

# Representation Theorem for Stacks

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**Summary.** In the paper the concept of stacks is formalized. As the main result the Theorem of Representation for Stacks is given. Formalization is done according to [13].

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The papers [6], [15], [14], [2], [4], [7], [16], [8], [9], [10], [5], [1], [17], [11], [19], [21], [20], [3], [18], and [12] provide the terminology and notation for this paper.

## 1. INTRODUCTIONS

In this paper  $i$  is a natural number and  $x$  is a set.

Let  $A$  be a set and let  $s_1, s_2$  be finite sequences of elements of  $A$ . Then  $s_1 \frown s_2$  is an element of  $A^*$ .

Let  $A$  be a set, let  $i$  be a natural number, and let  $s$  be a finite sequence of elements of  $A$ . Then  $s_{\upharpoonright i}$  is an element of  $A^*$ .

The following two propositions are true:

- (1)  $\emptyset_{\upharpoonright i} = \emptyset$ .
- (2) Let  $D$  be a non empty set and  $s$  be a finite sequence of elements of  $D$ . Suppose  $s \neq \emptyset$ . Then there exists a finite sequence  $w$  of elements of  $D$  and there exists an element  $n$  of  $D$  such that  $s = \langle n \rangle \frown w$ .

The scheme *IndSeqD* deals with a non empty set  $\mathcal{A}$  and a unary predicate  $\mathcal{P}$ , and states that:

For every finite sequence  $p$  of elements of  $\mathcal{A}$  holds  $\mathcal{P}[p]$  provided the following conditions are met:

- $\mathcal{P}[\varepsilon_{\mathcal{A}}]$ , and

- For every finite sequence  $p$  of elements of  $\mathcal{A}$  and for every element  $x$  of  $\mathcal{A}$  such that  $\mathcal{P}[p]$  holds  $\mathcal{P}[\langle x \rangle \cap p]$ .

Let  $C, D$  be non empty sets and let  $R$  be a binary relation. A function from  $C \times D$  into  $D$  is said to be a binary operation of  $C$  and  $D$  being congruence w.r.t.  $R$  if:

- (Def. 1) For every element  $x$  of  $C$  and for all elements  $y_1, y_2$  of  $D$  such that  $\langle y_1, y_2 \rangle \in R$  holds  $\langle \text{it}(x, y_1), \text{it}(x, y_2) \rangle \in R$ .

The scheme *LambdaD2* deals with non empty sets  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and a binary functor  $\mathcal{F}$  yielding an element of  $\mathcal{C}$ , and states that:

There exists a function  $M$  from  $\mathcal{A} \times \mathcal{B}$  into  $\mathcal{C}$  such that for every element  $i$  of  $\mathcal{A}$  and for every element  $j$  of  $\mathcal{B}$  holds  $M(i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

Let  $C, D$  be non empty sets, let  $R$  be an equivalence relation of  $D$ , and let  $b$  be a function from  $C \times D$  into  $D$ . Let us assume that  $b$  is a binary operation of  $C$  and  $D$  being congruence w.r.t.  $R$ . The functor  $b/R$  yielding a function from  $C \times \text{Classes } R$  into  $\text{Classes } R$  is defined as follows:

- (Def. 2) For all sets  $x, y, y_1$  such that  $x \in C$  and  $y \in \text{Classes } R$  and  $y_1 \in y$  holds  $b/R(x, y) = [b(x, y_1)]_R$ .

Let  $A, B$  be non empty sets, let  $C$  be a subset of  $A$ , let  $D$  be a subset of  $B$ , let  $f$  be a function from  $A$  into  $B$ , and let  $g$  be a function from  $C$  into  $D$ . Then  $f + \cdot g$  is a function from  $A$  into  $B$ .

## 2. STACK ALGEBRA

We introduce stack systems which are extensions of 2-sorted and are systems  $\langle \text{a carrier, a carrier}', \text{empty stacks, a push function, a pop function, a top function} \rangle$ ,

where the carrier is a set, the carrier' is a set, the empty stacks constitute a subset of the carrier', the push function is a function from the carrier  $\times$  the carrier' into the carrier', the pop function is a function from the carrier' into the carrier', and the top function is a function from the carrier' into the carrier.

Let  $a_1$  be a non empty set, let  $a_2$  be a set, let  $a_3$  be a subset of  $a_2$ , let  $a_4$  be a function from  $a_1 \times a_2$  into  $a_2$ , let  $a_5$  be a function from  $a_2$  into  $a_2$ , and let  $a_6$  be a function from  $a_2$  into  $a_1$ . Observe that stack system  $\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$  is non empty.

Let  $a_1$  be a set, let  $a_2$  be a non empty set, let  $a_3$  be a subset of  $a_2$ , let  $a_4$  be a function from  $a_1 \times a_2$  into  $a_2$ , let  $a_5$  be a function from  $a_2$  into  $a_2$ , and let  $a_6$  be a function from  $a_2$  into  $a_1$ . One can verify that stack system  $\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$  is non void.

Let us note that there exists a stack system which is non empty, non void, and strict.

Let  $X$  be a stack system. A stack of  $X$  is an element of the carrier' of  $X$ .

Let  $X$  be a non empty non void stack system and let  $s$  be a stack of  $X$ . The predicate  $\text{empty}(s)$  is defined by:

(Def. 3)  $s \in$  the empty stacks of  $X$ .

The functor  $\text{pop } s$  yields a stack of  $X$  and is defined by:

(Def. 4)  $\text{pop } s = (\text{the pop function of } X)(s)$ .

The functor  $\text{top } s$  yields an element of  $X$  and is defined by:

(Def. 5)  $\text{top } s = (\text{the top function of } X)(s)$ .

Let  $e$  be an element of  $X$ . The functor  $\text{push}(e, s)$  yields a stack of  $X$  and is defined by:

(Def. 6)  $\text{push}(e, s) = (\text{the push function of } X)(e, s)$ .

Let  $A$  be a non empty set. Standard stack system over  $A$  yielding a non empty non void strict stack system is defined by the conditions (Def. 7).

- (Def. 7)(i) The carrier of standard stack system over  $A = A$ ,  
 (ii) the carrier' of standard stack system over  $A = A^*$ , and  
 (iii) for every stack  $s$  of standard stack system over  $A$  holds  $\text{empty}(s)$  iff  $s$  is empty and for every finite sequence  $g$  such that  $g = s$  holds if not  $\text{empty}(s)$ , then  $\text{top } s = g(1)$  and  $\text{pop } s = g_{\uparrow 1}$  and if  $\text{empty}(s)$ , then  $\text{top } s =$  the element of standard stack system over  $A$  and  $\text{pop } s = \emptyset$  and for every element  $e$  of standard stack system over  $A$  holds  $\text{push}(e, s) = \langle e \rangle \frown g$ .

In the sequel  $A$  denotes a non empty set,  $c$  denotes an element of standard stack system over  $A$ , and  $m$  denotes a stack of standard stack system over  $A$ .

Let us consider  $A$ . Note that every stack of standard stack system over  $A$  is relation-like and function-like.

Let us consider  $A$ . Observe that every stack of standard stack system over  $A$  is finite sequence-like.

We adopt the following convention:  $X$  denotes a non empty non void stack system,  $s, s_1$  denote stacks of  $X$ , and  $e, e_1, e_2$  denote elements of  $X$ .

Let us consider  $X$ . We say that  $X$  is pop-finite if and only if the condition (Def. 8) is satisfied.

- (Def. 8) Let  $f$  be a function from  $\mathbb{N}$  into the carrier' of  $X$ . Then there exists a natural number  $i$  and there exists  $s$  such that  $f(i) = s$  and if not  $\text{empty}(s)$ , then  $f(i + 1) \neq \text{pop } s$ .

We say that  $X$  is push-pop if and only if:

- (Def. 9) If not  $\text{empty}(s)$ , then  $s = \text{push}(\text{top } s, \text{pop } s)$ .

We say that  $X$  is top-push if and only if:

- (Def. 10)  $e = \text{top push}(e, s)$ .

We say that  $X$  is pop-push if and only if:

- (Def. 11)  $s = \text{pop push}(e, s)$ .

We say that  $X$  is push-non-empty if and only if:

(Def. 12) not empty(push( $e, s$ )).

Let  $A$  be a non empty set. One can verify the following observations:

- \* standard stack system over  $A$  is pop-finite,
- \* standard stack system over  $A$  is push-pop,
- \* standard stack system over  $A$  is top-push,
- \* standard stack system over  $A$  is pop-push, and
- \* standard stack system over  $A$  is push-non-empty.

Let us observe that there exists a non empty non void stack system which is pop-finite, push-pop, top-push, pop-push, push-non-empty, and strict.

A stack algebra is a pop-finite push-pop top-push pop-push push-non-empty non empty non void stack system.

Next we state the proposition

- (3) For every non empty non void stack system  $X$  such that  $X$  is pop-finite there exists a stack  $s$  of  $X$  such that empty( $s$ ).

Let  $X$  be a pop-finite non empty non void stack system. Note that the empty stacks of  $X$  is non empty.

We now state two propositions:

- (4) If  $X$  is top-push and pop-push and push( $e_1, s_1$ ) = push( $e_2, s_2$ ), then  $e_1 = e_2$  and  $s_1 = s_2$ .
- (5) If  $X$  is push-pop and not empty( $s_1$ ) and not empty( $s_2$ ) and pop  $s_1$  = pop  $s_2$  and top  $s_1$  = top  $s_2$ , then  $s_1 = s_2$ .

### 3. SCHEMES OF INDUCTION

Now we present three schemes. The scheme *INDsch* deals with a stack algebra  $\mathcal{A}$ , a stack  $\mathcal{B}$  of  $\mathcal{A}$ , and a unary predicate  $\mathcal{P}$ , and states that:

$$\mathcal{P}[\mathcal{B}]$$

provided the following conditions are satisfied:

- For every stack  $s$  of  $\mathcal{A}$  such that empty( $s$ ) holds  $\mathcal{P}[s]$ , and
- For every stack  $s$  of  $\mathcal{A}$  and for every element  $e$  of  $\mathcal{A}$  such that  $\mathcal{P}[s]$  holds  $\mathcal{P}[\text{push}(e, s)]$ .

The scheme *EXsch* deals with a stack algebra  $\mathcal{A}$ , a stack  $\mathcal{B}$  of  $\mathcal{A}$ , a non empty set  $\mathcal{C}$ , an element  $\mathcal{D}$  of  $\mathcal{C}$ , and a binary functor  $\mathcal{F}$  yielding an element of  $\mathcal{C}$ , and states that:

There exists an element  $a$  of  $\mathcal{C}$  and there exists a function  $F$  from the carrier' of  $\mathcal{A}$  into  $\mathcal{C}$  such that

- (i)  $a = F(\mathcal{B})$ ,
- (ii) for every stack  $s_1$  of  $\mathcal{A}$  such that empty( $s_1$ ) holds  $F(s_1) = \mathcal{D}$ , and

- (iii) for every stack  $s_1$  of  $\mathcal{A}$  and for every element  $e$  of  $\mathcal{A}$  holds  
 $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$

for all values of the parameters.

The scheme *UNIQsch* deals with a stack algebra  $\mathcal{A}$ , a stack  $\mathcal{B}$  of  $\mathcal{A}$ , a non empty set  $\mathcal{C}$ , an element  $\mathcal{D}$  of  $\mathcal{C}$ , and a binary functor  $\mathcal{F}$  yielding an element of  $\mathcal{C}$ , and states that:

Let  $a_1, a_2$  be elements of  $\mathcal{C}$ . Suppose that

- (i) there exists a function  $F$  from the carrier' of  $\mathcal{A}$  into  $\mathcal{C}$  such that  $a_1 = F(\mathcal{B})$  and for every stack  $s_1$  of  $\mathcal{A}$  such that  $\text{empty}(s_1)$  holds  $F(s_1) = \mathcal{D}$  and for every stack  $s_1$  of  $\mathcal{A}$  and for every element  $e$  of  $\mathcal{A}$  holds  $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$ , and  
 (ii) there exists a function  $F$  from the carrier' of  $\mathcal{A}$  into  $\mathcal{C}$  such that  $a_2 = F(\mathcal{B})$  and for every stack  $s_1$  of  $\mathcal{A}$  such that  $\text{empty}(s_1)$  holds  $F(s_1) = \mathcal{D}$  and for every stack  $s_1$  of  $\mathcal{A}$  and for every element  $e$  of  $\mathcal{A}$  holds  $F(\text{push}(e, s_1)) = \mathcal{F}(e, F(s_1))$ .

Then  $a_1 = a_2$

for all values of the parameters.

#### 4. STACK CONGRUENCE

We adopt the following rules:  $X$  is a stack algebra,  $s, s_1, s_2, s_3$  are stacks of  $X$ , and  $e, e_1, e_2, e_3$  are elements of  $X$ .

Let us consider  $X, s$ . The functor  $|s|$  yielding an element of  $(\text{the carrier of } X)^*$  is defined by the condition (Def. 13).

- (Def. 13) There exists a function  $F$  from the carrier' of  $X$  into  $(\text{the carrier of } X)^*$  such that  $|s| = F(s)$  and for every  $s_1$  such that  $\text{empty}(s_1)$  holds  $F(s_1) = \emptyset$  and for all  $s_1, e$  holds  $F(\text{push}(e, s_1)) = \langle e \rangle \cap F(s_1)$ .

Next we state several propositions:

- (6) If  $\text{empty}(s)$ , then  $|s| = \emptyset$ .
- (7) If not  $\text{empty}(s)$ , then  $|s| = \langle \text{top } s \rangle \cap |\text{pop } s|$ .
- (8) If not  $\text{empty}(s)$ , then  $|\text{pop } s| = |s|_{|1}$ .
- (9)  $|\text{push}(e, s)| = \langle e \rangle \cap |s|$ .
- (10) If not  $\text{empty}(s)$ , then  $\text{top } s = |s|(1)$ .
- (11) If  $|s| = \emptyset$ , then  $\text{empty}(s)$ .
- (12) For every stack  $s$  of standard stack system over  $A$  holds  $|s| = s$ .
- (13) For every element  $x$  of  $(\text{the carrier of } X)^*$  there exists  $s$  such that  $|s| = x$ .

Let us consider  $X, s_1, s_2$ . The predicate  $s_1 =_G s_2$  is defined as follows:

- (Def. 14)  $|s_1| = |s_2|$ .

Let us notice that the predicate  $s_1 =_G s_2$  is reflexive and symmetric.

The following propositions are true:

- (14) If  $s_1 =_G s_2$  and  $s_2 =_G s_3$ , then  $s_1 =_G s_3$ .
- (15) If  $s_1 =_G s_2$  and  $\text{empty}(s_1)$ , then  $\text{empty}(s_2)$ .
- (16) If  $\text{empty}(s_1)$  and  $\text{empty}(s_2)$ , then  $s_1 =_G s_2$ .
- (17) If  $s_1 =_G s_2$ , then  $\text{push}(e, s_1) =_G \text{push}(e, s_2)$ .
- (18) If  $s_1 =_G s_2$  and  $\text{not empty}(s_1)$ , then  $\text{pop } s_1 =_G \text{pop } s_2$ .
- (19) If  $s_1 =_G s_2$  and  $\text{not empty}(s_1)$ , then  $\text{top } s_1 = \text{top } s_2$ .

Let us consider  $X$ . We say that  $X$  is proper for identity if and only if:

- (Def. 15) For all  $s_1, s_2$  such that  $s_1 =_G s_2$  holds  $s_1 = s_2$ .

Let us consider  $A$ . Observe that standard stack system over  $A$  is proper for identity.

Let us consider  $X$ . The functor  $==_X$  yields a binary relation on the carrier' of  $X$  and is defined as follows:

- (Def. 16)  $\langle s_1, s_2 \rangle \in ==_X$  iff  $s_1 =_G s_2$ .

Let us consider  $X$ . Observe that  $==_X$  is total, symmetric, and transitive.

One can prove the following proposition

- (20) If  $\text{empty}(s)$ , then  $[s]_{==_X}$  = the empty stacks of  $X$ .

Let us consider  $X, s$ . The functor  $\text{coset } s$  yielding a subset of the carrier' of  $X$  is defined by the conditions (Def. 17).

- (Def. 17)(i)  $s \in \text{coset } s$ ,
- (ii) for all  $e, s_1$  such that  $s_1 \in \text{coset } s$  holds  $\text{push}(e, s_1) \in \text{coset } s$  and if  $\text{not empty}(s_1)$ , then  $\text{pop } s_1 \in \text{coset } s$ , and
  - (iii) for every subset  $A$  of the carrier' of  $X$  such that  $s \in A$  and for all  $e, s_1$  such that  $s_1 \in A$  holds  $\text{push}(e, s_1) \in A$  and if  $\text{not empty}(s_1)$ , then  $\text{pop } s_1 \in A$  holds  $\text{coset } s \subseteq A$ .

Next we state three propositions:

- (21) If  $\text{push}(e, s) \in \text{coset } s_1$ , then  $s \in \text{coset } s_1$  and if  $\text{not empty}(s)$  and  $\text{pop } s \in \text{coset } s_1$ , then  $s \in \text{coset } s_1$ .
- (22)  $s \in \text{coset } \text{push}(e, s)$  and if  $\text{not empty}(s)$ , then  $s \in \text{coset } \text{pop } s$ .
- (23) There exists  $s_1$  such that  $\text{empty}(s_1)$  and  $s_1 \in \text{coset } s$ .

Let us consider  $A$  and let  $R$  be a binary relation on  $A$ . Note that there exists a reduction sequence w.r.t.  $R$  which is  $A$ -valued.

Let us consider  $X$ . The construction reduction  $X$  yielding a binary relation on the carrier' of  $X$  is defined as follows:

- (Def. 18)  $\langle s_1, s_2 \rangle \in \text{the construction reduction } X$  iff  $\text{not empty}(s_1)$  and  $s_2 = \text{pop } s_1$  or there exists  $e$  such that  $s_2 = \text{push}(e, s_1)$ .

Next we state the proposition

- (24) Let  $R$  be a binary relation on  $A$  and  $t$  be a reduction sequence w.r.t.  $R$ . Then  $t(1) \in A$  if and only if  $t$  is  $A$ -valued.

The scheme *PathIND* deals with a non empty set  $\mathcal{A}$ , elements  $\mathcal{B}, \mathcal{C}$  of  $\mathcal{A}$ , a binary relation  $\mathcal{D}$  on  $\mathcal{A}$ , and a unary predicate  $\mathcal{P}$ , and states that:

$$\mathcal{P}[\mathcal{C}]$$

provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{B}]$ ,
- $\mathcal{D}$  reduces  $\mathcal{B}$  to  $\mathcal{C}$ , and
- For all elements  $x, y$  of  $\mathcal{A}$  such that  $\mathcal{D}$  reduces  $\mathcal{B}$  to  $x$  and  $\langle x, y \rangle \in \mathcal{D}$  and  $\mathcal{P}[x]$  holds  $\mathcal{P}[y]$ .

One can prove the following propositions:

- (25) For every reduction sequence  $t$  w.r.t. the construction reduction  $X$  such that  $s = t(1)$  holds  $\text{rng } t \subseteq \text{coset } s$ .
- (26)  $\text{coset } s = \{s_1 : \text{the construction reduction } X \text{ reduces } s \text{ to } s_1\}$ .

Let us consider  $X, s$ . The functor  $\text{core } s$  yields a stack of  $X$  and is defined by the conditions (Def. 19).

- (Def. 19)(i)  $\text{empty}(\text{core } s)$ , and
- (ii) there exists a the carrier' of  $X$ -valued reduction sequence  $t$  w.r.t. the construction reduction  $X$  such that  $t(1) = s$  and  $t(\text{len } t) = \text{core } s$  and for every  $i$  such that  $1 \leq i < \text{len } t$  holds  $\text{not empty}(t_i)$  and  $t_{i+1} = \text{pop}(t_i)$ .

The following propositions are true:

- (27) If  $\text{empty}(s)$ , then  $\text{core } s = s$ .
- (28)  $\text{core push}(e, s) = \text{core } s$ .
- (29) If  $\text{not empty}(s)$ , then  $\text{core pop } s = \text{core } s$ .
- (30)  $\text{core } s \in \text{coset } s$ .
- (31) For every element  $x$  of  $(\text{the carrier of } X)^*$  there exists  $s_1$  such that  $|s_1| = x$  and  $s_1 \in \text{coset } s$ .
- (32) If  $s_1 \in \text{coset } s$ , then  $\text{core } s_1 = \text{core } s$ .
- (33) If  $s_1, s_2 \in \text{coset } s$  and  $|s_1| = |s_2|$ , then  $s_1 = s_2$ .
- (34) There exists  $s$  such that  $\text{coset } s_1 \cap [s_2]_{==_X} = \{s\}$ .

## 5. QUOTIENT STACK SYSTEM

Let us consider  $X$ . The functor  $X_{/==}$  yields a strict stack system and is defined by the conditions (Def. 20).

- (Def. 20)(i) The carrier of  $X_{/==}$  = the carrier of  $X$ ,
- (ii) the carrier' of  $X_{/==}$  =  $\text{Classes}_{==_X}$ ,
- (iii) the empty stacks of  $X_{/==}$  =  $\{\text{the empty stacks of } X\}$ ,
- (iv) the push function of  $X_{/==}$  =  $(\text{the push function of } X)_{/==_X}$ ,
- (v) the pop function of  $X_{/==}$  =  $((\text{the pop function of } X) + \cdot \text{id}_{\text{the empty stacks of } X})_{/==_X}$ , and

- (vi) for every choice function  $f$  of  $\text{Classes} =_X$  holds the top function of  $X_{/==} = (\text{the top function of } X) \cdot f + \cdot (\text{the empty stacks of } X, \text{ the element of the carrier of } X)$ .

Let us consider  $X$ . One can verify that  $X_{/==}$  is non empty and non void.

The following propositions are true:

- (35) For every stack  $S$  of  $X_{/==}$  there exists  $s$  such that  $S = [s]_{==_X}$ .
- (36)  $[s]_{==_X}$  is a stack of  $X_{/==}$ .
- (37) For every stack  $S$  of  $X_{/==}$  such that  $S = [s]_{==_X}$  holds  $\text{empty}(s)$  iff  $\text{empty}(S)$ .
- (38) For every stack  $S$  of  $X_{/==}$  holds  $\text{empty}(S)$  iff  $S = \text{the empty stacks of } X$ .
- (39) For every stack  $S$  of  $X_{/==}$  and for every element  $E$  of  $X_{/==}$  such that  $S = [s]_{==_X}$  and  $E = e$  holds  $\text{push}(e, s) \in \text{push}(E, S)$  and  $[\text{push}(e, s)]_{==_X} = \text{push}(E, S)$ .
- (40) For every stack  $S$  of  $X_{/==}$  such that  $S = [s]_{==_X}$  and not  $\text{empty}(s)$  holds  $\text{pop } s \in \text{pop } S$  and  $[\text{pop } s]_{==_X} = \text{pop } S$ .
- (41) For every stack  $S$  of  $X_{/==}$  such that  $S = [s]_{==_X}$  and not  $\text{empty}(s)$  holds  $\text{top } S = \text{top } s$ .

Let us consider  $X$ . One can verify the following observations:

- \*  $X_{/==}$  is pop-finite,
- \*  $X_{/==}$  is push-pop,
- \*  $X_{/==}$  is top-push,
- \*  $X_{/==}$  is pop-push, and
- \*  $X_{/==}$  is push-non-empty.

Next we state the proposition

- (42) For every stack  $S$  of  $X_{/==}$  such that  $S = [s]_{==_X}$  holds  $|S| = |s|$ .

Let us consider  $X$ . Note that  $X_{/==}$  is proper for identity.

Let us note that there exists a stack algebra which is proper for identity.

## 6. REPRESENTATION THEOREM FOR STACKS

Let  $X_1, X_2$  be stack algebras and let  $F, G$  be functions. We say that  $F$  and  $G$  form isomorphism between  $X_1$  and  $X_2$  if and only if the conditions (Def. 21) are satisfied.

- (Def. 21)  $\text{dom } F = \text{the carrier of } X_1$  and  $\text{rng } F = \text{the carrier of } X_2$  and  $F$  is one-to-one and  $\text{dom } G = \text{the carrier of } X_1$  and  $\text{rng } G = \text{the carrier of } X_2$  and  $G$  is one-to-one and for every stack  $s_1$  of  $X_1$  and for every stack  $s_2$  of  $X_2$  such that  $s_2 = G(s_1)$  holds  $\text{empty}(s_1)$  iff  $\text{empty}(s_2)$  and if not  $\text{empty}(s_1)$ , then  $\text{pop } s_2 = G(\text{pop } s_1)$  and  $\text{top } s_2 = F(\text{top } s_1)$  and for every element



$e_1$  of  $X_1$  and for every element  $e_2$  of  $X_2$  such that  $e_2 = F(e_1)$  holds  $\text{push}(e_2, s_2) = G(\text{push}(e_1, s_1))$ .

We use the following convention:  $X_1, X_2, X_3$  are stack algebras and  $F, F_1, F_2, G, G_1, G_2$  are functions.

The following propositions are true:

- (43)  $\text{id}_{\text{the carrier of } X}$  and  $\text{id}_{\text{the carrier}' \text{ of } X}$  form isomorphism between  $X$  and  $X$ .
- (44) If  $F$  and  $G$  form isomorphism between  $X_1$  and  $X_2$ , then  $F^{-1}$  and  $G^{-1}$  form isomorphism between  $X_2$  and  $X_1$ .
- (45) Suppose  $F_1$  and  $G_1$  form isomorphism between  $X_1$  and  $X_2$  and  $F_2$  and  $G_2$  form isomorphism between  $X_2$  and  $X_3$ . Then  $F_2 \cdot F_1$  and  $G_2 \cdot G_1$  form isomorphism between  $X_1$  and  $X_3$ .
- (46) Suppose  $F$  and  $G$  form isomorphism between  $X_1$  and  $X_2$ . Let  $s_1$  be a stack of  $X_1$  and  $s_2$  be a stack of  $X_2$ . If  $s_2 = G(s_1)$ , then  $|s_2| = F \cdot |s_1|$ .

Let  $X_1, X_2$  be stack algebras. We say that  $X_1$  and  $X_2$  are isomorphic if and only if:

- (Def. 22) There exist functions  $F, G$  such that  $F$  and  $G$  form isomorphism between  $X_1$  and  $X_2$ .

Let us notice that the predicate  $X_1$  and  $X_2$  are isomorphic is reflexive and symmetric.

We now state four propositions:

- (47) If  $X_1$  and  $X_2$  are isomorphic and  $X_2$  and  $X_3$  are isomorphic, then  $X_1$  and  $X_3$  are isomorphic.
- (48) If  $X_1$  and  $X_2$  are isomorphic and  $X_1$  is proper for identity, then  $X_2$  is proper for identity.
- (49) Let  $X$  be a proper for identity stack algebra. Then there exists  $G$  such that
  - (i) for every stack  $s$  of  $X$  holds  $G(s) = |s|$ , and
  - (ii)  $\text{id}_{\text{the carrier of } X}$  and  $G$  form isomorphism between  $X$  and standard stack system over the carrier of  $X$ .
- (50) Let  $X$  be a proper for identity stack algebra. Then  $X$  and standard stack system over the carrier of  $X$  are isomorphic.

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