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# First Order Languages: Further Syntax and Semantics[1]

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**Summary.** Third of a series of articles laying down the bases for classical first order model theory. Interpretation of a language in a universe set. Evaluation of a term in a universe. Truth evaluation of an atomic formula. Reassigning the value of a symbol in a given interpretation. Syntax and semantics of a non atomic formula are then defined concurrently (this point is explained in [16], 4.2.1). As a consequence, the evaluation of any w.f.f. string and the relation of logical implication are introduced. Depth of a formula. Definition of satisfaction and entailment (aka entailment or logical implication) relations, see [18] III.3.2 and III.4.1 respectively.

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The terminology and notation used in this paper have been introduced in the following papers: [7], [1], [23], [6], [8], [17], [14], [15], [22], [9], [10], [11], [2], [21], [26], [24], [5], [3], [4], [12], [27], [28], [19], [20], [25], and [13].

For simplicity, we follow the rules: $m$, $n$ denote natural numbers, $m_1$ denotes an element of $\mathbb{N}$, $A$, $B$, $X$, $Y$, $Z$, $x$, $y$ denote sets, $S$, $S_1$, $S_2$ denote languages, $s$ denotes an element of $S$, $w$, $w_1$, $w_2$ denote strings of $S$, $U$ denotes a non empty set, $f$, $g$ denote functions, and $p$, $p_2$ denote finite sequences.

Let us consider $S$. Then TheNorSymbOf $S$ is an element of $S$.

Let $U$ be a non empty set. The functor $U$-deltaInterpreter yielding a function from $U^2$ into *Boolean* is defined by:

(Def. 1) $U$-deltaInterpreter $= \chi_{(\text{the concatenation of } U)^\circ(\mathrm{id}_{U^1}), U^2}$.

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Let $X$ be a set. Then $\mathrm{id}_X$ is an equivalence relation of $X$.

Let $S$ be a language, let $U$ be a non empty set, and let $s$ be an of-atomic-formula element of $S$. Interpreter of $s$ and $U$ is defined as follows:

(Def. 2)(i) It is a function from $U^{|\mathrm{ar}\, s|}$ into *Boolean* if $s$ is relational,

(ii) it is a function from $U^{|\mathrm{ar}\, s|}$ into $U$, otherwise.

Let us consider $S$, $U$ and let $s$ be an of-atomic-formula element of $S$. We see that the interpreter of $s$ and $U$ is a function from $U^{|\mathrm{ar}\, s|}$ into $U \cup$ *Boolean* .

Let us consider $S$, $U$ and let $s$ be a termal element of $S$. One can verify that every interpreter of $s$ and $U$ is $U$-valued.

Let $S$ be a language. Note that every element of $S$ which is literal is also own.

Let us consider $S$, $U$. A function is called an interpreter of $S$ and $U$ if:

(Def. 3) For every own element $s$ of $S$ holds $\mathrm{it}(s)$ is an interpreter of $s$ and $U$.

Let us consider $S$, $U$, $f$. We say that $f$ is $(S, U)$-interpreter-like if and only if:

(Def. 4) $f$ is an interpreter of $S$ and $U$ and function yielding.

Let us consider $S$ and let $U$ be a non empty set. One can verify that every function which is $(S, U)$-interpreter-like is also function yielding.

Let us consider $S$, $U$ and let $s$ be an own element of $S$. Observe that every interpreter of $s$ and $U$ is non empty.

Let $S$ be a language and let $U$ be a non empty set. Note that there exists a function which is $(S, U)$-interpreter-like.

Let us consider $S$, $U$, let $I$ be an $(S, U)$-interpreter-like function, and let $s$ be an own element of $S$. Then $I(s)$ is an interpreter of $s$ and $U$.

Let $S$ be a language, let $U$ be a non empty set, let $I$ be an $(S, U)$-interpreter-like function, let $x$ be an own element of $S$, and let $f$ be an interpreter of $x$ and $U$. One can check that $I + \cdot (x \dot{\longmapsto} f)$ is $(S, U)$-interpreter-like.

Let us consider $f$, $x$, $y$. The functor $(x, y)\,\mathrm{ReassignIn}\, f$ yields a function and is defined by:

(Def. 5) $(x, y)\,\mathrm{ReassignIn}\, f = f + \cdot (x \dot{\longmapsto} (\emptyset \dot{\longmapsto} y))$.

Let $S$ be a language, let $U$ be a non empty set, let $I$ be an $(S, U)$-interpreter-like function, let $x$ be a literal element of $S$, and let $u$ be an element of $U$. One can verify that $(x, u)\,\mathrm{ReassignIn}\, I$ is $(S, U)$-interpreter-like.

Let $S$ be a language. One can check that $\mathrm{AllSymbolsOf}\, S$ is non empty.

Let $Y$ be a set and let $X$, $Z$ be non empty sets. Observe that every function from $X$ into $Z^Y$ is function yielding.

Let $X$, $Y$, $Z$ be non empty sets. One can verify that there exists a function from $X$ into $Z^Y$ which is function yielding.

Let $f$ be a function yielding function and let $g$ be a function. The functor $[g, f]$ yields a function and is defined by:

(Def. 6) $\operatorname{dom}[g, f] = \operatorname{dom} f$ and for every $x$ such that $x \in \operatorname{dom} f$ holds $[g, f](x) = g \cdot f(x)$.

Let $f$ be an empty function and let $g$ be a function. One can verify that $[g, f]$ is empty.

Let $f$ be a function yielding function and let $g$ be a function. The functor $[f, g]$ yielding a function is defined as follows:

(Def. 7) $\operatorname{dom}[f, g] = \operatorname{dom} f \cap \operatorname{dom} g$ and for every set $x$ such that $x \in \operatorname{dom}[f, g]$ holds $[f, g](x) = f(x)(g(x))$.

Let $f$ be a function yielding function and let $g$ be an empty function. One can check that $[f, g]$ is empty.

Let $X$ be a finite sequence-membered set. Observe that every function which is $X$-valued is also function yielding.

Let $E$, $D$ be non empty sets, let $p$ be a $D$-valued finite sequence, and let $h$ be a function from $D$ into $E$. Note that $h \cdot p$ is $\operatorname{len} p$-element.

Let $X$, $Y$ be non empty sets, let $f$ be a function from $X$ into $Y$, and let $p$ be an $X$-valued finite sequence. One can verify that $f \cdot p$ is finite sequence-like.

Let $E$, $D$ be non empty sets, let $n$ be a natural number, let $p$ be an $n$-element $D$-valued finite sequence, and let $h$ be a function from $D$ into $E$. Observe that $h \cdot p$ is $n$-element.

We now state the proposition

(1) For every 0-termal string $t_0$ of $S$ holds $t_0 = \langle S\text{-firstChar}(t_0) \rangle$.

Let us consider $S$, let $U$ be a non empty set, let $u$ be an element of $U$, and let $I$ be an $(S, U)$-interpreter-like function. The functor $(I, u)$-TermEval yields a function from $\mathbb{N}$ into $U^{\text{AllTermsOf}\, S}$ and is defined as follows:

(Def. 8) $(I, u)\text{-TermEval}(0) = \text{AllTermsOf}\, S \longmapsto u$ and for every $m_1$ holds $(I, u)\text{-TermEval}(m_1 + 1) = [I \cdot S\text{-firstChar}, [((I, u)\text{-TermEval}(m_1)$ **qua** function$), S\text{-subTerms}]]$.

Let us consider $S$, $U$, let $I$ be an $(S, U)$-interpreter-like function, and let $t$ be an element of AllTermsOf $S$. The functor $I$-TermEval $t$ yields an element of $U$ and is defined as follows:

(Def. 9) For every element $u_1$ of $U$ and for every $m_1$ such that $t \in S\text{-termsOfMaxDepth}(m_1)$ holds $I\text{-TermEval}\, t = (I, u_1)\text{-TermEval}(m_1 + 1)(t)$.

Let us consider $S$, $U$ and let $I$ be an $(S, U)$-interpreter-like function. The functor $I$-TermEval yielding a function from AllTermsOf $S$ into $U$ is defined by:

(Def. 10) For every element $t$ of AllTermsOf $S$ holds $I\text{-TermEval}(t) = I\text{-TermEval}\, t$.

Let us consider $S$, $U$ and let $I$ be an $(S, U)$-interpreter-like function. The functor $I$ === yielding a function is defined as follows:

(Def. 11) $I ==== I {+}\cdot (\text{TheEqSymbOf}\, S \dot{\longmapsto} U\text{-deltaInterpreter})$.

Let us consider $S$, $U$, let $I$ be an $(S,U)$-interpreter-like function, and let $x$ be a set. We say that $x$ is $I$-extension if and only if:

(Def. 12) $x = I ===$ .

Let us consider $S$, $U$ and let $I$ be an $(S,U)$-interpreter-like function. Note that $I ===$ is $I$-extension and every set which is $I$-extension is also function-like. Observe that there exists a function which is $I$-extension. Observe that $I ===$ is $(S,U)$-interpreter-like.

Let $f$ be an $I$-extension function, and let $s$ be an of-atomic-formula element of $S$. Then $f(s)$ is an interpreter of $s$ and $U$.

Let $p_1$ be a 0-w.f.f. string of $S$. The functor $I$-AtomicEval $p_1$ is defined as follows:

(Def. 13) $I\text{-AtomicEval}\,p_1 = (I === (S\text{-firstChar}(p_1)))(I\text{-TermEval} \cdot \text{SubTerms}\,p_1)$.

Let us consider $S, U$, let $I$ be an $(S,U)$-interpreter-like function, and let $p_1$ be a 0-w.f.f. string of $S$. Then $I$-AtomicEval $p_1$ is an element of *Boolean*. Note that $I{\restriction}\,\text{OwnSymbolsOf}\,S$ is $(U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))$-valued and $I{\restriction}\,\text{OwnSymbolsOf}\,S$ is $(S,U)$-interpreter-like.

Let us consider $S, U$ and let $I$ be an $(S,U)$-interpreter-like function. Observe that $I{\restriction}\,\text{OwnSymbolsOf}\,S$ is total.

Let us consider $S$, $U$. The functor $U$-InterpretersOf $S$ is defined by:

(Def. 14) $U\text{-InterpretersOf}\,S = \{f \in (U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))^{\text{OwnSymbolsOf}\,S}\colon f$ is $(S,U)$-interpreter-like$\}$.

Let us consider $S$, $U$. Then $U$-InterpretersOf $S$ is a subset of $(U^* \dot{\rightarrow} (U \cup \mathit{Boolean}))^{\text{OwnSymbolsOf}\,S}$. Observe that $U$-InterpretersOf $S$ is non empty. One can verify that every element of $U$-InterpretersOf $S$ is $(S,U)$-interpreter-like. The functor $S$-TruthEval $U$ yields a function from

$(U\text{-InterpretersOf}\,S) \times \text{AtomicFormulasOf}\,S$ into *Boolean* and is defined by:

(Def. 15) For every element $I$ of $U$-InterpretersOf $S$ and for every element $p_1$ of AtomicFormulasOf $S$ holds $(S\text{-TruthEval}\,U)(I, p_1) = I\text{-AtomicEval}\,p_1$.

Let us consider $S$, $U$, let $I$ be an element of $U$-InterpretersOf $S$, let $f$ be a partial function from $(U\text{-InterpretersOf}\,S) \times ((\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\})$ to *Boolean*, and let $p_1$ be an element of $(\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}$. The functor $f$-ExFunctor$(I,p_1)$ yielding an element of *Boolean* is defined as follows:

(Def. 16) $$f\text{-ExFunctor}(I,p_1) = \begin{cases} \mathit{true}, & \text{if there exists an element } u \text{ of } U \text{ and there exists a literal element } v \text{ of } S \text{ such that } p_1(1) = v \text{ and } f((v,u)\,\text{ReassignIn}\,I,\ (p_1)_{\lfloor 1}) = \mathit{true}, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Let us consider $S$, $U$ and let $g$ be an element of $(U\text{-InterpretersOf}\,S) \times ((\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$. The functor ExIterator $g$ yields a partial function from $(U\text{-InterpretersOf}\,S) \times ((\text{AllSymbolsOf}\,S)^* \setminus \{\emptyset\})$ to *Boolean* and is defined by the conditions (Def. 17).

(Def. 17)(i) For every element $x$ of $U$-InterpretersOf $S$ and for every element $y$ of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ holds $\langle x, y\rangle \in \text{dom ExIterator } g$ iff there exists a literal element $v$ of $S$ and there exists a string $w$ of $S$ such that $\langle x, w\rangle \in \text{dom } g$ and $y = \langle v\rangle \frown w$, and

(ii) for every element $x$ of $U$-InterpretersOf $S$ and for every element $y$ of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $\langle x, y\rangle \in \text{dom ExIterator } g$ holds $(\text{ExIterator } g)(x, y) = g\text{-ExFunctor}(x, y)$.

Let us consider $S$, $U$, let $f$ be a partial function from $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$ to *Boolean*, let $I$ be an element of $U$-InterpretersOf $S$, and let $p_1$ be an element of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$.

The functor $f$-NorFunctor$(I, p_1)$ yielding an element of *Boolean* is defined by:

(Def. 18)
$$f\text{-NorFunctor}(I, p_1) = \begin{cases} \mathit{true}, & \text{if there exist elements } w_1,\ w_2 \text{ of } (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} \text{ such that } \langle I, w_1\rangle \in \text{dom } f \text{ and } f(I, w_1) = \mathit{false} \text{ and } f(I, w_2) = \mathit{false} \text{ and } p_1 = \langle \text{TheNorSymbOf } S\rangle \frown w_1 \frown w_2, \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Let us consider $S$, $U$ and let $g$ be an element of $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$. The functor NorIterator $g$ yielding a partial function from $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$ to *Boolean* is defined by the conditions (Def. 19).

(Def. 19)(i) For every element $x$ of $U$-InterpretersOf $S$ and for every element $y$ of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ holds $\langle x, y\rangle \in \text{dom NorIterator } g$ iff there exist elements $p_3$, $p_4$ of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $y = \langle \text{TheNorSymbOf } S\rangle \frown p_3 \frown p_4$ and $\langle x, p_3\rangle$, $\langle x, p_4\rangle \in \text{dom } g$, and

(ii) for every element $x$ of $U$-InterpretersOf $S$ and for every element $y$ of $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ such that $\langle x, y\rangle \in \text{dom NorIterator } g$ holds $(\text{NorIterator } g)(x, y) = g\text{-NorFunctor}(x, y)$.

Let us consider $S$, $U$. The functor $(S, U)$-TruthEval yields a function from $\mathbb{N}$ into $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$ and is defined as follows:

(Def. 20) $(S, U)\text{-TruthEval}(0) = S\text{-TruthEval } U$ and for every $m_1$ holds $(S, U)\text{-TruthEval}(m_1+1) = \text{ExIterator}(S, U)\text{-TruthEval}(m_1) {+\cdot}\ \text{NorIterator}\ (S, U)\text{-TruthEval}(m_1) {+\cdot} (S, U)\text{-TruthEval}(m_1)$.

Next we state the proposition

(2) For every $(S, U)$-interpreter-like function $I$ holds $I \restriction \text{OwnSymbolsOf } S \in U\text{-InterpretersOf } S$.

Let $S$ be a language, let $m$ be a natural number, and let $U$ be a non empty set.

The functor $(S,U)$-TruthEval $m$ yielding an element of ($U$-InterpretersOf $S$) $\times$ ((AllSymbolsOf $S)^* \setminus \{\emptyset\}) \dot{\rightarrow}$ *Boolean* is defined as follows:

(Def. 21) For every $m_1$ such that $m = m_1$ holds $(S,U)$-TruthEval $m$ = $(S,U)$-TruthEval$(m_1)$.

Let us consider $S$, $U$, $m$ and let $I$ be an element of $U$-InterpretersOf $S$. The functor $(I,m)$-TruthEval yields an element of ((AllSymbolsOf $S)^* \setminus \{\emptyset\}) \dot{\rightarrow}$ *Boolean* and is defined by:

(Def. 22) $(I,m)$-TruthEval = (curry(($S,U$)-TruthEval $m$))($I$).

Let us consider $S$, $m$. The functor $S$-formulasOfMaxDepth $m$ yielding a subset of (AllSymbolsOf $S)^* \setminus \{\emptyset\}$ is defined as follows:

(Def. 23) For every non empty set $U$ and for every element $I$ of $U$-InterpretersOf $S$ and for every element $m_1$ of $\mathbb{N}$ such that $m = m_1$ holds $S$-formulasOfMaxDepth $m$ = dom($(I,m_1)$-TruthEval).

Let us consider $S$, $m$, $w$. We say that $w$ is $m$-w.f.f. if and only if:

(Def. 24) $w \in S$-formulasOfMaxDepth $m$.

Let us consider $S$, $w$. We say that $w$ is w.f.f. if and only if:

(Def. 25) There exists $m$ such that $w$ is $m$-w.f.f..

Let us consider $S$. Note that every string of $S$ which is 0-w.f.f. is also 0-w.f.f. and every string of $S$ which is 0-w.f.f. is also 0-w.f.f.. Let us consider $m$. One can check that every string of $S$ which is $m$-w.f.f. is also w.f.f.. Let us consider $n$. One can check that every string of $S$ which is $m+0\cdot n$-w.f.f. is also $m+n$-w.f.f..

Let us consider $S$, $m$. Observe that there exists a string of $S$ which is $m$-w.f.f.. Note that $S$-formulasOfMaxDepth $m$ is non empty. One can verify that there exists a string of $S$ which is w.f.f..

Let us consider $S$, $U$, let $I$ be an element of $U$-InterpretersOf $S$, and let $w$ be a w.f.f. string of $S$. The functor $I$-TruthEval $w$ yields an element of *Boolean* and is defined as follows:

(Def. 26) For every natural number $m$ such that $w$ is $m$-w.f.f. holds $I$-TruthEval $w = (I,m)$-TruthEval$(w)$.

Let us consider $S$. The functor AllFormulasOf $S$ is defined by:

(Def. 27) AllFormulasOf $S = \{w; w$ ranges over strings of $S$: $\bigvee_m$ $w$ is $m$-w.f.f.$\}$.

Let us consider $S$. One can check that AllFormulasOf $S$ is non empty.

For simplicity, we follow the rules: $u$, $u_1$, $u_2$ are elements of $U$, $t$ is a termal string of $S$, $I$ is an $(S,U)$-interpreter-like function, $l$, $l_1$, $l_2$ are literal elements of $S$, $m_2$, $n_1$ are non zero natural numbers, $p_0$ is a 0-w.f.f. string of $S$, and $p_5$, $p_1$, $p_3$, $p_4$ are w.f.f. strings of $S$.

The following propositions are true:

(3) $(I,u)$-TermEval$(m+1)(t) = I(S$-firstChar$(t))((I,u)$-TermEval$(m) \cdot$ SubTerms $t)$ and if $t$ is 0-termal, then $(I,u)$-TermEval$(m+1)(t) = I(S$-firstChar$(t))(\emptyset)$.

(4) For every $m$-termal string $t$ of $S$ holds $(I, u_1)$-TermEval$(m+1)(t) = (I, u_2)$-TermEval$(m+1+n)(t)$.

(5) curry($(S, U)$-TruthEval $m$) is a function from $U$-InterpretersOf $S$ into $((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \dot{\rightarrow} \mathit{Boolean}$.

(6) $x \in X \cup Y \cup Z$ iff $x \in X$ or $x \in Y$ or $x \in Z$.

(7) $S$-formulasOfMaxDepth 0 = AtomicFormulasOf $S$.

Let us consider $S$, $m$. Then $S$-formulasOfMaxDepth $m$ can be characterized by the condition:

(Def. 28) For every non empty set $U$ and for every element $I$ of $U$-InterpretersOf $S$ holds $S$-formulasOfMaxDepth $m$ = dom($(I, m)$-TruthEval).

Next we state the proposition

(8) $(S, U)$-TruthEval $m \in \mathit{Boolean}^{(U\text{-InterpretersOf } S) \times (S\text{-formulasOfMaxDepth } m)}$ and $(S, U)$-TruthEval$(m) \in \mathit{Boolean}^{(U\text{-InterpretersOf } S) \times (S\text{-formulasOfMaxDepth } m)}$.

Let us consider $S$, $m$. The functor $m$-ExFormulasOf $S$ is defined by:

(Def. 29) $m$-ExFormulasOf $S = \{\langle v \rangle \frown p_1 : v$ ranges over elements of LettersOf $S, p_1$ ranges over elements of $S$-formulasOfMaxDepth $m\}$.

The functor $m$-NorFormulasOf $S$ is defined as follows:

(Def. 30) $m$-NorFormulasOf $S = \{\langle$TheNorSymbOf $S\rangle \frown p_3 \frown p_4 : p_3$ ranges over elements of $S$-formulasOfMaxDepth $m, p_4$ ranges over elements of $S$-formulasOfMaxDepth $m\}$.

Let us consider $S$ and let $w_1$, $w_2$ be strings of $S$. Then $w_1 \frown w_2$ is a string of $S$.

Let us consider $S$, $s$. Then $\langle s \rangle$ is a string of $S$.

One can prove the following two propositions:

(9) $S$-formulasOfMaxDepth$(m+1)$ = ($m$-ExFormulasOf $S$)$\cup$($m$-NorFormulasOf $S$)$\cup$($S$-formulasOfMaxDepth $m$).

(10) AtomicFormulasOf $S$ is $S$-prefix.

Let us consider $S$. Note that AtomicFormulasOf $S$ is $S$-prefix. Observe that $S$-formulasOfMaxDepth 0 is $S$-prefix.

Let us consider $p_1$. The functor Depth $p_1$ yielding a natural number is defined by:

(Def. 31) $p_1$ is Depth $p_1$-w.f.f. and for every $n$ such that $p_1$ is $n$-w.f.f. holds Depth $p_1 \leq n$.

Let us consider $S$, $m$ and let $p_3$, $p_4$ be $m$-w.f.f. strings of $S$. Note that $\langle$TheNorSymbOf $S\rangle \frown p_3 \frown p_4$ is $m+1$-w.f.f..

Let us consider $S$, $p_3$, $p_4$. Observe that $\langle$TheNorSymbOf $S\rangle \frown p_3 \frown p_4$ is w.f.f..

Let us consider $S$, $m$, let $p_1$ be an $m$-w.f.f. string of $S$, and let $v$ be a literal element of $S$. Note that $\langle v \rangle \frown p_1$ is $m+1$-w.f.f..

Let us consider $S$, $l$, $p_1$. Note that $\langle l\rangle \frown p_1$ is w.f.f..

Let us consider $S$, $w$ and let $s$ be a non relational element of $S$. One can check that $\langle s\rangle \frown w$ is non 0-w.f.f..

Let us consider $S$, $w_1$, $w_2$ and let $s$ be a non relational element of $S$. Observe that $\langle s\rangle \frown w_1 \frown w_2$ is non 0-w.f.f..

Let us consider $S$. Observe that TheNorSymbOf $S$ is non relational.

Let us consider $S$, $w$. Observe that $\langle \mathrm{TheNorSymbOf}\, S\rangle \frown w$ is non 0-w.f.f..

Let us consider $S$, $l$, $w$. Note that $\langle l\rangle \frown w$ is non 0-w.f.f..

Let us consider $S$, $w$. We say that $w$ is exal if and only if:

(Def. 32) $S$-firstChar$(w)$ is literal.

Let us consider $S$, $w$, $l$. One can verify that $\langle l\rangle \frown w$ is exal.

Let us consider $S$, $m_2$. Observe that there exists an $m_2$-w.f.f. string of $S$ which is exal.

Let us consider $S$. Note that every string of $S$ which is exal is also non 0-w.f.f..

Let us consider $S$, $m_2$. One can check that there exists an exal $m_2$-w.f.f. string of $S$ which is non 0-w.f.f..

Let us consider $S$. One can verify that there exists an exal w.f.f. string of $S$ which is non 0-w.f.f..

Let us consider $S$ and let $p_1$ be a non 0-w.f.f. w.f.f. string of $S$. Note that Depth $p_1$ is non zero.

Let us consider $S$ and let $w$ be a non 0-w.f.f. w.f.f. string of $S$. Observe that $S$-firstChar$(w)$ is non relational.

Let us consider $S$, $m$. Observe that $S$-formulasOfMaxDepth $m$ is $S$-prefix. Then AllFormulasOf $S$ is a subset of $(\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$. Observe that every element of AllFormulasOf $S$ is w.f.f.. Note that AllFormulasOf $S$ is $S$-prefix.

We now state three propositions:

(11) $\mathrm{dom}\,\mathrm{NorIterator}((S,U)\text{-TruthEval}\, m) = (U\text{-InterpretersOf}\, S) \times (m\text{-NorFormulasOf}\, S)$.

(12) $\mathrm{dom}\,\mathrm{ExIterator}((S,U)\text{-TruthEval}\, m) = (U\text{-InterpretersOf}\, S) \times (m\text{-ExFormulasOf}\, S)$.

(13) $U\text{-deltaInterpreter}^{-1}(\{1\}) = \{\langle u,u\rangle : u$ ranges over elements of $U\}$.

Let us consider $S$. Then TheEqSymbOf $S$ is an element of $S$.

Let us consider $S$. One can verify that $\mathrm{ar}\,\mathrm{TheEqSymbOf}\, S + 2$ is zero and $|\mathrm{ar}\,\mathrm{TheEqSymbOf}\, S| - 2$ is zero.

We now state two propositions:

(14) Let $p_1$ be a 0-w.f.f. string of $S$ and $I$ be an $(S,U)$-interpreter-like function. Then

(i) if $S$-firstChar$(p_1) \neq \mathrm{TheEqSymbOf}\, S$, then $I\text{-AtomicEval}\, p_1 = I(S\text{-firstChar}(p_1))(I\text{-TermEval} \cdot \mathrm{SubTerms}\, p_1)$, and

(ii) if $S$-firstChar$(p_1)$ = TheEqSymbOf $S$, then $I$-AtomicEval $p_1$ = $U$-deltaInterpreter($I$-TermEval $\cdot$ SubTerms $p_1$).

(15) Let $I$ be an $(S, U)$-interpreter-like function and $p_1$ be a 0-w.f.f. string of $S$. If $S$-firstChar$(p_1)$ = TheEqSymbOf $S$, then $I$-AtomicEval $p_1 = 1$ iff $I$-TermEval((SubTerms $p_1$)(1)) = $I$-TermEval((SubTerms $p_1$)(2)).

Let us consider $S$, $m$. One can check that $m$-ExFormulasOf $S$ is non empty. Note that $m$-NorFormulasOf $S$ is non empty. Then $m$-NorFormulasOf $S$ is a subset of (AllSymbolsOf $S)^* \setminus \{\emptyset\}$.

Let us consider $S$ and let $w$ be an exal string of $S$. One can verify that $S$-firstChar$(w)$ is literal.

Let us consider $S$, $m$. Observe that every element of $m$-NorFormulasOf $S$ is non exal. Then $m$-ExFormulasOf $S$ is a subset of (AllSymbolsOf $S)^* \setminus \{\emptyset\}$.

Let us consider $S$, $m$. One can check that every element of $m$-ExFormulasOf $S$ is exal.

Let us consider $S$. One can check that there exists an element of $S$ which is non literal.

Let us consider $S$, $w$ and let $s$ be a non literal element of $S$. Note that $\langle s \rangle \frown w$ is non exal.

Let us consider $S$, $w_1$, $w_2$ and let $s$ be a non literal element of $S$. Observe that $\langle s \rangle \frown w_1 \frown w_2$ is non exal.

Let us consider $S$. Note that TheNorSymbOf $S$ is non literal.

Next we state the proposition

(16) $p_1 \in$ AllFormulasOf $S$.

Let us consider $S$, $m$, $w$. We introduce $w$ is $m$-non-w.f.f. as an antonym of $w$ is $m$-w.f.f..

Let us consider $m$, $S$. One can verify that every string of $S$ which is non $m$-w.f.f. is also $m$-non-w.f.f..

Let us consider $S$, $p_3$, $p_4$. Observe that $\langle$TheNorSymbOf $S\rangle \frown p_3 \frown p_4$ is max(Depth $p_3$, Depth $p_4$)-non-w.f.f..

Let us consider $S$, $p_1$, $l$. Note that $\langle l \rangle \frown p_1$ is Depth $p_1$-non-w.f.f..

Let us consider $S$, $p_1$, $l$. One can check that $\langle l \rangle \frown p_1$ is $1 +$ Depth $p_1$-w.f.f..

Let us consider $S$, $U$. Observe that every element of $U$-InterpretersOf $S$ is OwnSymbolsOf $S$-defined.

Let us consider $S$, $U$. Note that there exists an element of $U$-InterpretersOf $S$ which is OwnSymbolsOf $S$-defined.

Let us consider $S$, $U$. Note that every OwnSymbolsOf $S$-defined element of $U$-InterpretersOf $S$ is total.

Let us consider $S$, $U$, let $I$ be an element of $U$-InterpretersOf $S$, let $x$ be a literal element of $S$, and let $u$ be an element of $U$. Then $(x, u)$ ReassignIn $I$ is an element of $U$-InterpretersOf $S$.

In the sequel $I$ denotes an element of $U$-InterpretersOf $S$.

Let us consider $S$, $w$. The functor $\operatorname{xnot} w$ yields a string of $S$ and is defined as follows:

(Def. 33) $\operatorname{xnot} w = \langle \text{TheNorSymbOf}\, S\rangle \mathbin{^\frown} w \mathbin{^\frown} w$.

Let us consider $S$, $m$ and let $p_1$ be an $m$-w.f.f. string of $S$. Observe that $\operatorname{xnot} p_1$ is $m+1$-w.f.f..

Let us consider $S$, $p_1$. Note that $\operatorname{xnot} p_1$ is w.f.f..

Let us consider $S$. One can verify that TheEqSymbOf $S$ is non own.

Let us consider $S$, $X$. We say that $X$ is $S$-mincover if and only if:

(Def. 34) For every $p_1$ holds $p_1 \in X$ iff $\operatorname{xnot} p_1 \notin X$.

One can prove the following propositions:

(17) $\text{Depth}(\langle \text{TheNorSymbOf}\, S\rangle \mathbin{^\frown} p_3 \mathbin{^\frown} p_4) = 1 + \max(\text{Depth}\, p_3, \text{Depth}\, p_4)$ and $\text{Depth}(\langle l\rangle \mathbin{^\frown} p_3) = \text{Depth}\, p_3 + 1$.

(18) If $\text{Depth}\, p_1 = m + 1$, then $p_1$ is exal iff $p_1 \in m$-ExFormulasOf $S$ and $p_1$ is non exal iff $p_1 \in m$-NorFormulasOf $S$.

(19) $I$-TruthEval$\langle l\rangle \mathbin{^\frown} p_1 = \mathit{true}$ iff there exists $u$ such that $((l, u)\, \text{ReassignIn}\, I)$-TruthEval $p_1 = 1$ and $I$-TruthEval$\langle \text{TheNorSymbOf}\, S\rangle \mathbin{^\frown} p_3 \mathbin{^\frown} p_4 = \mathit{true}$ iff $I$-TruthEval $p_3 = \mathit{false}$ and $I$-TruthEval $p_4 = \mathit{false}$.

In the sequel $I$ denotes an $(S, U)$-interpreter-like function.

One can prove the following two propositions:

(20) $(I, u)$-TermEval$(m + 1) \restriction S$-termsOfMaxDepth$(m) =$ $I$-TermEval $\restriction S$-termsOfMaxDepth$(m)$.

(21) $I$-TermEval$(t) = I(S\text{-firstChar}(t))(I\text{-TermEval} \cdot \text{SubTerms}\, t)$.

Let us consider $S$, $p_1$. The functor SubWffsOf $p_1$ is defined as follows:

(Def. 35)(i) There exist $p_3$, $p$ such that $p$ is AllSymbolsOf $S$-valued and SubWffsOf $p_1 = \langle p_3, p\rangle$ and $p_1 = \langle S\text{-firstChar}(p_1)\rangle \mathbin{^\frown} p_3 \mathbin{^\frown} p$ if $p_1$ is non 0-w.f.f.,

(ii) SubWffsOf $p_1 = \langle p_1, \emptyset\rangle$, otherwise.

Let us consider $S$, $p_1$. The functor head $p_1$ yields a w.f.f. string of $S$ and is defined as follows:

(Def. 36) head $p_1 = (\text{SubWffsOf}\, p_1)_{\mathbf{1}}$.

The functor tail $p_1$ yields an element of $(\text{AllSymbolsOf}\, S)^*$ and is defined by:

(Def. 37) tail $p_1 = (\text{SubWffsOf}\, p_1)_{\mathbf{2}}$.

Let us consider $S$, $m$. One can verify that $(S\text{-formulasOfMaxDepth}\, m) \setminus$ AllFormulasOf $S$ is empty.

Let us consider $S$. Observe that AtomicFormulasOf $S \setminus$ AllFormulasOf $S$ is empty.

We now state two propositions:

(22) $\text{Depth}(\langle l\rangle \mathbin{^\frown} p_3) > \text{Depth}\, p_3$ and $\text{Depth}(\langle \text{TheNorSymbOf}\, S\rangle \mathbin{^\frown} p_3 \mathbin{^\frown} p_4) > \text{Depth}\, p_3$ and $\text{Depth}(\langle \text{TheNorSymbOf}\, S\rangle \mathbin{^\frown} p_3 \mathbin{^\frown} p_4) > \text{Depth}\, p_4$.

(23) If $p_1$ is not 0-w.f.f., then $p_1 = \langle x \rangle \mathbin{^\frown} p_4 \mathbin{^\frown} p_2$ iff $x = S$-firstChar$(p_1)$ and $p_4 = \operatorname{head} p_1$ and $p_2 = \operatorname{tail} p_1$.

Let us consider $S$, $m_2$. Observe that there exists a non 0-w.f.f. $m_2$-w.f.f. string of $S$ which is non exal.

Let us consider $S$ and let $p_1$ be an exal w.f.f. string of $S$. One can verify that $\operatorname{tail} p_1$ is empty.

Let us consider $S$ and let $p_1$ be a non exal non 0-w.f.f. w.f.f. string of $S$. Then $\operatorname{tail} p_1$ is a w.f.f. string of $S$.

Let us consider $S$ and let $p_1$ be a non exal non 0-w.f.f. w.f.f. string of $S$. One can check that $\operatorname{tail} p_1$ is w.f.f..

Let us consider $S$ and let $p_1$ be a non 0-w.f.f. non exal w.f.f. string of $S$. One can verify that $S$-firstChar$(p_1) \dot{-}$ TheNorSymbOf $S$ is empty.

Let us consider $m$, $S$ and let $p_1$ be an $m+1$-w.f.f. string of $S$. Note that $\operatorname{head} p_1$ is $m$-w.f.f..

Let us consider $m$, $S$ and let $p_1$ be an $m+1$-w.f.f. non exal non 0-w.f.f. string of $S$. Observe that $\operatorname{tail} p_1$ is $m$-w.f.f..

One can prove the following proposition

(24) For every element $I$ of $U$-InterpretersOf $S$ holds $(I, m)$-TruthEval $\in$ $Boolean^{S\text{-formulasOfMaxDepth}\, m}$.

Let us consider $S$. One can check that there exists an of-atomic-formula element of $S$ which is non literal.

One can prove the following proposition

(25) If $l_2 \notin \operatorname{rng} p$, then $((l_2, u)\operatorname{ReassignIn} I)$-TermEval$(p) = I$-TermEval$(p)$.

Let us consider $X$, $S$, $s$. We say that $s$ is $X$-occurring if and only if:

(Def. 38) $s \in \text{SymbolsOf}(((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \cap X)$.

Let us consider $S$, $s$ and let us consider $X$. We say that $X$ is $s$-containing if and only if:

(Def. 39) $s \in \text{SymbolsOf}((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} \cap X)$.

Let us consider $X$, $S$, $s$. We introduce $s$ is $X$-absent as an antonym of $s$ is $X$-occurring.

Let us consider $S$, $s$, $X$. We introduce $X$ is $s$-free as an antonym of $X$ is $s$-containing.

Let $X$ be a finite set and let us consider $S$. Observe that there exists a literal element of $S$ which is $X$-absent.

Let us consider $S$, $t$. Note that $\operatorname{rng} t \cap \text{LettersOf } S$ is non empty.

Let us consider $S$, $p_1$. One can verify that $\operatorname{rng} p_1 \cap \text{LettersOf } S$ is non empty.

Let us consider $B$, $S$ and let $A$ be a subset of $B$. Note that every element of $S$ which is $A$-occurring is also $B$-occurring.

Let us consider $A$, $B$, $S$. Observe that every element of $S$ which is $A \operatorname{null} B$-absent is also $A \cap B$-absent.

Let $F$ be a finite set and let us consider $A$, $S$. Note that every $F$-absent element of $S$ which is $A$-absent is also $A \cup F$-absent.

Let us consider $S$, $U$ and let $I$ be an $(S,U)$-interpreter-like function. One can check that $\mathrm{OwnSymbolsOf}\, S \setminus \mathrm{dom}\, I$ is empty.

One can prove the following proposition

(26) There exists $u$ such that $u = I(l)(\emptyset)$ and $(l,u)\, \mathrm{ReassignIn}\, I = I$.

Let us consider $S$, $X$. We say that $X$ is $S$-covering if and only if:

(Def. 40) For every $p_1$ holds $p_1 \in X$ or $\mathrm{xnot}\, p_1 \in X$.

Let us consider $S$. One can check that every set which is $S$-mincover is also $S$-covering.

Let us consider $U$, let $p_1$ be a non 0-w.f.f. non exal w.f.f. string of $S$, and let $I$ be an element of $U$-InterpretersOf $S$.

One can verify that $(I\text{-TruthEval}\, p_1) \dot{-} ((I\text{-TruthEval}\, \mathrm{head}\, p_1)$ $'\mathrm{nor}'$ $(I\text{-TruthEval}\, \mathrm{tail}\, p_1))$ is empty.

The functor ExFormulasOf $S$ yielding a subset of $(\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}$ is defined by:

(Def. 41) $\mathrm{ExFormulasOf}\, S = \{p_1; p_1$ ranges over strings of $S$: $p_1$ is w.f.f. $\wedge$ $p_1$ is exal$\}$.

Let us consider $S$. Note that ExFormulasOf $S$ is non empty.

Let us consider $S$. One can check that every element of ExFormulasOf $S$ is exal and w.f.f..

Let us consider $S$. Note that every element of ExFormulasOf $S$ is w.f.f..

Let us consider $S$. Observe that every element of ExFormulasOf $S$ is exal.

Let us consider $S$. Observe that $\mathrm{ExFormulasOf}\, S \setminus \mathrm{AllFormulasOf}\, S$ is empty.

Let us consider $U$, $S_1$ and let $S_2$ be an $S_1$-extending language. Note that every function which is $(S_2,U)$-interpreter-like is also $(S_1,U)$-interpreter-like.

Let us consider $U$, $S_1$, let $S_2$ be an $S_1$-extending language, and let $I$ be an $(S_2,U)$-interpreter-like function. Observe that $I \restriction \mathrm{OwnSymbolsOf}\, S_1$ is $(S_1,U)$-interpreter-like.

Let us consider $U$, $S_1$, let $S_2$ be an $S_1$-extending language, let $I_1$ be an element of $U$-InterpretersOf $S_1$, and let $I_2$ be an $(S_2,U)$-interpreter-like function. Note that $I_2 {+\cdot} I_1$ is $(S_2,U)$-interpreter-like.

Let us consider $U$, $S$, let $I$ be an element of $U$-InterpretersOf $S$, and let us consider $X$. We say that $X$ is $I$-satisfied if and only if:

(Def. 42) For every $p_1$ such that $p_1 \in X$ holds $I\text{-TruthEval}\, p_1 = 1$.

Let us consider $S$, $U$, $X$ and let $I$ be an element of $U$-InterpretersOf $S$. We say that $I$ satisfies $X$ if and only if:

(Def. 43) $X$ is $I$-satisfied.

Let us consider $U$, $S$, let $e$ be an empty set, and let $I$ be an element of $U$-InterpretersOf $S$. Observe that $e\, \mathrm{null}\, I$ is $I$-satisfied.

Let us consider $X$, $U$, $S$ and let $I$ be an element of $U$-InterpretersOf $S$. Observe that there exists a subset of $X$ which is $I$-satisfied.

Let us consider $U$, $S$ and let $I$ be an element of $U$-InterpretersOf $S$. One can check that there exists a set which is $I$-satisfied.

Let us consider $U$, $S$, let $I$ be an element of $U$-InterpretersOf $S$, and let $X$ be an $I$-satisfied set. One can check that every subset of $X$ is $I$-satisfied.

Let us consider $U$, $S$, let $I$ be an element of $U$-InterpretersOf $S$, and let $X$, $Y$ be $I$-satisfied sets. One can verify that $X \cup Y$ is $I$-satisfied.

Let us consider $U$, $S$, let $I$ be an element of $U$-InterpretersOf $S$, and let $X$ be an $I$-satisfied set. Observe that $I$ null $X$ satisfies $X$ .

Let us consider $S$, $X$. We say that $X$ is $S$-correct if and only if the condition (Def. 44) is satisfied.

(Def. 44) Let $U$ be a non empty set, $I$ be an element of $U$-InterpretersOf $S$, $x$ be an $I$-satisfied set, and given $p_1$. If $\langle x, p_1 \rangle \in X$, then $I$-TruthEval $p_1 = 1$.

Let us consider $S$. Note that $\emptyset$ null $S$ is $S$-correct.

Let us consider $S$, $X$. Observe that there exists a subset of $X$ which is $S$-correct.

Next we state two propositions:

(27) For every element $I$ of $U$-InterpretersOf $S$ holds $I$-TruthEval $p_1 = 1$ iff $\{p_1\}$ is $I$-satisfied.

(28) $s$ is $\{w\}$-occurring iff $s \in \mathrm{rng}\, w$.

Let us consider $U$, $S$, let us consider $p_3$, $p_4$, and let $I$ be an element of $U$-InterpretersOf $S$. Observe that ($I$-TruthEval$\langle$TheNorSymbOf $S\rangle \frown p_3 \frown p_4) \dot{-}$ (($I$-TruthEval $p_3$) 'nor' ($I$-TruthEval $p_4$)) is empty.

Let us consider $S$, $p_1$, $U$ and let $I$ be an element of $U$-InterpretersOf $S$. Note that ($I$-TruthEval xnot $p_1) \dot{-} \neg$($I$-TruthEval $p_1$) is empty.

Let us consider $X$, $S$, $p_1$. We say that $p_1$ is $X$-implied if and only if:

(Def. 45) For every non empty set $U$ and for every element $I$ of $U$-InterpretersOf $S$ such that $X$ is $I$-satisfied holds $I$-TruthEval $p_1 = 1$.

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