FORMALIZED MATHEMATICS
Vol. 20, No. 2, Pages 97–104, 2012
DOI: 10.2478/v10037-012-0013-1

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Fundamental Group of n-sphere for $n \geq 2$

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Summary. Triviality of fundamental groups of spheres of dimension greater than 1 is proven, [17].

MML identifier: `TOPALG_6`, version: `7.12.02 4.176.1140`

The notation and terminology used in this paper have been introduced in the following papers: [4], [11], [12], [19], [9], [3], [5], [6], [21], [22], [1], [2], [7], [18], [20], [24], [25], [23], [16], [13], [14], [10], [15], and [8].

1. Preliminaries

In this paper T, U are non empty topological spaces, t is a point of T, and n is a natural number.

Let S be a topological space and let T be a non empty topological space. Note that every function from S into T which is constant is also continuous.

The following two propositions are true:

(1) $\mathrm{L}_{01}(0, 1, 0, 1) = \mathrm{id}_{[0,1]_{\mathrm{T}}}$.

(2) For all real numbers r_1, r_2, r_3, r_4, r_5, r_6 such that $r_1 < r_2$ and $r_3 \leq r_4$ and $r_5 < r_6$ holds $\mathrm{L}_{01}(r_1, r_2, r_3, r_4) \cdot \mathrm{L}_{01}(r_5, r_6, r_1, r_2) = \mathrm{L}_{01}(r_5, r_6, r_3, r_4)$.

Let n be a positive natural number. Observe that $\mathcal{E}^n_{\mathrm{T}}$ is infinite and every non empty topological space which is n-locally Euclidean is also infinite.

The following propositions are true:

[1] This work has been supported by the Polish Ministry of Science and Higher Education project "Managing a Large Repository of Computer-verified Mathematical Knowledge" (N N519 385136).

ISSN 1426–2630(p), 1898-9934(e)

(3) For every point p of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1)$ holds $-p \in \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1) \setminus \{p\}$.

(4) Let T be a non empty topological structure, t_1, t_2 be points of T, and p be a path from t_1 to t_2. Then $\inf \mathrm{dom}\, p = 0$ and $\sup \mathrm{dom}\, p = 1$.

(5) For all constant loops C_1, C_2 of t holds C_1, C_2 are homotopic.

(6) Let S be a non empty subspace of T, t_1, t_2 be points of T, s_1, s_2 be points of S, A, B be paths from t_1 to t_2, and C, D be paths from s_1 to s_2. Suppose s_1, s_2 are connected and t_1, t_2 are connected and $A = C$ and $B = D$ and C, D are homotopic. Then A, B are homotopic.

(7) Let S be a non empty subspace of T, t_1, t_2 be points of T, s_1, s_2 be points of S, A, B be paths from t_1 to t_2, and C, D be paths from s_1 to s_2. Suppose s_1, s_2 are connected and t_1, t_2 are connected and $A = C$ and $B = D$ and $[C]_{\mathrm{EqRel}(S,s_1,s_2)} = [D]_{\mathrm{EqRel}(S,s_1,s_2)}$. Then $[A]_{\mathrm{EqRel}(T,t_1,t_2)} = [B]_{\mathrm{EqRel}(T,t_1,t_2)}$.

(8) Let T be a trivial non empty topological space, t be a point of T, and L be a loop of t. Then the carrier of $\pi_1(T,t) = \{[L]_{\mathrm{EqRel}(T,t)}\}$.

(9) For every point p of $\mathcal{E}^n_{\mathrm{T}}$ and for every subset S of $\mathcal{E}^n_{\mathrm{T}}$ such that $n \geq 2$ and $S = \Omega_{\mathcal{E}^n_{\mathrm{T}}} \setminus \{p\}$ holds $\mathcal{E}^n_{\mathrm{T}} {\restriction} S$ is pathwise connected.

(10) Let S be a non empty subset of T. Suppose $n \geq 2$ and $S = \Omega_T \setminus \{t\}$ and $\mathcal{E}^n_{\mathrm{T}}$ and T are homeomorphic. Then $T {\restriction} S$ is pathwise connected.

Let n be an element of $\mathbb{N}$ and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. Observe that $\mathrm{TPlane}(p,q)$ is convex.

2. Fundamental Groups

Let us consider T. We say that T has trivial fundamental group if and only if:

(Def. 1) For every point t of T holds $\pi_1(T,t)$ is trivial.

Let us consider T. We say that T is simply connected if and only if:

(Def. 2) T is pathwise connected and has trivial fundamental group.

One can verify that every non empty topological space which is simply connected is also pathwise connected and has trivial fundamental group and every non empty topological space which is pathwise connected and has trivial fundamental group is also simply connected.

The following proposition is true

(11) If T has trivial fundamental group, then for every point t of T and for all loops P, Q of t holds P, Q are homotopic.

Let n be a natural number. Note that $\mathcal{E}^n_{\mathrm{T}}$ has trivial fundamental group.

Let us note that every non empty topological space which is trivial also has trivial fundamental group.

The following proposition is true

(12) T is simply connected if and only if for all points t_1, t_2 of T holds t_1, t_2 are connected and for all paths P, Q from t_1 to t_2 holds $[P]_{\mathrm{EqRel}(T,t_1,t_2)} = [Q]_{\mathrm{EqRel}(T,t_1,t_2)}$.

Let T be a non empty topological space with trivial fundamental group and let t be a point of T. One can check that $\pi_1(T,t)$ is trivial.

Next we state three propositions:

(13) Let S, T be non empty topological spaces. Suppose S and T are homeomorphic. If S has trivial fundamental group, then T has trivial fundamental group.

(14) Let S, T be non empty topological spaces. Suppose S and T are homeomorphic. If S is simply connected, then T is simply connected.

(15) Let T be a non empty topological space with trivial fundamental group, t be a point of T, and P_1, P_2 be loops of t. Then P_1, P_2 are homotopic.

Let us consider T, t and let l be a loop of t. We say that l is null-homotopic if and only if:

(Def. 3) There exists a constant loop c of t such that l, c are homotopic.

Let us consider T, t. Observe that every loop of t which is constant is also null-homotopic.

Let us consider T, t. Note that there exists a loop of t which is constant.

The following proposition is true

(16) Let f be a loop of t and g be a continuous function from T into U. If f is null-homotopic, then $g \cdot f$ is null-homotopic.

Let T, U be non empty topological spaces, let t be a point of T, let f be a null-homotopic loop of t, and let g be a continuous function from T into U. Note that $g \cdot f$ is null-homotopic.

Let T be a non empty topological space with trivial fundamental group and let t be a point of T. Note that every loop of t is null-homotopic.

One can prove the following proposition

(17) If for every point t of T holds every loop of t is null-homotopic, then T has trivial fundamental group.

Let n be an element of $\mathbb{N}$ and let p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. Note that $\mathrm{TPlane}(p,q)$ has trivial fundamental group.

We now state the proposition

(18) Let S be a non empty subspace of T, s be a point of S, f be a loop of t, and g be a loop of s. If $t = s$ and $f = g$ and g is null-homotopic, then f is null-homotopic.

3. Curves

In the sequel T is a topological structure and f is a partial function from $\mathbb{R}^{\mathbf{1}}$ to T.

Let us consider T, f. We say that f is parametrized curve if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) dom f is an interval subset of $\mathbb{R}$, and

(ii) there exists a subspace S of $\mathbb{R}^{\mathbf{1}}$ and there exists a function g from S into T such that $f = g$ and $S = \mathbb{R}^{\mathbf{1}} \restriction \operatorname{dom} f$ and g is continuous.

Let us consider T. Observe that there exists a partial function from $\mathbb{R}^{\mathbf{1}}$ to T which is parametrized curve.

One can prove the following proposition

(19) $\emptyset$ is a parametrized curve partial function from $\mathbb{R}^{\mathbf{1}}$ to T.

Let us consider T. The functor T-Curves yields a subset of $\mathbb{R} \dot{\rightarrow} \Omega_T$ and is defined as follows:

(Def. 5) T-Curves $= \{f \in \mathbb{R} \dot{\rightarrow} \Omega_T : f$ is a parametrized curve partial function from $\mathbb{R}^{\mathbf{1}}$ to $T\}$.

Let us consider T. One can check that T-Curves is non empty.

Let us consider T. A curve of T is an element of T-Curves.

In the sequel c is a curve of T.

We now state several propositions:

(20) Every parametrized curve partial function from $\mathbb{R}^{\mathbf{1}}$ to T is a curve of T.

(21) $\emptyset$ is a curve of T.

(22) Let t_1, t_2 be points of T and p be a path from t_1 to t_2. If t_1, t_2 are connected, then p is a curve of T.

(23) c is a parametrized curve partial function from $\mathbb{R}^{\mathbf{1}}$ to T.

(24) $\operatorname{dom} c \subseteq \mathbb{R}$ and $\operatorname{rng} c \subseteq \Omega_T$.

Let us consider T, c. One can verify that dom c is real-membered.

Let us consider T, c. We say that c has first point if and only if:

(Def. 6) dom c is left-ended.

We say that c has last point if and only if:

(Def. 7) dom c is right-ended.

Let us consider T, c. We say that c has endpoints if and only if:

(Def. 8) c has first point and last point.

Let us consider T. One can check that every curve of T which has first point and last point also has endpoints and every curve of T which has endpoints also has first point and last point.

In the sequel T denotes a non empty topological structure.

Let us consider T. Note that there exists a curve of T which has endpoints.

Let us consider T and let c be a curve of T with first point. Note that $\operatorname{dom} c$ is non empty and $\inf \operatorname{dom} c$ is real.

Let us consider T and let c be a curve of T with last point. Note that $\operatorname{dom} c$ is non empty and $\sup \operatorname{dom} c$ is real.

Let us consider T. Observe that every curve of T which has first point is also non empty and every curve of T which has last point is also non empty.

Let us consider T and let c be a curve of T with first point. The first point of c yielding a point of T is defined by:

(Def. 9) The first point of $c = c(\inf \operatorname{dom} c)$.

Let us consider T and let c be a curve of T with last point. The last point of c yielding a point of T is defined by:

(Def. 10) The last point of $c = c(\sup \operatorname{dom} c)$.

The following propositions are true:

(25) Let t_1, t_2 be points of T and p be a path from t_1 to t_2. If t_1, t_2 are connected, then p is a curve of T with endpoints.

(26) For every curve c of T and for all real numbers r_1, r_2 holds $c{\restriction}[r_1, r_2]$ is a curve of T.

(27) For every curve c of T with endpoints holds $\operatorname{dom} c = [\inf \operatorname{dom} c, \sup \operatorname{dom} c]$.

(28) Let c be a curve of T with endpoints. Suppose $\operatorname{dom} c = [0, 1]$. Then c is a path from the first point of c to the last point of c.

(29) Let c be a curve of T with endpoints. Then $c \cdot \mathrm{L}_{01}(0, 1, \inf \operatorname{dom} c, \sup \operatorname{dom} c)$ is a path from the first point of c to the last point of c.

(30) Let c be a curve of T with endpoints and t_1, t_2 be points of T. Suppose $c \cdot \mathrm{L}_{01}(0, 1, \inf \operatorname{dom} c, \sup \operatorname{dom} c)$ is a path from t_1 to t_2 and t_1, t_2 are connected. Then $t_1 =$ the first point of c and $t_2 =$ the last point of c.

(31) For every curve c of T with endpoints holds the first point of $c \in \operatorname{rng} c$ and the last point of $c \in \operatorname{rng} c$.

(32) Let r_1, r_2 be real numbers, t_1, t_2 be points of T, and p_1 be a path from t_1 to t_2. Suppose t_1, t_2 are connected and $r_1 < r_2$. Then $p_1 \cdot \mathrm{L}_{01}(r_1, r_2, 0, 1)$ is a curve of T with endpoints.

(33) For every curve c of T with endpoints holds the first point of c, the last point of c are connected.

Let T be a non empty topological structure and let c_1, c_2 be curves of T with endpoints. We say that c_1, c_2 are homotopic if and only if the condition (Def. 11) is satisfied.

(Def. 11) There exist points a, b of T and there exist paths p_1, p_2 from a to b such that $p_1 = c_1 \cdot \mathrm{L}_{01}(0, 1, \inf \operatorname{dom} c_1, \sup \operatorname{dom} c_1)$ and $p_2 = c_2 \cdot \mathrm{L}_{01}(0, 1, \inf \operatorname{dom} c_2, \sup \operatorname{dom} c_2)$ and p_1, p_2 are homotopic.

Let us note that the predicate c_1, c_2 are homotopic is symmetric.

Let T be a non empty topological space and let c_1, c_2 be curves of T with endpoints. Let us notice that the predicate c_1, c_2 are homotopic is reflexive and symmetric.

The following three propositions are true:

(34) Let T be a non empty topological structure, c_1, c_2 be curves of T with endpoints, a, b be points of T, and p_1, p_2 be paths from a to b. Suppose $c_1 = p_1$ and $c_2 = p_2$ and a, b are connected. Then c_1, c_2 are homotopic if and only if p_1, p_2 are homotopic.

(35) Let c_1, c_2 be curves of T with endpoints. Suppose c_1, c_2 are homotopic. Then the first point of $c_1 =$ the first point of c_2 and the last point of $c_1 =$ the last point of c_2.

(36) Let T be a non empty topological space, c_1, c_2 be curves of T with endpoints, and S be a subset of $\mathbb{R}^{\mathbf{1}}$. Suppose $\operatorname{dom} c_1 = \operatorname{dom} c_2$ and $S = \operatorname{dom} c_1$. Then c_1, c_2 are homotopic if and only if there exists a function f from $(\mathbb{R}^{\mathbf{1}} \restriction S) \times \mathbb{I}$ into T and there exist points a, b of T such that f is continuous and for every point t of $\mathbb{R}^{\mathbf{1}} \restriction S$ holds $f(t, 0) = c_1(t)$ and $f(t, 1) = c_2(t)$ and for every point t of $\mathbb{I}$ holds $f(\inf S, t) = a$ and $f(\sup S, t) = b$.

Let T be a topological structure and let c_1, c_2 be curves of T. The functor $c_1 + c_2$ yielding a curve of T is defined as follows:

(Def. 12) $$c_1 + c_2 = \begin{cases} c_1 \cup c_2, \text{ if } c_1 \cup c_2 \text{ is a curve of } T, \\ \emptyset, \text{ otherwise.} \end{cases}$$

One can prove the following three propositions:

(37) Let c be a curve of T with endpoints and r be a real number. Then there exist elements c_1, c_2 of T-Curves such that $c = c_1 + c_2$ and $c_1 = c \restriction [\inf \operatorname{dom} c, r]$ and $c_2 = c \restriction [r, \sup \operatorname{dom} c]$.

(38) Let T be a non empty topological space and c_1, c_2 be curves of T with endpoints. Suppose $\sup \operatorname{dom} c_1 = \inf \operatorname{dom} c_2$ and the last point of $c_1 =$ the first point of c_2. Then $c_1 + c_2$ has endpoints and $\operatorname{dom}(c_1 + c_2) = [\inf \operatorname{dom} c_1, \sup \operatorname{dom} c_2]$ and $(c_1 + c_2)(\inf \operatorname{dom} c_1) =$ the first point of c_1 and $(c_1 + c_2)(\sup \operatorname{dom} c_2) =$ the last point of c_2.

(39) Let T be a non empty topological space and c_1, c_2, c_3, c_4, c_5, c_6 be curves of T with endpoints. Suppose that c_1, c_2 are homotopic and $\operatorname{dom} c_1 = \operatorname{dom} c_2$ and c_3, c_4 are homotopic and $\operatorname{dom} c_3 = \operatorname{dom} c_4$ and $c_5 = c_1 + c_3$ and $c_6 = c_2 + c_4$ and the last point of $c_1 =$ the first point of c_3 and $\sup \operatorname{dom} c_1 = \inf \operatorname{dom} c_3$. Then c_5, c_6 are homotopic.

Let T be a topological structure and let f be a finite sequence of elements of T-Curves. The functor $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ yielding a finite sequence of elements of T-Curves is defined as follows:

(Def. 13) $\operatorname{len} f = \operatorname{len}((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}})$ and $f(1) = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(1)$ and for every natural number i such that $1 \leq i < \operatorname{len} f$ holds $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(i +$

$1) = ((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}})_i + f_{i+1}$.

Let T be a topological structure and let f be a finite sequence of elements of T-Curves. The functor $\sum f$ yields a curve of T and is defined as follows:

(Def. 14) $$\sum f = \begin{cases} (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(\operatorname{len} f), & \text{if } \operatorname{len} f > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state several propositions:

(40) For every curve c of T holds $\sum \langle c \rangle = c$.

(41) For every curve c of T and for every finite sequence f of elements of T-Curves holds $\sum (f \frown \langle c \rangle) = \sum f + c$.

(42) Let X be a set and f be a finite sequence of elements of T-Curves. Suppose that for every natural number i such that $1 \leq i \leq \operatorname{len} f$ holds $\operatorname{rng}(f_i) \subseteq X$. Then $\operatorname{rng} \sum f \subseteq X$.

(43) Let T be a non empty topological space and f be a finite sequence of elements of T-Curves. Suppose that

(i) $\operatorname{len} f > 0$,

(ii) for every natural number i such that $1 \leq i < \operatorname{len} f$ holds $f_i(\sup \operatorname{dom}(f_i)) = f_{i+1}(\inf \operatorname{dom}(f_{i+1}))$ and $\sup \operatorname{dom}(f_i) = \inf \operatorname{dom}(f_{i+1})$, and

(iii) for every natural number i such that $1 \leq i \leq \operatorname{len} f$ holds f_i has endpoints.

Then there exists a curve c of T with endpoints such that $\sum f = c$ and $\operatorname{dom} c = [\inf \operatorname{dom}(f_1), \sup \operatorname{dom}(f_{\operatorname{len} f})]$ and the first point of $c = f_1(\inf \operatorname{dom}(f_1))$ and the last point of $c = f_{\operatorname{len} f}(\sup \operatorname{dom}(f_{\operatorname{len} f}))$.

(44) Let T be a non empty topological space, f_1, f_2 be finite sequences of elements of T-Curves, and c_1, c_2 be curves of T with endpoints. Suppose that $\operatorname{len} f_1 > 0$ and $\operatorname{len} f_1 = \operatorname{len} f_2$ and $\sum f_1 = c_1$ and $\sum f_2 = c_2$ and for every natural number i such that $1 \leq i < \operatorname{len} f_1$ holds $(f_1)_i(\sup \operatorname{dom}((f_1)_i)) = (f_1)_{i+1}(\inf \operatorname{dom}((f_1)_{i+1}))$ and $\sup \operatorname{dom}((f_1)_i) = \inf \operatorname{dom}((f_1)_{i+1})$ and for every natural number i such that $1 \leq i < \operatorname{len} f_2$ holds $(f_2)_i(\sup \operatorname{dom}((f_2)_i)) = (f_2)_{i+1}(\inf \operatorname{dom}((f_2)_{i+1}))$ and $\sup \operatorname{dom}((f_2)_i) = \inf \operatorname{dom}((f_2)_{i+1})$ and for every natural number i such that $1 \leq i \leq \operatorname{len} f_1$ there exist curves c_3, c_4 of T with endpoints such that $c_3 = (f_1)_i$ and $c_4 = (f_2)_i$ and c_3, c_4 are homotopic and $\operatorname{dom} c_3 = \operatorname{dom} c_4$. Then c_1, c_2 are homotopic.

(45) Let c be a curve of T with endpoints and h be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{len} h \geq 2$ and $h(1) = \inf \operatorname{dom} c$ and $h(\operatorname{len} h) = \sup \operatorname{dom} c$ and h is increasing. Then there exists a finite sequence f of elements of T-Curves such that $\operatorname{len} f = \operatorname{len} h - 1$ and $c = \sum f$ and for every natural number i such that $1 \leq i \leq \operatorname{len} f$ holds $f_i = c \restriction [h_i, h_{i+1}]$.

(46) If $n \geq 2$, then $\mathbb{S}^n$ has trivial fundamental group.

(47) Let n be a non empty natural number, r be a positive real number, and x be a point of $\mathcal{E}^n_{\mathrm{T}}$. If $n \geq 3$, then $\mathrm{Tcircle}(x, r)$ has trivial fundamental group.

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Received September 20, 2011