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More on Continuous Functions on Normed Linear Spaces

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Summary. In this article we formalize the definition and some facts about continuous functions from $\mathbb{R}$ into normed linear spaces [14].

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The terminology and notation used in this paper have been introduced in the following papers: [2], [12], [3], [4], [10], [11], [1], [5], [13], [7], [17], [18], [15], [9], [8], [16], [19], and [6].

1. Preliminaries

For simplicity, we adopt the following rules: n denotes an element of $\mathbb{N}$, X, X_1 denote sets, r, p denote real numbers, s, x_0, x_1, x_2 denote real numbers, S, T denote real normed spaces, f, f_1, f_2 denote partial functions from $\mathbb{R}$ to the carrier of S, s_1 denotes a sequence of real numbers, and Y denotes a subset of $\mathbb{R}$.

The following propositions are true:

(1) Let s_2 be a sequence of real numbers and h be a partial function from $\mathbb{R}$ to the carrier of S. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$, then $s_2(n) \in \operatorname{dom} h$.

(2) Let h_1, h_2 be partial functions from $\mathbb{R}$ to the carrier of S and s_2 be a sequence of real numbers. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1+h_2)_* s_2 = (h_{1*} s_2) + (h_{2*} s_2)$ and $(h_1 - h_2)_* s_2 = (h_{1*} s_2) - (h_{2*} s_2)$.

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(3) For every sequence h of S and for every real number r holds $r\,h = r \cdot h$.

(4) Let h be a partial function from $\mathbb{R}$ to the carrier of S, s_2 be a sequence of real numbers, and r be a real number. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$, then $r\,h_* s_2 = r \cdot (h_* s_2)$.

(5) Let h be a partial function from $\mathbb{R}$ to the carrier of S and s_2 be a sequence of real numbers. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$, then $\|h_* s_2\| = \|h\|_* s_2$ and $-(h_* s_2) = -h_* s_2$.

2. Continuous Real Functions into Normed Linear Spaces

Let us consider S, f, x_0. We say that f is continuous in x_0 if and only if:

(Def. 1) $x_0 \in \operatorname{dom} f$ and for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f_* s_1$ is convergent and $f_{x_0} = \lim(f_* s_1)$.

Next we state a number of propositions:

(6) If $x_0 \in X$ and f is continuous in x_0, then $f{\restriction}X$ is continuous in x_0.

(7) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every s_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ and for every n holds $s_1(n) \neq x_0$ holds $f_* s_1$ is convergent and $f_{x_0} = \lim(f_* s_1)$.

(8) f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(9) Let given S, f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_1 of f_{x_0} there exists a neighbourhood N of x_0 such that for every x_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_1$.

(10) Let given S, f, x_0. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_1 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^{\circ} N \subseteq N_1$.

(11) If there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$, then f is continuous in x_0.

(12) If $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and f_1 is continuous in x_0 and f_2 is continuous in x_0, then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(13) If f is continuous in x_0, then $r\,f$ is continuous in x_0.

(14) If $x_0 \in \operatorname{dom} f$ and f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

(15) Let f_1 be a partial function from $\mathbb{R}$ to the carrier of S and f_2 be a partial function from the carrier of S to the carrier of T. Suppose $x_0 \in \operatorname{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0.

Let us consider S, f. We say that f is continuous if and only if:

(Def. 2) For every x_0 such that $x_0 \in \operatorname{dom} f$ holds f is continuous in x_0.

Next we state two propositions:

(16) Let given X, f. Suppose $X \subseteq \operatorname{dom} f$. Then $f \restriction X$ is continuous if and only if for every s_1 such that $\operatorname{rng} s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 \in X$ holds $f_* s_1$ is convergent and $f_{\lim s_1} = \lim(f_* s_1)$.

(17) Suppose $X \subseteq \operatorname{dom} f$. Then $f \restriction X$ is continuous if and only if for all x_0, r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every x_1 such that $x_1 \in X$ and $|x_1 - x_0| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

Let us consider S. One can check that every partial function from $\mathbb{R}$ to the carrier of S which is constant is also continuous.

Let us consider S. Note that there exists a partial function from $\mathbb{R}$ to the carrier of S which is continuous.

Let us consider S, let f be a continuous partial function from $\mathbb{R}$ to the carrier of S, and let X be a set. Observe that $f \restriction X$ is continuous.

Next we state the proposition

(18) If $f \restriction X$ is continuous and $X_1 \subseteq X$, then $f \restriction X_1$ is continuous.

Let us consider S. Observe that every partial function from $\mathbb{R}$ to the carrier of S which is empty is also continuous.

Let us consider S, f and let X be a trivial set. Observe that $f \restriction X$ is continuous.

Let us consider S and let f_1, f_2 be continuous partial functions from $\mathbb{R}$ to the carrier of S. Observe that $f_1 + f_2$ is continuous and $f_1 - f_2$ is continuous.

The following two propositions are true:

(19) Let given X, f_1, f_2. Suppose $X \subseteq \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $f_1 \restriction X$ is continuous and $f_2 \restriction X$ is continuous. Then $(f_1 + f_2) \restriction X$ is continuous and $(f_1 - f_2) \restriction X$ is continuous.

(20) Let given X, X_1, f_1, f_2. Suppose $X \subseteq \operatorname{dom} f_1$ and $X_1 \subseteq \operatorname{dom} f_2$ and $f_1 \restriction X$ is continuous and $f_2 \restriction X_1$ is continuous. Then $(f_1 + f_2) \restriction (X \cap X_1)$ is continuous and $(f_1 - f_2) \restriction (X \cap X_1)$ is continuous.

Let us consider S, let f be a continuous partial function from $\mathbb{R}$ to the carrier of S, and let us consider r. One can check that $r\, f$ is continuous.

We now state several propositions:

(21) If $X \subseteq \operatorname{dom} f$ and $f \restriction X$ is continuous, then $(r\, f) \restriction X$ is continuous.

(22) If $X \subseteq \operatorname{dom} f$ and $f{\restriction}X$ is continuous, then $\|f\|{\restriction}X$ is continuous and $(-f){\restriction}X$ is continuous.

(23) If f is total and for all x_1, x_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists x_0 such that f is continuous in x_0, then $f{\restriction}\mathbb{R}$ is continuous.

(24) If $\operatorname{dom} f$ is compact and $f{\restriction}\operatorname{dom} f$ is continuous, then $\operatorname{rng} f$ is compact.

(25) If $Y \subseteq \operatorname{dom} f$ and Y is compact and $f{\restriction}Y$ is continuous, then $f^{\circ}Y$ is compact.

3. Lipschitz Continuity

Let us consider S, f. We say that f is Lipschitzian if and only if:

(Def. 3) There exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} f$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$.

The following proposition is true

(26) $f{\restriction}X$ is Lipschitzian if and only if there exists a real number r such that $0 < r$ and for all x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}(f{\restriction}X)$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$.

Let us consider S. Observe that every partial function from $\mathbb{R}$ to the carrier of S which is empty is also Lipschitzian.

Let us consider S. One can verify that there exists a partial function from $\mathbb{R}$ to the carrier of S which is empty.

Let us consider S, let f be a Lipschitzian partial function from $\mathbb{R}$ to the carrier of S, and let X be a set. One can check that $f{\restriction}X$ is Lipschitzian.

The following proposition is true

(27) If $f{\restriction}X$ is Lipschitzian and $X_1 \subseteq X$, then $f{\restriction}X_1$ is Lipschitzian.

Let us consider S and let f_1, f_2 be Lipschitzian partial functions from $\mathbb{R}$ to the carrier of S. One can check that $f_1 + f_2$ is Lipschitzian and $f_1 - f_2$ is Lipschitzian.

One can prove the following propositions:

(28) If $f_1{\restriction}X$ is Lipschitzian and $f_2{\restriction}X_1$ is Lipschitzian, then $(f_1+f_2){\restriction}(X \cap X_1)$ is Lipschitzian.

(29) If $f_1{\restriction}X$ is Lipschitzian and $f_2{\restriction}X_1$ is Lipschitzian, then $(f_1-f_2){\restriction}(X \cap X_1)$ is Lipschitzian.

Let us consider S, let f be a Lipschitzian partial function from $\mathbb{R}$ to the carrier of S, and let us consider p. Note that $p\,f$ is Lipschitzian.

Next we state the proposition

(30) If $f{\restriction}X$ is Lipschitzian and $X \subseteq \operatorname{dom} f$, then $(p\,f){\restriction}X$ is Lipschitzian.

Let us consider S and let f be a Lipschitzian partial function from $\mathbb{R}$ to the carrier of S. Note that $\|f\|$ is Lipschitzian.

One can prove the following proposition

(31) If $f{\restriction}X$ is Lipschitzian, then $-f{\restriction}X$ is Lipschitzian and $(-f){\restriction}X$ is Lipschitzian and $\|f\|{\restriction}X$ is Lipschitzian.

Let us consider S. One can verify that every partial function from $\mathbb{R}$ to the carrier of S which is constant is also Lipschitzian.

Let us consider S. Observe that every partial function from $\mathbb{R}$ to the carrier of S which is Lipschitzian is also continuous.

Next we state two propositions:

(32) If there exists a point r of S such that $\operatorname{rng} f = \{r\}$, then f is continuous.

(33) For all points r, p of S such that for every x_0 such that $x_0 \in X$ holds $f_{x_0} = x_0 \cdot r + p$ holds $f{\restriction}X$ is continuous.

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