Sorting by Exchanging

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Summary. We show that exchanging of pairs in an array which are in incorrect order leads to sorted array. It justifies correctness of Bubble Sort, Insertion Sort, and Quicksort.

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The notation and terminology used here have been introduced in the following papers: [20], [6], [11], [1], [8], [16], [12], [13], [10], [9], [17], [18], [3], [4], [2], [7], [14], [21], [22], [19], [5], and [15].

1. PRELIMINARIES

We adopt the following convention: $\alpha, \beta, \gamma, \delta$ denote ordinal numbers, k denotes a natural number, and x, y, z, t, X, Y, Z denote sets.

The following propositions are true:

- (1) $x \in (\alpha + \beta) \setminus \alpha$ iff there exists γ such that $x = \alpha + \gamma$ and $\gamma \in \beta$.
- (2) Suppose $\alpha \in \beta$ and $\gamma \in \delta$. Then $\gamma \neq \alpha$ and $\gamma \neq \beta$ and $\delta \neq \alpha$ and $\delta \neq \beta$ or $\gamma \in \alpha$ and $\delta = \alpha$ or $\gamma \in \alpha$ and $\delta = \beta$ or $\gamma = \alpha$ and $\delta \in \beta$ or $\gamma = \alpha$ and $\delta = \beta$ or $\gamma = \alpha$ and $\beta \in \delta$ or $\alpha \in \gamma$ and $\delta = \beta$ or $\gamma = \beta$ and $\beta \in \delta$.
- (3) If $x \notin y$, then $(y \cup \{x\}) \setminus y = \{x\}$.
- (4) $\text{succ } x \setminus x = \{x\}$.
- (5) Let f be a function, r be a binary relation, and given x . Then $x \in f^\circ r$ if and only if there exist y, z such that $\langle y, z \rangle \in r$ and $\langle y, z \rangle \in \text{dom } f$ and $f(y, z) = x$.
- (6) If $\alpha \setminus \beta \neq \emptyset$, then $\inf(\alpha \setminus \beta) = \beta$ and $\sup(\alpha \setminus \beta) = \alpha$ and $\bigcup(\alpha \setminus \beta) = \bigcup \alpha$.

- (7) If $\alpha \setminus \beta$ is non empty and finite, then there exists a natural number n such that $\alpha = \beta + n$.

2. ARRAYS

Let f be a set. We say that f is segmental if and only if:

- (Def. 1) There exist α, β such that $\pi_1(f) = \alpha \setminus \beta$.

In the sequel f, g denote functions.

The following two propositions are true:

- (8) If $\text{dom } f = \text{dom } g$ and f is segmental, then g is segmental.
 (9) If f is segmental, then for all α, β, γ such that $\alpha \subseteq \beta \subseteq \gamma$ and $\alpha, \gamma \in \text{dom } f$ holds $\beta \in \text{dom } f$.

Let us observe that every function which is transfinite sequence-like is also segmental and every function which is finite sequence-like is also segmental.

Let us consider α and let s be a set. We say that s is α -based if and only if:

- (Def. 2) If $\beta \in \pi_1(s)$, then $\alpha \in \pi_1(s)$ and $\alpha \subseteq \beta$.

We say that s is α -limited if and only if:

- (Def. 3) $\alpha = \sup \pi_1(s)$.

Next we state two propositions:

- (10) f is α -based and segmental iff there exists β such that $\text{dom } f = \beta \setminus \alpha$ and $\alpha \subseteq \beta$.
 (11) f is β -limited, non empty, and segmental iff there exists α such that $\text{dom } f = \beta \setminus \alpha$ and $\alpha \in \beta$.

Let us observe that every function which is transfinite sequence-like is also 0-based and every function which is finite sequence-like is also 1-based.

The following three propositions are true:

- (12) f is $\inf \text{dom } f$ -based.
 (13) f is $\sup \text{dom } f$ -limited.
 (14) If f is β -limited and $\alpha \in \text{dom } f$, then $\alpha \in \beta$.

Let us consider f . The functor base f yielding an ordinal number is defined as follows:

- (Def. 4)(i) f is base f -based if there exists α such that $\alpha \in \text{dom } f$,
 (ii) base $f = 0$, otherwise.

The functor limit f yields an ordinal number and is defined as follows:

- (Def. 5)(i) f is limit f -limited if there exists α such that $\alpha \in \text{dom } f$,
 (ii) limit $f = 0$, otherwise.

Let us consider f . The functor length f yielding an ordinal number is defined as follows:

(Def. 6) $\text{length } f = \text{limit } f - \text{base } f$.

We now state four propositions:

- (15) $\text{base } \emptyset = 0$ and $\text{limit } \emptyset = 0$ and $\text{length } \emptyset = 0$.
- (16) $\text{limit } f = \sup \text{dom } f$.
- (17) f is limit f -limited.
- (18) Every empty set is α -based.

Let us consider α, X, Y . Note that there exists a transfinite sequence which is Y -defined, X -valued, α -based, segmental, finite, and empty.

An array is a segmental function.

Let A be an array. Observe that $\text{dom } A$ is ordinal-membered.

We now state the proposition

- (19) For every array f holds f is 0-limited iff f is empty.

Let us mention that every array which is 0-based is also transfinite sequence-like.

Let us consider X . An array of X is a X -valued array.

Let X be a 1-sorted structure. An array of X is an array of the carrier of X .

Let us consider α, X . An array of α, X is a α -defined array of X .

In the sequel A, B, C denote arrays.

Next we state several propositions:

- (20) $\text{base } f = \inf \text{dom } f$.
- (21) f is base f -based.
- (22) $\text{dom } A = \text{limit } A \setminus \text{base } A$.
- (23) If $\text{dom } A = \alpha \setminus \beta$ and A is non empty, then $\text{base } A = \beta$ and $\text{limit } A = \alpha$.
- (24) For every transfinite sequence f holds $\text{base } f = 0$ and $\text{limit } f = \text{dom } f$ and $\text{length } f = \text{dom } f$.

Let us consider α, β, X . Note that there exists an array of α, X which is β -based, natural-valued, integer-valued, real-valued, complex-valued, and finite.

Let us consider α, x . Note that $\{\langle \alpha, x \rangle\}$ is segmental.

Let us consider α and let x be a natural number. Observe that $\{\langle \alpha, x \rangle\}$ is natural-valued.

Let us consider α and let x be a real number. One can verify that $\{\langle \alpha, x \rangle\}$ is real-valued.

Let us consider α , let X be a non empty set, and let x be an element of X . One can check that $\{\langle \alpha, x \rangle\}$ is X -valued.

Let us consider α, x . One can check that $\{\langle \alpha, x \rangle\}$ is α -based and $\text{succ } \alpha$ -limited.

Let us consider β . Note that there exists an array which is non empty, β -based, natural-valued, integer-valued, real-valued, complex-valued, and finite. Let X be a non empty set. Note that there exists an array which is non empty, β -based, finite, and X -valued.

Let s be a transfinite sequence. We introduce s last as a synonym of last s .

Let A be an array. The functor last A is defined by:

(Def. 7) last $A = A(\bigcup \text{dom } A)$.

3. DESCENDING SEQUENCES

Let f be a function. We say that f is descending if and only if:

(Def. 8) For all α, β such that $\alpha, \beta \in \text{dom } f$ and $\alpha \in \beta$ holds $f(\beta) \subset f(\alpha)$.

We now state four propositions:

- (25) For every finite array f such that for every α such that $\alpha, \text{succ } \alpha \in \text{dom } f$ holds $f(\text{succ } \alpha) \subset f(\alpha)$ holds f is descending.
- (26) For every array f such that $\text{length } f = \omega$ and for every α such that $\alpha, \text{succ } \alpha \in \text{dom } f$ holds $f(\text{succ } \alpha) \subset f(\alpha)$ holds f is descending.
- (27) For every transfinite sequence f such that f is descending and $f(0)$ is finite holds f is finite.
- (28) Let f be a transfinite sequence. Suppose f is descending and $f(0)$ is finite and for every α such that $f(\alpha) \neq \emptyset$ holds $\text{succ } \alpha \in \text{dom } f$. Then last $f = \emptyset$.

The scheme A deals with a transfinite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ is finite

provided the parameters meet the following requirements:

- $\mathcal{F}(\mathcal{A}(0))$ is finite, and
- For every α such that $\text{succ } \alpha \in \text{dom } \mathcal{A}$ and $\mathcal{F}(\mathcal{A}(\alpha))$ is finite holds $\mathcal{F}(\mathcal{A}(\text{succ } \alpha)) \subset \mathcal{F}(\mathcal{A}(\alpha))$.

4. SWAP

Let us consider X , let f be a X -defined function, and let α, β be sets. Note that $\text{Swap}(f, \alpha, \beta)$ is X -defined.

Let X be a set, let f be a X -valued function, and let x, y be sets. Note that $\text{Swap}(f, x, y)$ is X -valued.

The following propositions are true:

- (29) If $x, y \in \text{dom } f$, then $(\text{Swap}(f, x, y))(x) = f(y)$.
- (30) For every array f of X such that $x, y \in \text{dom } f$ holds $(\text{Swap}(f, x, y))_x = f_y$.
- (31) If $x, y \in \text{dom } f$, then $(\text{Swap}(f, x, y))(y) = f(x)$.
- (32) For every array f of X such that $x, y \in \text{dom } f$ holds $(\text{Swap}(f, x, y))_y = f_x$.

- (33) If $z \neq x$ and $z \neq y$, then $(\text{Swap}(f, x, y))(z) = f(z)$.
- (34) For every array f of X such that $z \in \text{dom } f$ and $z \neq x$ and $z \neq y$ holds $(\text{Swap}(f, x, y))_z = f_z$.
- (35) If $x, y \in \text{dom } f$, then $\text{Swap}(f, x, y) = \text{Swap}(f, y, x)$.

Let X be a non empty set. Observe that there exists a X -valued non empty function which is onto.

Let X be a non empty set, let f be an onto X -valued non empty function, and let x, y be elements of $\text{dom } f$. Note that $\text{Swap}(f, x, y)$ is onto.

Let us consider A and let us consider x, y . Note that $\text{Swap}(A, x, y)$ is segmental.

We now state the proposition

- (36) If $x, y \in \text{dom } A$, then $\text{Swap}(\text{Swap}(A, x, y), x, y) = A$.

Let A be a real-valued array and let us consider x, y . One can verify that $\text{Swap}(A, x, y)$ is real-valued.

5. PERMUTATIONS

Let A be an array. An array is called a permutation of A if:

(Def. 9) There exists a permutation f of $\text{dom } A$ such that it $= A \cdot f$.

We now state several propositions:

- (37) For every permutation B of A holds $\text{dom } B = \text{dom } A$ and $\text{rng } B = \text{rng } A$.
- (38) A is a permutation of A .
- (39) If A is a permutation of B , then B is a permutation of A .
- (40) If A is a permutation of B and B is a permutation of C , then A is a permutation of C .
- (41) $\text{Swap}(\text{id}_X, x, y)$ is one-to-one.

Let X be a non empty set and let x, y be elements of X .

Note that $\text{Swap}(\text{id}_X, x, y)$ is one-to-one, X -valued, and X -defined.

Let X be a non empty set and let x, y be elements of X .

Note that $\text{Swap}(\text{id}_X, x, y)$ is onto and total.

Let X, Y be non empty sets, let f be a function from X into Y , and let x, y be elements of X . Then $\text{Swap}(f, x, y)$ is a function from X into Y .

Next we state three propositions:

- (42) If $x, y \in X$ and $f = \text{Swap}(\text{id}_X, x, y)$ and $X = \text{dom } A$, then $\text{Swap}(A, x, y) = A \cdot f$.
- (43) If $x, y \in \text{dom } A$, then $\text{Swap}(A, x, y)$ is a permutation of A and A is a permutation of $\text{Swap}(A, x, y)$.
- (44) Suppose $x, y \in \text{dom } A$ and A is a permutation of B . Then $\text{Swap}(A, x, y)$ is a permutation of B and A is a permutation of $\text{Swap}(B, x, y)$.

6. EXCHANGING

Let O be a relational structure and let A be an array of O . We say that A is ascending if and only if:

(Def. 10) For all α, β such that $\alpha, \beta \in \text{dom } A$ and $\alpha \in \beta$ holds $A_\alpha \leq A_\beta$.

The functor $\text{inversions } A$ is defined by:

(Def. 11) $\text{inversions } A = \{\langle \alpha, \beta \rangle; \alpha \text{ ranges over elements of } \text{dom } A, \beta \text{ ranges over elements of } \text{dom } A : \alpha \in \beta \wedge A_\alpha \not\leq A_\beta\}$.

Let O be a relational structure. One can verify that every empty array of O is ascending. Let A be an empty array of O . One can verify that $\text{inversions } A$ is empty.

In the sequel O is a connected non empty poset and R, Q are arrays of O .

We now state the proposition

(45) For every O and for all elements x, y of O holds $x > y$ iff $x \not\leq y$.

Let us consider O, R . Then $\text{inversions } R$ can be characterized by the condition:

(Def. 12) $\text{inversions } R = \{\langle \alpha, \beta \rangle; \alpha \text{ ranges over elements of } \text{dom } R, \beta \text{ ranges over elements of } \text{dom } R : \alpha \in \beta \wedge R_\alpha > R_\beta\}$.

Next we state two propositions:

(46) $\langle x, y \rangle \in \text{inversions } R$ iff $x, y \in \text{dom } R$ and $x \in y$ and $R_x > R_y$.

(47) $\text{inversions } R \subseteq \text{dom } R \times \text{dom } R$.

Let us consider O and let R be a finite array of O . Observe that $\text{inversions } R$ is finite.

We now state three propositions:

(48) R is ascending iff $\text{inversions } R = \emptyset$.

(49) If $\langle x, y \rangle \in \text{inversions } R$, then $\langle y, x \rangle \notin \text{inversions } R$.

(50) If $\langle x, y \rangle, \langle y, z \rangle \in \text{inversions } R$, then $\langle x, z \rangle \in \text{inversions } R$.

Let us consider O, R . Note that $\text{inversions } R$ is relation-like.

Let us consider O, R . Note that $\text{inversions } R$ is asymmetric and transitive.

Let us consider O and let α, β be elements of O . Let us note that the predicate $\alpha < \beta$ is antisymmetric.

Next we state several propositions:

(51) If $\langle x, y \rangle \in \text{inversions } R$, then $\langle x, y \rangle \notin \text{inversions } \text{Swap}(R, x, y)$.

(52) If $x, y \in \text{dom } R$ and $z \neq x$ and $z \neq y$ and $t \neq x$ and $t \neq y$, then $\langle z, t \rangle \in \text{inversions } R$ iff $\langle z, t \rangle \in \text{inversions } \text{Swap}(R, x, y)$.

(53) If $\langle x, y \rangle \in \text{inversions } R$, then $\langle z, y \rangle \in \text{inversions } R$ and $z \in x$ iff $\langle z, x \rangle \in \text{inversions } \text{Swap}(R, x, y)$.

(54) If $\langle x, y \rangle \in \text{inversions } R$, then $\langle z, x \rangle \in \text{inversions } R$ iff $z \in x$ and $\langle z, y \rangle \in \text{inversions } \text{Swap}(R, x, y)$.

- (55) If $\langle x, y \rangle \in \text{inversions } R$ and $z \in y$ and $\langle x, z \rangle \in \text{inversions } \text{Swap}(R, x, y)$, then $\langle x, z \rangle \in \text{inversions } R$.
- (56) If $\langle x, y \rangle \in \text{inversions } R$ and $x \in z$ and $\langle z, y \rangle \in \text{inversions } \text{Swap}(R, x, y)$, then $\langle z, y \rangle \in \text{inversions } R$.
- (57) If $\langle x, y \rangle \in \text{inversions } R$ and $y \in z$ and $\langle x, z \rangle \in \text{inversions } \text{Swap}(R, x, y)$, then $\langle y, z \rangle \in \text{inversions } R$.
- (58) If $\langle x, y \rangle \in \text{inversions } R$, then $y \in z$ and $\langle x, z \rangle \in \text{inversions } R$ iff $\langle y, z \rangle \in \text{inversions } \text{Swap}(R, x, y)$.

Let us consider O, R, x, y . The functor $\subseteq_{x,y}^R$ yields a function and is defined by:

(Def. 13) $\subseteq_{x,y}^R = \text{Swap}(\text{id}_{\text{dom } R}, x, y) \times \text{Swap}(\text{id}_{\text{dom } R}, x, y) + \text{id}_{\{x\} \times (\text{succ } y \setminus x) \cup (\text{succ } y \setminus x) \times \{y\}}$.

Next we state the proposition

- (59) $\gamma \in \text{succ } \beta \setminus \alpha$ iff $\alpha \subseteq \gamma \subseteq \beta$.

We adopt the following convention: T is a non empty array of O and p, q, r, s are elements of $\text{dom } T$.

The following propositions are true:

- (60) $\text{succ } q \setminus p \subseteq \text{dom } T$.
- (61) $\text{dom } \subseteq_{p,q}^T = \text{dom } T \times \text{dom } T$ and $\text{rng } \subseteq_{p,q}^T \subseteq \text{dom } T \times \text{dom } T$.
- (62) If $p \subseteq r \subseteq q$, then $(\subseteq_{p,q}^T)(p, r) = \langle p, r \rangle$ and $(\subseteq_{p,q}^T)(r, q) = \langle r, q \rangle$.
- (63) If $r \neq p$ and $s \neq q$ and $f = \text{Swap}(\text{id}_{\text{dom } T}, p, q)$, then $(\subseteq_{p,q}^T)(r, s) = \langle f(r), f(s) \rangle$.
- (64) If $r \in p$ and $f = \text{Swap}(\text{id}_{\text{dom } T}, p, q)$, then $(\subseteq_{p,q}^T)(r, q) = \langle f(r), f(q) \rangle$ and $(\subseteq_{p,q}^T)(r, p) = \langle f(r), f(p) \rangle$.
- (65) If $q \in r$ and $f = \text{Swap}(\text{id}_{\text{dom } T}, p, q)$, then $(\subseteq_{p,q}^T)(p, r) = \langle f(p), f(r) \rangle$ and $(\subseteq_{p,q}^T)(q, r) = \langle f(q), f(r) \rangle$.
- (66) If $p \in q$, then $(\subseteq_{p,q}^T)(p, q) = \langle p, q \rangle$.
- (67) If $p \in q$ and $r \neq p$ and $r \neq q$ and $s \neq p$ and $s \neq q$, then $(\subseteq_{p,q}^T)(r, s) = \langle r, s \rangle$.
- (68) If $r \in p$ and $p \in q$, then $(\subseteq_{p,q}^T)(r, p) = \langle r, q \rangle$ and $(\subseteq_{p,q}^T)(r, q) = \langle r, p \rangle$.
- (69) If $p \in s$ and $s \in q$, then $(\subseteq_{p,q}^T)(p, s) = \langle p, s \rangle$ and $(\subseteq_{p,q}^T)(s, q) = \langle s, q \rangle$.
- (70) If $p \in q$ and $q \in s$, then $(\subseteq_{p,q}^T)(p, s) = \langle q, s \rangle$ and $(\subseteq_{p,q}^T)(q, s) = \langle p, s \rangle$.
- (71) If $p \in q$, then $\subseteq_{p,q}^T \upharpoonright (\text{inversions } \text{Swap}(T, p, q) \text{ qua set})$ is one-to-one.

Let us consider O, R, x, y, z . Note that $(\subseteq_{x,y}^R)^\circ z$ is relation-like.

7. CORRECTNESS OF SORTING BY EXCHANGING

The following proposition is true

- (72) If $\langle x, y \rangle \in \text{inversions } R$, then $(\subseteq_{x,y}^R)^\circ \text{inversions } \text{Swap}(R, x, y) \subset \text{inversions } R$.

Let R be a finite function and let us consider x, y . One can check that $\text{Swap}(R, x, y)$ is finite.

Next we state two propositions:

- (73) For every array R of O such that $\langle x, y \rangle \in \text{inversions } R$ and $\text{inversions } R$ is finite holds $\overline{\text{inversions } \text{Swap}(R, x, y)} \in \overline{\text{inversions } R}$.
- (74) For every finite array R of O such that $\langle x, y \rangle \in \text{inversions } R$ holds $\overline{\text{inversions } \text{Swap}(R, x, y)} < \overline{\text{inversions } R}$.

Let us consider O, R . A non empty array is called a computation of R if it satisfies the conditions (Def. 14).

- (Def. 14)(i) $\text{It}(\text{base it}) = R$,
- (ii) for every α such that $\alpha \in \text{dom it}$ holds $\text{it}(\alpha)$ is an array of O , and
- (iii) for every α such that $\alpha, \text{succ } \alpha \in \text{dom it}$ there exist R, x, y such that $\langle x, y \rangle \in \text{inversions } R$ and $\text{it}(\alpha) = R$ and $\text{it}(\text{succ } \alpha) = \text{Swap}(R, x, y)$.

We now state the proposition

- (75) $\{\langle \alpha, R \rangle\}$ is a computation of R .

Let us consider O, R, α . One can check that there exists a computation of R which is α -based and finite.

Let us consider O, R , let C be a computation of R , and let us consider x . One can check that $C(x)$ is segmental, function-like, and relation-like.

Let us consider O, R , let C be a computation of R , and let us consider x . Observe that $C(x)$ is the carrier of O -valued.

Let us consider O, R and let C be a computation of R . Observe that last C is segmental, relation-like, and function-like.

Let us consider O, R and let C be a computation of R . Observe that last C is the carrier of O -valued.

Let us consider O, R and let C be a computation of R . We say that C is complete if and only if:

- (Def. 15) last C is ascending.

One can prove the following three propositions:

- (76) For every 0-based computation C of R such that R is a finite array of O holds C is finite.
- (77) Let C be a 0-based computation of R . Suppose R is a finite array of O and for every α such that $\text{inversions } C(\alpha) \neq \emptyset$ holds $\text{succ } \alpha \in \text{dom } C$. Then C is complete.

- (78) Let C be a finite computation of R . Then $\text{last } C$ is a permutation of R and for every α such that $\alpha \in \text{dom } C$ holds $C(\alpha)$ is a permutation of R .

8. EXISTENCE OF COMPLETE COMPUTATIONS

Next we state three propositions:

- (79) For every 0-based finite array A of X such that $A \neq \emptyset$ holds $\text{last } A \in X$.

- (80) $\text{last} \langle x \rangle = x$.

- (81) For every 0-based finite array A holds $\text{last}(A \cap \langle x \rangle) = x$.

Let X be a set. Observe that every element of X^ω is X -valued.

The scheme A deals with a unary functor $\mathcal{F}$ yielding a set, a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a finite 0-based non empty array f and there exists an element k of $\mathcal{A}$ such that

- (i) $k = \text{last } f$,
- (ii) $\mathcal{F}(k) = \emptyset$,
- (iii) $f(0) = \mathcal{B}$, and
- (iv) for every α such that $\text{succ } \alpha \in \text{dom } f$ there exist elements x, y of $\mathcal{A}$ such that $x = f(\alpha)$ and $y = f(\text{succ } \alpha)$ and $\mathcal{P}[x, y]$

provided the following requirements are met:

- $\mathcal{B} \in \mathcal{A}$,
- $\mathcal{F}(\mathcal{B})$ is finite, and
- For every element x of $\mathcal{A}$ such that $\mathcal{F}(x) \neq \emptyset$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{F}(y) \subset \mathcal{F}(x)$.

In the sequel A is an array and B is a permutation of A .

We now state the proposition

- (82) $B \in (\text{rng } A)^{\text{dom } A}$.

Let A be a real-valued array. One can verify that every permutation of A is real-valued.

Let us consider α and let X be a non empty set. Observe that every element of X^α is transfinite sequence-like.

Let us consider X and let Y be a real-membered non empty set. One can check that every element of Y^X is real-valued.

Let us consider X and let A be an array of X . One can check that every permutation of A is X -valued.

Let X be a set, let Z be a set, and let Y be a subset of Z . Note that every element of Y^X is Z -valued.

One can prove the following propositions:

- (83) Every X -defined Y -valued binary relation is a relation between X and Y .

- (84) For every finite ordinal number α and for every x such that $x \in \alpha$ holds $x = 0$ or there exists β such that $x = \text{succ } \beta$.
- (85) For every 0-based finite non empty array A of O holds there exists a 0-based computation of A which is complete.
- (86) For every 0-based finite non empty array A of O holds there exists a permutation of A which is ascending.

Let us consider O and let A be a 0-based finite array of O . Observe that there exists a permutation of A which is ascending.

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