

The Geometric Interior in Real Linear Spaces

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Summary. We introduce the notions of the geometric interior and the centre of mass for subsets of real linear spaces. We prove a number of theorems concerning these notions which are used in the theory of abstract simplicial complexes.

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The papers [1], [6], [11], [2], [5], [3], [4], [13], [7], [16], [10], [14], [12], [8], [9], and [15] provide the terminology and notation for this paper.

1. PRELIMINARIES

For simplicity, we adopt the following convention: x denotes a set, r, s denote real numbers, n denotes a natural number, V denotes a real linear space, v, u, w, p denote vectors of V , A, B denote subsets of V , A_1 denotes a finite subset of V , I denotes an affinely independent subset of V , I_1 denotes a finite affinely independent subset of V , F denotes a family of subsets of V , and L_1, L_2 denote linear combinations of V .

Next we state four propositions:

- (1) Let L be a linear combination of A . Suppose L is convex and $v \neq \sum L$ and $L(v) \neq 0$. Then there exists p such that $p \in \text{conv } A \setminus \{v\}$ and $\sum L = L(v) \cdot v + (1 - L(v)) \cdot p$ and $\frac{1}{L(v)} \cdot \sum L + (1 - \frac{1}{L(v)}) \cdot p = v$.
- (2) Let p_1, p_2, w_1, w_2 be elements of V . Suppose that $v, u \in \text{conv } I$ and $u \notin \text{conv } I \setminus \{p_1\}$ and $u \notin \text{conv } I \setminus \{p_2\}$ and $w_1 \in \text{conv } I \setminus \{p_1\}$ and

- $w_2 \in \text{conv } I \setminus \{p_2\}$ and $r \cdot u + (1 - r) \cdot w_1 = v$ and $s \cdot u + (1 - s) \cdot w_2 = v$ and $r < 1$ and $s < 1$. Then $w_1 = w_2$ and $r = s$.
- (3) Let L be a linear combination of A_1 . Suppose $A_1 \subseteq \text{conv } I_1$ and $\text{sum } L = 1$. Then
- (i) $\sum L \in \text{Affin } I_1$, and
 - (ii) for every element x of V there exists a finite sequence F of elements of \mathbb{R} and there exists a finite sequence G of elements of V such that $(\sum L \rightarrow I_1)(x) = \sum F$ and $\text{len } G = \text{len } F$ and G is one-to-one and $\text{rng } G = \text{the support of } L$ and for every n such that $n \in \text{dom } F$ holds $F(n) = L(G(n)) \cdot (G(n) \rightarrow I_1)(x)$.
- (4) For every subset A_2 of V such that A_2 is affine and $\text{conv } A \cap \text{conv } B \subseteq A_2$ and $\text{conv } A \setminus \{v\} \subseteq A_2$ and $v \notin A_2$ holds $\text{conv } A \setminus \{v\} \cap \text{conv } B = \text{conv } A \cap \text{conv } B$.

2. THE GEOMETRIC INTERIOR

Let V be a non empty RLS structure and let A be a subset of V . The functor $\text{Int } A$ yields a subset of V and is defined by:

- (Def. 1) $x \in \text{Int } A$ iff $x \in \text{conv } A$ and it is not true that there exists a subset B of V such that $B \subset A$ and $x \in \text{conv } B$.

Let V be a non empty RLS structure and let A be an empty subset of V . Observe that $\text{Int } A$ is empty.

We now state a number of propositions:

- (5) For every non empty RLS structure V and for every subset A of V holds $\text{Int } A \subseteq \text{conv } A$.
- (6) Let V be a real linear space-like non empty RLS structure and A be a subset of V . Then $\text{Int } A = A$ if and only if A is trivial.
- (7) If $A \subset B$, then $\text{conv } A$ misses $\text{Int } B$.
- (8) $\text{conv } A = \bigcup \{\text{Int } B : B \subseteq A\}$.
- (9) $\text{conv } A = \text{Int } A \cup \bigcup \{\text{conv } A \setminus \{v\} : v \in A\}$.
- (10) If $x \in \text{Int } A$, then there exists a linear combination L of A such that L is convex and $x = \sum L$.
- (11) For every linear combination L of A such that L is convex and $\sum L \in \text{Int } A$ holds the support of $L = A$.
- (12) For every linear combination L of I such that L is convex and the support of $L = I$ holds $\sum L \in \text{Int } I$.
- (13) If $\text{Int } A$ is non empty, then A is finite.
- (14) If $v \in I$ and $u \in \text{Int } I$ and $p \in \text{conv } I \setminus \{v\}$ and $r \cdot v + (1 - r) \cdot p = u$, then $p \in \text{Int}(I \setminus \{v\})$.

3. THE CENTER OF MASS

Let us consider V . The center of mass of V yielding a function from $2_+^{\text{the carrier of } V}$ into the carrier of V is defined by the conditions (Def. 2).

- (Def. 2)(i) For every non empty finite subset A of V holds (the center of mass of V)(A) = $\frac{1}{\#A} \cdot \sum A$, and
(ii) for every A such that A is infinite holds (the center of mass of V)(A) = 0_V .

One can prove the following propositions:

- (15) There exists a linear combination L of A_1 such that $\sum L = r \cdot \sum A_1$ and $\text{sum } L = r \cdot \overline{A_1}$ and $L = \mathbf{0}_{LC_V} + \cdot (A_1 \mapsto r)$.
- (16) If A_1 is non empty, then (the center of mass of V)(A_1) $\in \text{conv } A_1$.
- (17) If $\bigcup F$ is finite, then (the center of mass of V) $^\circ F \subseteq \text{conv } \bigcup F$.
- (18) If $v \in I_1$, then ((the center of mass of V)(I_1) $\rightarrow I_1$)(v) = $\frac{1}{\#I_1}$.
- (19) (The center of mass of V)(I_1) $\in I_1$ iff $\overline{I_1} = 1$.
- (20) If I_1 is non empty, then (the center of mass of V)(I_1) $\in \text{Int } I_1$.
- (21) If $A \subseteq I_1$ and (the center of mass of V)(I_1) $\in \text{Affin } A$, then $I_1 = A$.
- (22) If $v \in A_1$ and $A_1 \setminus \{v\}$ is non empty, then (the center of mass of V)(A_1) = $(1 - \frac{1}{\#A_1}) \cdot (\text{the center of mass of } V)_{A_1 \setminus \{v\}} + \frac{1}{\#A_1} \cdot v$.
- (23) If $\text{conv } A \subseteq \text{conv } I_1$ and I_1 is non empty and $\text{conv } A$ misses $\text{Int } I_1$, then there exists a subset B of V such that $B \subset I_1$ and $\text{conv } A \subseteq \text{conv } B$.
- (24) If $\sum L_1 \neq \sum L_2$ and $\text{sum } L_1 = \text{sum } L_2$, then there exists v such that $L_1(v) > L_2(v)$.
- (25) Let p be a real number. Suppose $(r \cdot L_1 + (1 - r) \cdot L_2)(v) \leq p \leq (s \cdot L_1 + (1 - s) \cdot L_2)(v)$. Then there exists a real number r_1 such that $(r_1 \cdot L_1 + (1 - r_1) \cdot L_2)(v) = p$ and if $r \leq s$, then $r \leq r_1 \leq s$ and if $s \leq r$, then $s \leq r_1 \leq r$.
- (26) If $v, u \in \text{conv } A$ and $v \neq u$, then there exist p, w, r such that $p \in A$ and $w \in \text{conv } A \setminus \{p\}$ and $0 \leq r < 1$ and $r \cdot u + (1 - r) \cdot w = v$.
- (27) $A \cup \{v\}$ is affinely independent iff A is affinely independent but $v \in A$ or $v \notin \text{Affin } A$.
- (28) If $A_1 \subseteq I$ and $v \in A_1$, then $(I \setminus \{v\}) \cup \{(\text{the center of mass of } V)(A_1)\}$ is an affinely independent subset of V .
- (29) Let F be a \subseteq -linear family of subsets of V . Suppose $\bigcup F$ is finite and affinely independent. Then (the center of mass of V) $^\circ F$ is an affinely independent subset of V .
- (30) Let F be a \subseteq -linear family of subsets of V . Suppose $\bigcup F$ is affinely independent and finite. Then $\text{Int}((\text{the center of mass of } V)^\circ F) \subseteq \text{Int } \bigcup F$.

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