More on Continuous Functions on Normed Linear Spaces

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Summary. In this article we formalize the definition and some facts about continuous functions from \( \mathbb{R} \) into normed linear spaces [14].

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The terminology and notation used in this paper have been introduced in the following papers: [2], [12], [3], [4], [10], [11], [1], [5], [13], [7], [17], [18], [15], [9], [8], [16], [19], and [6].

1. Preliminaries

For simplicity, we adopt the following rules: \( n \) denotes an element of \( \mathbb{N} \), \( X \), \( X_1 \) denote sets, \( r, p \) denote real numbers, \( s, x_0, x_1, x_2 \) denote real numbers, \( S, T \) denote real normed spaces, \( f, f_1, f_2 \) denote partial functions from \( \mathbb{R} \) to the carrier of \( S \), \( s_1 \) denotes a sequence of real numbers, and \( Y \) denotes a subset of \( \mathbb{R} \).

The following propositions are true:

1. Let \( s_2 \) be a sequence of real numbers and \( h \) be a partial function from \( \mathbb{R} \) to the carrier of \( S \). If \( \text{rng} \ s_2 \subseteq \text{dom} \ h \), then \( s_2(n) \in \text{dom} \ h \).
2. Let \( h_1, h_2 \) be partial functions from \( \mathbb{R} \) to the carrier of \( S \) and \( s_2 \) be a sequence of real numbers. If \( \text{rng} \ s_2 \subseteq \text{dom} \ h_1 \cap \text{dom} \ h_2 \), then \( (h_1 + h_2) \ast s_2 = (h_1 \ast s_2) + (h_2 \ast s_2) \) and \( (h_1 - h_2) \ast s_2 = (h_1 \ast s_2) - (h_2 \ast s_2) \).
(3) For every sequence $h$ of $S$ and for every real number $r$ holds $r \cdot h = r \cdot h$.
(4) Let $h$ be a partial function from $\mathbb{R}$ to the carrier of $S$, $s_2$ be a sequence of real numbers, and $r$ be a real number. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$, then $r \cdot (h \cdot s_2) = r \cdot (h \cdot s_2)$.
(5) Let $h$ be a partial function from $\mathbb{R}$ to the carrier of $S$ and $s_2$ be a sequence of real numbers. If $\operatorname{rng} s_2 \subseteq \operatorname{dom} h$, then $\|h \cdot s_2\| = \|h \cdot s_2\|$ and $-(h \cdot s_2) = -h \cdot s_2$.

2. Continuous Real Functions into Normed Linear Spaces

Let us consider $S$, $f$, $x_0$. We say that $f$ is continuous in $x_0$ if and only if:

(Def. 1) $x_0 \in \operatorname{dom} f$ and for every $s_1$ such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $s_1$ is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Next we state a number of propositions:

(6) If $x_0 \in X$ and $f$ is continuous in $x_0$, then $f|X$ is continuous in $x_0$.
(7) $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:
   (i) $x_0 \in \operatorname{dom} f$, and
   (ii) for every $s_1$ such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $s_1$ is convergent and $\lim s_1 = x_0$ and for every $n$ holds $s_1(n) \neq x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.
(8) $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:
   (i) $x_0 \in \operatorname{dom} f$, and
   (ii) for every $r$ such that $0 < r$ there exists $s$ such that $0 < s$ and for every $x_1$ such that $x_1 \in \operatorname{dom} f$ and $|x_1 - x_0| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.
(9) Let given $S$, $f$, $x_0$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:
   (i) $x_0 \in \operatorname{dom} f$, and
   (ii) for every neighbourhood $N_1$ of $f_{x_0}$ there exists a neighbourhood $N$ of $x_0$ such that for every $x_1$ such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_1$.
(10) Let given $S$, $f$, $x_0$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:
    (i) $x_0 \in \operatorname{dom} f$, and
    (ii) for every neighbourhood $N_1$ of $f_{x_0}$ there exists a neighbourhood $N$ of $x_0$ such that $f^0 N \subseteq N_1$.
(11) If there exists a neighbourhood $N$ of $x_0$ such that $\operatorname{dom} f \cap N = \{x_0\}$, then $f$ is continuous in $x_0$.
(12) If $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $x_0$, then $f_1 + f_2$ is continuous in $x_0$ and $f_1 - f_2$ is continuous in $x_0$.
(13) If $f$ is continuous in $x_0$, then $r \cdot f$ is continuous in $x_0$. 
(14) If \( x_0 \in \text{dom} \ f \) and \( f \) is continuous in \( x_0 \), then \( ||f|| \) is continuous in \( x_0 \) and \(-f\) is continuous in \( x_0 \).

(15) Let \( f_1 \) be a partial function from \( \mathbb{R} \) to the carrier of \( S \) and \( f_2 \) be a partial function from the carrier of \( S \) to the carrier of \( T \). Suppose \( x_0 \in \text{dom}(f_2 \cdot f_1) \) and \( f_1 \) is continuous in \( x_0 \) and \( f_2 \) is continuous in \((f_1)_{x_0}\). Then \( f_2 \cdot f_1 \) is continuous in \( x_0 \).

Let us consider \( S, f \). We say that \( f \) is continuous if and only if:

(Def. 2) For every \( x_0 \) such that \( x_0 \in \text{dom} \ f \) holds \( f \) is continuous in \( x_0 \).

Next we state two propositions:

(16) Let given \( X, f \). Suppose \( X \subseteq \text{dom} \ f \). Then \( f \mid X \) is continuous if and only if for every \( s_1 \) such that \( \text{rng} \ s_1 \subseteq X \) and \( s_1 \) is convergent and \( \lim s_1 \in X \) holds \( f \cdot s_1 \) is convergent and \( \lim(f \cdot s_1) = \lim(f \cdot s_1) \).

(17) Suppose \( X \subseteq \text{dom} \ f \). Then \( f \mid X \) is continuous if and only if for all \( x_0, r \) such that \( x_0 \in X \) and \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for every \( x_1 \) such that \( x_1 \in X \) and \( |x_1 - x_0| < s \) holds \( ||f_{x_1} - f_{x_0}|| < r \).

Let us consider \( S \). One can check that every partial function from \( \mathbb{R} \) to the carrier of \( S \) which is constant is also continuous.

Let us consider \( S \). Note that there exists a partial function from \( \mathbb{R} \) to the carrier of \( S \) which is continuous.

Let us consider \( S \), let \( f \) be a continuous partial function from \( \mathbb{R} \) to the carrier of \( S \), and let \( X \) be a set. Observe that \( f \mid X \) is continuous.

Next we state the proposition

(18) If \( f \mid X \) is continuous and \( X_1 \subseteq X \), then \( f \mid X_1 \) is continuous.

Let us consider \( S \). Observe that every partial function from \( \mathbb{R} \) to the carrier of \( S \) which is empty is also continuous.

Let us consider \( S, f \) and let \( X \) be a trivial set. Observe that \( f \mid X \) is continuous.

Let us consider \( S \) and let \( f_1, f_2 \) be continuous partial functions from \( \mathbb{R} \) to the carrier of \( S \). Observe that \( f_1 + f_2 \) is continuous and \( f_1 - f_2 \) is continuous.

The following two propositions are true:

(19) Let given \( X, f_1, f_2 \). Suppose \( X \subseteq \text{dom} f_1 \cap \text{dom} f_2 \) and \( f_1 \mid X \) is continuous and \( f_2 \mid X \) is continuous. Then \((f_1 + f_2)\mid X \) is continuous and \((f_1 - f_2)\mid X \) is continuous.

(20) Let given \( X, X_1, f_1, f_2 \). Suppose \( X \subseteq \text{dom} f_1 \) and \( X_1 \subseteq \text{dom} f_2 \) and \( f_1 \mid X \) is continuous and \( f_2 \mid X_1 \) is continuous. Then \((f_1 + f_2)\mid (X \cap X_1) \) is continuous and \((f_1 - f_2)\mid (X \cap X_1) \) is continuous.

Let us consider \( S \), let \( f \) be a continuous partial function from \( \mathbb{R} \) to the carrier of \( S \), and let us consider \( r \). One can check that \( r \ f \) is continuous.

We now state several propositions:

(21) If \( X \subseteq \text{dom} \ f \) and \( f \mid X \) is continuous, then \((r f)\mid X \) is continuous.
(22) If $X \subseteq \text{dom } f$ and $f|X$ is continuous, then $\|f\||X$ is continuous and $(-f)|X$ is continuous.

(23) If $f$ is total and for all $x_1, x_2$ holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists $x_0$ such that $f$ is continuous in $x_0$, then $f|\mathbb{R}$ is continuous.

(24) If $\text{dom } f$ is compact and $f|\text{dom } f$ is continuous, then $\text{rng } f$ is compact.

(25) If $Y \subseteq \text{dom } f$ and $Y$ is compact and $f|Y$ is continuous, then $f \circ Y$ is compact.

3. Lipschitz Continuity

Let us consider $S, f$. We say that $f$ is Lipschitzian if and only if:

(Def. 3) There exists a real number $r$ such that $0 < r$ and for all $x_1, x_2$ such that $x_1, x_2 \in \text{dom } f$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$.

The following proposition is true

(26) $f|X$ is Lipschitzian if and only if there exists a real number $r$ such that $0 < r$ and for all $x_1, x_2$ such that $x_1, x_2 \in \text{dom } (f|X)$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot |x_1 - x_2|$.

Let us consider $S$. Observe that every partial function from $\mathbb{R}$ to the carrier of $S$ which is empty is also Lipschitzian.

Let us consider $S, f$. One can verify that there exists a partial function from $\mathbb{R}$ to the carrier of $S$ which is empty.

Let us consider $S, f$. One can check that $f|X$ is Lipschitzian.

The following proposition is true

(27) If $f|X$ is Lipschitzian and $X_1 \subseteq X$, then $f|X_1$ is Lipschitzian.

Let us consider $S$ and let $f_1, f_2$ be Lipschitzian partial functions from $\mathbb{R}$ to the carrier of $S$. One can check that $f_1 + f_2$ is Lipschitzian and $f_1 - f_2$ is Lipschitzian.

One can prove the following propositions:

(28) If $f_1|X$ is Lipschitzian and $f_2|X_1$ is Lipschitzian, then $(f_1 + f_2)|(X \cap X_1)$ is Lipschitzian.

(29) If $f_1|X$ is Lipschitzian and $f_2|X_1$ is Lipschitzian, then $(f_1 - f_2)|(X \cap X_1)$ is Lipschitzian.

Let us consider $S$, let $f$ be a Lipschitzian partial function from $\mathbb{R}$ to the carrier of $S$, and let us consider $p$. Note that $p f$ is Lipschitzian.

Next we state the proposition

(30) If $f|X$ is Lipschitzian and $X \subseteq \text{dom } f$, then $(p f)|X$ is Lipschitzian.

Let us consider $S$, let $f$ be a Lipschitzian partial function from $\mathbb{R}$ to the carrier of $S$. Note that $\|f\|$ is Lipschitzian.
One can prove the following proposition

(31) If $f|X$ is Lipschitzian, then $-f|X$ is Lipschitzian and $(-f)|X$ is Lipschitzian and $\|f||X$ is Lipschitzian.

Let us consider $S$. One can verify that every partial function from $\mathbb{R}$ to the carrier of $S$ which is constant is also Lipschitzian.

Let us consider $S$. Observe that every partial function from $\mathbb{R}$ to the carrier of $S$ which is Lipschitzian is also continuous.

Next we state two propositions:

(32) If there exists a point $r$ of $S$ such that rng $f = \{r\}$, then $f$ is continuous.

(33) For all points $r, p$ of $S$ such that for every $x_0$ such that $x_0 \in X$ holds $f_{x_0} = x_0 \cdot r + p$ holds $f|X$ is continuous.

References


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