = \mathcal{X}(f_i)$.
- (10) $(\text{conjunction}(\varepsilon_{(\text{the LTLB-WFF})}))_{\text{len conjunction}(\varepsilon_{(\text{the LTLB-WFF})})} = \top_t$.
- (11) $(\text{conjunction}\langle A \rangle)_{\text{len conjunction}\langle A \rangle} = A$.
- (12) For every k such that $n \leq k$ holds $(\text{conjunction } f)(n) = (\text{conjunction}(f \upharpoonright k))(n)$.
- (13) For every k such that $n \leq k$ and $1 \leq n \leq \text{len } f$ holds $(\text{conjunction } f)_n = (\text{conjunction}(f \upharpoonright k))_n$.
- (14) $\text{negation}\langle A \rangle = \langle \neg A \rangle$.
- (15) $\text{negation}(f \wedge \langle A \rangle) = (\text{negation } f) \wedge \langle \neg A \rangle$.
- (16) $\text{negation}(f \wedge f_1) = (\text{negation } f) \wedge \text{negation } f_1$.
- (17) $\text{VAL}_g((\text{conjunction}(f \wedge f_1))_{\text{len conjunction}(f \wedge f_1)}) = \text{VAL}_g((\text{conjunction } f)_{\text{len conjunction } f}) \wedge \text{VAL}_g((\text{conjunction } f_1)_{\text{len conjunction } f_1})$.
- (18) If $n \in \text{dom } f$, then $\text{VAL}_g((\text{conjunction } f)_{\text{len conjunction } f}) = \text{VAL}_g((\text{conjunction}(f \upharpoonright (n-1)))_{\text{len conjunction}(f \upharpoonright (n-1))}) \wedge \text{VAL}_g(f_n)$
 $\text{VAL}_g((\text{conjunction}(f \upharpoonright n))_{\text{len conjunction}(f \upharpoonright n)})$.
- (19) $\text{VAL}_g((\text{conjunction } f)_{\text{len conjunction } f}) = 1$ iff for every natural number i such that $i \in \text{dom } f$ holds $\text{VAL}_g(f_i) = 1$.
- (20) $\text{VAL}_g(\neg((\text{conjunction negation } f)_{\text{len conjunction negation } f})) = 0$ iff for every natural number i such that $i \in \text{dom } f$ holds $\text{VAL}_g(f_i) = 0$.
- (21) If $\text{rng } f = \text{rng } f_1$, then $\text{VAL}_g((\text{conjunction } f)_{\text{len conjunction } f}) = \text{VAL}_g((\text{conjunction } f_1)_{\text{len conjunction } f_1})$.

3. CLASSICAL TAUTOLOGIES OF TEMPORAL LANGUAGE

Next we state a number of propositions:

- (22) $p \Rightarrow \top_t$ is tautologically valid.
- (23) $\neg \top_t \Rightarrow p$ is tautologically valid.
- (24) $p \Rightarrow p$ is tautologically valid.
- (25) $\neg \neg p \Rightarrow p$ is tautologically valid.
- (26) $p \Rightarrow \neg \neg p$ is tautologically valid.
- (27) $p \&\& q \Rightarrow p$ is tautologically valid.
- (28) $p \&\& q \Rightarrow q$ is tautologically valid.
- (29) For every natural number k such that $k \in \text{dom } f$ holds $f_k \Rightarrow \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f})$ is tautologically valid.
- (30) If $\text{rng } f \subseteq \text{rng } f_1$, then $\neg((\text{conjunction negation } f)_{\text{len conjunction negation } f}) \Rightarrow \neg((\text{conjunction negation } f_1)_{\text{len conjunction negation } f_1})$ is tautologically valid.

- (31) $\neg(p \Rightarrow q) \Rightarrow p$ is tautologically valid.
- (32) $\neg(p \Rightarrow q) \Rightarrow \neg q$ is tautologically valid.
- (33) $p \Rightarrow (q \Rightarrow p)$ is tautologically valid.
- (34) $p \Rightarrow (q \Rightarrow (p \Rightarrow q))$ is tautologically valid.
- (35) $\neg(p \&\& q) \Rightarrow \neg p \vee \neg q$ is tautologically valid.
- (36) $\neg(p \vee q) \Rightarrow \neg p \&\& \neg q$ is tautologically valid.
- (37) $\neg(p \&\& q) \Rightarrow (p \Rightarrow \neg q)$ is tautologically valid.
- (38) $\neg(\top_t \&\& \neg A) \Rightarrow A$ is tautologically valid.
- (39) $\neg(s \&\& q) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow \neg s))$ is tautologically valid.
- (40) $(p \Rightarrow r) \Rightarrow ((p \Rightarrow s) \Rightarrow (p \Rightarrow r \&\& s))$ is tautologically valid.
- (41) $\neg(p \&\& s) \Rightarrow \neg(r \&\& s \&\& (p \&\& q))$ is tautologically valid.
- (42) $\neg(p \&\& s) \Rightarrow \neg(p \&\& q \&\& (r \&\& s))$ is tautologically valid.
- (43) $(p \Rightarrow q \&\& \neg q) \Rightarrow \neg p$ is tautologically valid.
- (44) $(q \Rightarrow p \&\& r) \Rightarrow ((p \Rightarrow s) \Rightarrow (q \Rightarrow s \&\& r))$ is tautologically valid.
- (45) $(p \Rightarrow q) \Rightarrow ((r \Rightarrow s) \Rightarrow (p \&\& r \Rightarrow q \&\& s))$ is tautologically valid.
- (46) $(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q)))$ is tautologically valid.
- (47) $(p \Rightarrow q) \Rightarrow ((p \Rightarrow \neg r) \Rightarrow (p \Rightarrow \neg(q \Rightarrow r)))$ is tautologically valid.
- (48) $(p \Rightarrow q \vee r) \Rightarrow ((r \Rightarrow s) \Rightarrow (p \Rightarrow q \vee s))$ is tautologically valid.
- (49) $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$ is tautologically valid.
- (50) $(r \Rightarrow \text{UN}(p, q)) \Rightarrow ((r \Rightarrow \neg p \&\& \neg q) \Rightarrow \neg r)$ is tautologically valid.
- (51) $(r \Rightarrow \text{UN}(p, q)) \Rightarrow ((r \Rightarrow \neg q \&\& \neg(p \mathcal{U} q)) \Rightarrow \neg r)$ is tautologically valid.

4. THE DERIVATIONS OF TEMPORAL LOGIC FORMULAS WITHIN CLASSICAL LOGIC

One can prove the following propositions:

- (52) If $X \vdash p \Rightarrow q$ and $X \vdash p \Rightarrow r$, then $X \vdash p \Rightarrow q \&\& r$.
- (53) If $X \vdash p \Rightarrow q$ and $X \vdash r \Rightarrow s$, then $X \vdash p \&\& r \Rightarrow q \&\& s$.
- (54) If $X \vdash p \Rightarrow q$ and $X \vdash p \Rightarrow r$ and $X \vdash r \Rightarrow p$, then $X \vdash r \Rightarrow q$.
- (55) If $X \vdash p \Rightarrow q \&\& \neg q$, then $X \vdash \neg p$.
- (56) If for every natural number i such that $i \in \text{dom } f$ holds
 $\emptyset_{\text{the LTLB-WFF}} \vdash p \Rightarrow f_i$, then
 $\emptyset_{\text{the LTLB-WFF}} \vdash p \Rightarrow (\text{conjunction } f)_{\text{len conjunction } f}$.
- (57) If for every natural number i such that $i \in \text{dom } f$ holds
 $\emptyset_{\text{the LTLB-WFF}} \vdash f_i \Rightarrow p$, then
 $\emptyset_{\text{the LTLB-WFF}} \vdash \neg((\text{conjunction negation } f)_{\text{len conjunction negation } f}) \Rightarrow p$.

5. THE DERIVATIONS OF TEMPORAL LOGIC FORMULAS

Next we state several propositions:

- (58) $X \vdash (\mathcal{X} p \Rightarrow \mathcal{X} q) \Rightarrow \mathcal{X}(p \Rightarrow q).$
- (59) $X \vdash \mathcal{X}(p \&\& q) \Rightarrow \mathcal{X} p \&\& \mathcal{X} q.$
- (60) $\emptyset_{\text{the LTLB-WFF}} \vdash (\text{conjunction next } f)_{\text{len conjunction next } f} \Rightarrow \mathcal{X}((\text{conjunction } f)_{\text{len conjunction } f}).$
- (61) $X \vdash \mathcal{X} p \vee \mathcal{X} q \Rightarrow \mathcal{X}(p \vee q).$
- (62) $X \vdash \mathcal{X}(p \vee q) \Rightarrow \mathcal{X} p \vee \mathcal{X} q.$
- (63) $X \vdash \neg(A \mathcal{U} B) \Rightarrow \mathcal{X} \neg \text{UN}(A, B).$

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