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On L^p Space Formed by Real-Valued Partial Functions

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Summary. This article is the continuation of [31]. We define the set of L^p integrable functions – the set of all partial functions whose absolute value raised to the p-th power is integrable. We show that L^p integrable functions form the L^p space. We also prove Minkowski's inequality, Hölder's inequality and that L^p space is Banach space ([15], [27]).

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The notation and terminology used in this paper have been introduced in the following papers: [7], [8], [9], [10], [4], [1], [31], [6], [19], [20], [13], [28], [14], [2], [24], [3], [11], [25], [22], [21], [16], [32], [29], [23], [18], [17], [26], [30], [5], and [12].

1. Preliminaries on Powers of Numbers and Operations on Real Sequences

For simplicity, we follow the rules: X denotes a non empty set, x denotes an element of X, S denotes a σ-field of subsets of X, M denotes a σ-measure on S, f, g, f_1, g_1 denote partial functions from X to $\mathbb{R}$, and a, b, c denote real numbers.

The following propositions are true:

(1) For all positive real numbers m, n such that $\frac{1}{m} + \frac{1}{n} = 1$ holds $m > 1$.

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(2) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then $\int f \, \mathrm{d}M \in \mathbb{R}$ if and only if f is integrable on M.

Let r be a real number. We say that r is great or equal to 1 if and only if:

(Def. 1) $1 \leq r$.

Let us note that every real number which is great or equal to 1 is also positive.

One can verify that there exists a real number which is great or equal to 1.

In the sequel k denotes a positive real number.

We now state several propositions:

(3) For all real numbers a, b, p such that $0 < p$ and $0 \leq a < b$ holds $a^p < b^p$.

(4) If $a \geq 0$ and $b > 0$, then $a^b \geq 0$.

(5) If $a \geq 0$ and $b \geq 0$ and $c > 0$, then $(a \cdot b)^c = a^c \cdot b^c$.

(6) For all real numbers a, b and for every f such that f is non-negative and $a > 0$ and $b > 0$ holds $(f^a)^b = f^{a \cdot b}$.

(7) For all real numbers a, b and for every f such that f is non-negative and $a > 0$ and $b > 0$ holds $f^a \, f^b = f^{a+b}$.

(8) $f^1 = f$.

(9) Let s_1, s_2 be sequences of real numbers and k be a positive real number. Suppose that for every element n of $\mathbb{N}$ holds $s_1(n) = s_2(n)^k$ and $s_2(n) \geq 0$. Then s_1 is convergent if and only if s_2 is convergent.

(10) Let s_3 be a sequence of real numbers and n, m be elements of $\mathbb{N}$. If $m \leq n$, then $|(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa \in \mathbb{N}}(m)| \leq (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa \in \mathbb{N}}(m)$ and $|(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa \in \mathbb{N}}(m)| \leq (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(11) Let s_3, s_2 be sequences of real numbers and k be a positive real number. Suppose s_3 is convergent and for every element n of $\mathbb{N}$ holds $s_2(n) = |\lim s_3 - s_3(n)|^k$. Then s_2 is convergent and $\lim s_2 = 0$.

2. Real Linear Space of L^p Integrable Functions

Next we state two propositions:

(12) For every positive real number k and for every non empty set X holds $(X \longmapsto 0)^k = X \longmapsto 0$.

(13) For every partial function f from X to $\mathbb{R}$ and for every set D holds $|f{\restriction}D| = |f|{\restriction}D$.

Let us consider X and let f be a partial function from X to $\mathbb{R}$. Observe that $|f|$ is non-negative.

One can prove the following two propositions:

(14) For every partial function f from X to $\mathbb{R}$ such that f is non-negative holds $|f| = f$.

(15) If $X = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $0 = f(x)$, then f is integrable on M and $\int f \, \mathrm{d}M = 0$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p functions(M, k) yielding a non empty subset of $\mathrm{PFunct}_{\mathrm{RLS}}\, X$ is defined by the condition (Def. 2).

(Def. 2) L^p functions$(M, k) = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $\bigvee_{E_1 : \text{element of } S} (M({E_1}^{\mathrm{c}}) = 0 \ \wedge\ \operatorname{dom} f = E_1 \ \wedge\ f$ is measurable on $E_1 \ \wedge\ |f|^k$ is integrable on $M)\}$.

Next we state a number of propositions:

(16) For all real numbers a, b, k such that $k > 0$ holds $|a + b|^k \leq (|a| + |b|)^k$ and $(|a| + |b|)^k \leq (2 \cdot \max(|a|, |b|))^k$ and $|a + b|^k \leq (2 \cdot \max(|a|, |b|))^k$.

(17) For all real numbers a, b, k such that $a \geq 0$ and $b \geq 0$ and $k > 0$ holds $(\max(a, b))^k \leq a^k + b^k$.

(18) For every partial function f from X to $\mathbb{R}$ and for all real numbers a, b such that $b > 0$ holds $|a|^b \, |f|^b = |a \, f|^b$.

(19) Let f be a partial function from X to $\mathbb{R}$ and a, b be real numbers. If $a > 0$ and $b > 0$, then $a^b \, |f|^b = (a \, |f|)^b$.

(20) For every partial function f from X to $\mathbb{R}$ and for every real number k and for every set E holds $(f {\restriction} E)^k = f^k {\restriction} E$.

(21) For all real numbers a, b, k such that $k > 0$ holds $|a+b|^k \leq 2^k \cdot (|a|^k + |b|^k)$.

(22) Let k be a positive real number and f, g be partial functions from X to $\mathbb{R}$. Suppose f, $g \in L^p$ functions(M, k). Then $|f|^k$ is integrable on M and $|g|^k$ is integrable on M and $|f|^k + |g|^k$ is integrable on M.

(23) $X \longmapsto 0$ is a partial function from X to $\mathbb{R}$ and $X \longmapsto 0 \in L^p$ functions(M, k).

(24) Let k be a real number. Suppose $k > 0$. Let f, g be partial functions from X to $\mathbb{R}$ and x be an element of X. If $x \in \operatorname{dom} f \cap \operatorname{dom} g$, then $|f + g|^k(x) \leq (2^k \, (|f|^k + |g|^k))(x)$.

(25) If f, $g \in L^p$ functions(M, k), then $f + g \in L^p$ functions(M, k).

(26) If $f \in L^p$ functions(M, k), then $a \, f \in L^p$ functions(M, k).

(27) If f, $g \in L^p$ functions(M, k), then $f - g \in L^p$ functions(M, k).

(28) If $f \in L^p$ functions(M, k), then $|f| \in L^p$ functions(M, k).

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Note that L^p functions(M, k) is multiplicatively-closed and add closed.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. One can check that $\langle L^p\,\mathrm{functions}(M,k), 0_{\mathrm{PFunct}_{\mathrm{RLS}}\,X}(\in L^p\,\mathrm{functions}(M,k)), \mathrm{add}\,|(L^p\,\mathrm{functions}(M,k), \mathrm{PFunct}_{\mathrm{RLS}}\,X), \cdot_{L^p\,\mathrm{functions}(M,k)}\rangle$ is Abelian, add-associative, and real linear space-like.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor RLSp LpFunct(M,k) yields a strict Abelian add-associative real linear space-like non empty RLS structure and is defined by:

(Def. 3) RLSp LpFunct$(M,k) = \langle L^p\,\mathrm{functions}(M,k), 0_{\mathrm{PFunct}_{\mathrm{RLS}}\,X}(\in L^p\,\mathrm{functions}(M,k)), \mathrm{add}\,|(L^p\,\mathrm{functions}(M,k), \mathrm{PFunct}_{\mathrm{RLS}}\,X), \cdot_{L^p\,\mathrm{functions}(M,k)}\rangle$.

3. Preliminaries on Real Normed Space of L^p Integrable Functions

In the sequel v, u are vectors of RLSp LpFunct(M,k).

We now state three propositions:

(29) $(v) + (u) = v + u$.

(30) $a\,(u) = a \cdot u$.

(31) Suppose $f = u$. Then

(i) $u + (-1) \cdot u = (X \longmapsto 0)\restriction \mathrm{dom}\, f$, and

(ii) there exist partial functions v, g from X to $\mathbb{R}$ such that $v, g \in L^p\,\mathrm{functions}(M,k)$ and $v = u + (-1) \cdot u$ and $g = X \longmapsto 0$ and $v =^{M}_{\mathrm{a.e.}} g$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor AlmostZeroLpFunctions(M,k) yielding a non empty subset of RLSp LpFunct(M,k) is defined by:

(Def. 4) AlmostZeroLpFunctions$(M,k) = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $f \in L^p\,\mathrm{functions}(M,k) \ \wedge\ f =^{M}_{\mathrm{a.e.}} X \longmapsto 0\}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. One can check that AlmostZeroLpFunctions(M,k) is add closed and multiplicatively-closed.

Next we state the proposition

(32) $0_{\mathrm{RLSp\ LpFunct}(M,k)} = X \longmapsto 0$ and $0_{\mathrm{RLSp\ LpFunct}(M,k)} \in$ AlmostZeroLpFunctions(M,k).

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor RLSpAlmostZeroLpFunctions(M,k) yielding a non empty RLS structure is defined by:

(Def. 5) RLSpAlmostZeroLpFunctions$(M,k) = \langle$AlmostZeroLpFunctions(M,k), $0_{\mathrm{RLSp\ LpFunct}(M,k)}(\in$ AlmostZeroLpFunctions$(M,k))$, add $|($AlmostZeroLp

Functions(M,k), RLSp LpFunct$(M,k))$, $\cdot_{\text{AlmostZeroLpFunctions}(M,k)}\rangle$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that RLSp LpFunct(M,k) is strict, Abelian, add-associative, right zeroed, and real linear space-like.

In the sequel v, u are vectors of RLSpAlmostZeroLpFunctions(M,k).

One can prove the following two propositions:

(33) $(v)+(u)=v+u$.

(34) $a\,(u)=a\cdot u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\mathbb{R}$, and let k be a positive real number. The functor a.e-eq-class $L^p(f,M,k)$ yields a subset of L^p functions(M,k) and is defined as follows:

(Def. 6) a.e-eq-class $L^p(f,M,k)=\{h;h$ ranges over partial functions from X to $\mathbb{R}$: $h\in L^p$ functions$(M,k)\ \wedge\ f=^{M}_{\text{a.e.}}h\}$.

Next we state a number of propositions:

(35) If $f\in L^p$ functions(M,k), then there exists an element E of S such that $M(E^{\rm c})=0$ and $\operatorname{dom}f=E$ and f is measurable on E.

(36) If $g\in L^p$ functions(M,k) and $g=^{M}_{\text{a.e.}}f$, then $g\in$ a.e-eq-class $L^p(f,M,k)$.

(37) Suppose there exists an element E of S such that $M(E^{\rm c})=0$ and $E=\operatorname{dom}f$ and f is measurable on E and $g\in$ a.e-eq-class $L^p(f,M,k)$. Then $g=^{M}_{\text{a.e.}}f$ and $f\in L^p$ functions(M,k).

(38) If $f\in L^p$ functions(M,k), then $f\in$ a.e-eq-class $L^p(f,M,k)$.

(39) Suppose there exists an element E of S such that $M(E^{\rm c})=0$ and $E=\operatorname{dom}g$ and g is measurable on E and a.e-eq-class $L^p(f,M,k)\neq\emptyset$ and a.e-eq-class $L^p(f,M,k)=$ a.e-eq-class $L^p(g,M,k)$. Then $f=^{M}_{\text{a.e.}}g$.

(40) Suppose $f\in L^p$ functions(M,k) and there exists an element E of S such that $M(E^{\rm c})=0$ and $E=\operatorname{dom}g$ and g is measurable on E and a.e-eq-class $L^p(f,M,k)=$ a.e-eq-class $L^p(g,M,k)$. Then $f=^{M}_{\text{a.e.}}g$.

(41) If $f=^{M}_{\text{a.e.}}g$, then a.e-eq-class $L^p(f,M,k)=$ a.e-eq-class $L^p(g,M,k)$.

(42) If $f=^{M}_{\text{a.e.}}g$, then a.e-eq-class $L^p(f,M,k)=$ a.e-eq-class $L^p(g,M,k)$.

(43) If $f\in L^p$ functions(M,k) and $g\in$ a.e-eq-class $L^p(f,M,k)$, then a.e-eq-class $L^p(f,M,k)=$ a.e-eq-class $L^p(g,M,k)$.

(44) Suppose that there exists an element E of S such that $M(E^{\rm c})=0$ and $E=\operatorname{dom}f$ and f is measurable on E and there exists an element E of S such that $M(E^{\rm c})=0$ and $E=\operatorname{dom}f_1$ and f_1 is measurable on E and there exists an element E of S such that $M(E^{\rm c})=0$ and $E=\operatorname{dom}g$ and g is measurable on E and there exists an element E of S such that $M(E^{\rm c})=0$ and $E=\operatorname{dom}g_1$ and g_1 is measurable on

E and a.e-eq-class $L^p(f, M, k)$ is non empty and a.e-eq-class $L^p(g, M, k)$ is non empty and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(f_1, M, k)$ and a.e-eq-class $L^p(g, M, k) =$ a.e-eq-class $L^p(g_1, M, k)$. Then a.e-eq-class $L^p(f + g, M, k) =$ a.e-eq-class $L^p(f_1 + g_1, M, k)$.

(45) If f, f_1, g, $g_1 \in L^p$ functions(M, k) and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(f_1, M, k)$ and a.e-eq-class $L^p(g, M, k) =$ a.e-eq-class $L^p(g_1, M, k)$, then a.e-eq-class $L^p(f + g, M, k) =$ a.e-eq-class $L^p(f_1 + g_1, M, k)$.

(46) Suppose that

(i) there exists an element E of S such that $M(E^c) = 0$ and dom $f = E$ and f is measurable on E,

(ii) there exists an element E of S such that $M(E^c) = 0$ and dom $g = E$ and g is measurable on E,

(iii) a.e-eq-class $L^p(f, M, k)$ is non empty, and

(iv) a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$.

Then a.e-eq-class $L^p(a\, f, M, k) =$ a.e-eq-class $L^p(a\, g, M, k)$.

(47) If f, $g \in L^p$ functions(M, k) and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$, then a.e-eq-class $L^p(a\, f, M, k) =$ a.e-eq-class $L^p(a\, g, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor CosetSet(M, k) yielding a non empty family of subsets of L^p functions(M, k) is defined by:

(Def. 7) CosetSet$(M, k) = \{$a.e-eq-class $L^p(f, M, k); f$ ranges over partial functions from X to $\mathbb{R}$: $f \in L^p$ functions$(M, k)\}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor addCoset(M, k) yields a binary operation on CosetSet(M, k) and is defined by the condition (Def. 8).

(Def. 8) Let A, B be elements of CosetSet(M, k) and a, b be partial functions from X to $\mathbb{R}$. If $a \in A$ and $b \in B$, then (addCoset$(M, k))(A, B) =$ a.e-eq-class $L^p(a + b, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor zeroCoset(M, k) yields an element of CosetSet(M, k) and is defined as follows:

(Def. 9) zeroCoset$(M, k) =$ a.e-eq-class $L^p(X \longmapsto 0, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor lmultCoset(M, k) yielding a function from $\mathbb{R} \times$ CosetSet(M, k) into CosetSet(M, k) is defined by the condition (Def. 10).

(Def. 10) Let z be an element of $\mathbb{R}$, A be an element of $\mathrm{CosetSet}(M,k)$, and f be a partial function from X to $\mathbb{R}$. If $f \in A$, then $(\mathrm{lmultCoset}(M,k))(z, A) = \text{a.e-eq-class}\, L^p(z\, f, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor Pre- L^p -Space(M,k) yielding a strict RLS structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of Pre- L^p -Space$(M,k) = \mathrm{CosetSet}(M,k)$,

(ii) the addition of Pre- L^p -Space$(M,k) = \mathrm{addCoset}(M,k)$,

(iii) $0_{\text{Pre- } L^p \text{-Space}(M,k)} = \mathrm{zeroCoset}(M,k)$, and

(iv) the external multiplication of Pre- L^p -Space$(M,k) = \mathrm{lmultCoset}(M,k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that Pre- L^p -Space(M,k) is non empty.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that Pre- L^p -Space(M,k) is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

4. Real Normed Space of L^p Integrable Functions

The following propositions are true:

(48) If f, $g \in L^p\,\mathrm{functions}(M,k)$ and $f =^{M}_{\text{a.e.}} g$, then $\int |f|^k \,\mathrm{d}M = \int |g|^k \,\mathrm{d}M$.

(49) If $f \in L^p\,\mathrm{functions}(M,k)$, then $\int |f|^k \,\mathrm{d}M \in \mathbb{R}$ and $0 \leq \int |f|^k \,\mathrm{d}M$.

(50) If there exists a vector x of Pre- L^p -Space(M,k) such that f, $g \in x$, then $f =^{M}_{\text{a.e.}} g$ and f, $g \in L^p\,\mathrm{functions}(M,k)$.

(51) Let k be a positive real number. Then there exists a function N_1 from the carrier of Pre- L^p -Space(M,k) into $\mathbb{R}$ such that for every point x of Pre- L^p -Space(M,k) holds there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and there exists a real number r such that $r = \int |f|^k \,\mathrm{d}M$ and $N_1(x) = r^{\frac{1}{k}}$.

In the sequel x denotes a point of Pre- L^p -Space(M,k).

We now state two propositions:

(52) If $f \in x$, then $|f|^k$ is integrable on M and $f \in L^p\,\mathrm{functions}(M,k)$.

(53) If f, $g \in x$, then $f =^{M}_{\text{a.e.}} g$ and $\int |f|^k \,\mathrm{d}M = \int |g|^k \,\mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p -Norm(M,k) yielding a function from the carrier of Pre- L^p -Space(M,k) into $\mathbb{R}$ is defined by the condition (Def. 12).

(Def. 12) Let x be a point of Pre- L^p -Space(M, k). Then there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and there exists a real number r such that $r = \int |f|^k \, \mathrm{d}M$ and $(L^p\text{-Norm}(M, k))(x) = r^{\frac{1}{k}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p -Space(M, k) yields a non empty normed structure and is defined by:

(Def. 13) L^p -Space(M, k) = $\langle$the carrier of Pre- L^p -Space(M, k), the zero of Pre- L^p -Space(M, k), the addition of Pre- L^p -Space(M, k), the external multiplication of Pre- L^p -Space(M, k), L^p -Norm$(M, k)\rangle$.

In the sequel x, y denote points of L^p -Space(M, k).

One can prove the following propositions:

(54)(i) There exists a partial function f from X to $\mathbb{R}$ such that $f \in L^p$ functions(M, k) and $x =$ a.e-eq-class $L^p(f, M, k)$, and

(ii) for every partial function f from X to $\mathbb{R}$ such that $f \in x$ there exists a real number r such that $0 \leq r = \int |f|^k \, \mathrm{d}M$ and $\|x\| = r^{\frac{1}{k}}$.

(55) If $f \in x$ and $g \in y$, then $f + g \in x + y$ and if $f \in x$, then $a\, f \in a \cdot x$.

(56) If $f \in x$, then $x =$ a.e-eq-class $L^p(f, M, k)$ and there exists a real number r such that $0 \leq r = \int |f|^k \, \mathrm{d}M$ and $\|x\| = r^{\frac{1}{k}}$.

(57) $X \longmapsto 0 \in$ the L^1 functions of M.

(58) If $f \in L^p$ functions(M, k) and $\int |f|^k \, \mathrm{d}M = 0$, then $f =^{M}_{\text{a.e.}} X \longmapsto 0$.

(59) $\int |X \longmapsto 0|^k \, \mathrm{d}M = 0$.

(60) Let m, n be positive real numbers. Suppose $\frac{1}{m} + \frac{1}{n} = 1$ and $f \in L^p$ functions(M, m) and $g \in L^p$ functions(M, n). Then $f\, g \in$ the L^1 functions of M and $f\, g$ is integrable on M.

(61) Let m, n be positive real numbers. Suppose $\frac{1}{m} + \frac{1}{n} = 1$ and $f \in L^p$ functions(M, m) and $g \in L^p$ functions(M, n). Then there exists a real number r_1 such that $r_1 = \int |f|^m \, \mathrm{d}M$ and there exists a real number r_2 such that $r_2 = \int |g|^n \, \mathrm{d}M$ and $\int |f\, g| \, \mathrm{d}M \leq {r_1}^{\frac{1}{m}} \cdot {r_2}^{\frac{1}{n}}$.

(62) Let m be a positive real number and r_1, r_2, r_3 be elements of $\mathbb{R}$. Suppose $1 \leq m$ and f, $g \in L^p$ functions(M, m) and $r_1 = \int |f|^m \, \mathrm{d}M$ and $r_2 = \int |g|^m \, \mathrm{d}M$ and $r_3 = \int |f + g|^m \, \mathrm{d}M$. Then ${r_3}^{\frac{1}{m}} \leq {r_1}^{\frac{1}{m}} + {r_2}^{\frac{1}{m}}$.

Let k be a great or equal to 1 real number, let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Note that L^p -Space(M, k) is reflexive, discernible, real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

5. Preliminaries on Completeness of L^p Space

The following propositions are true:

(63) Let S_1 be a sequence of L^p-Space(M, k). Then there exists a sequence F_1 of partial functions from X into $\mathbb{R}$ such that for every element n of $\mathbb{N}$ holds
$F_1(n) \in L^p$ functions(M, k) and $F_1(n) \in S_1(n)$ and $S_1(n) =$ a.e-eq-class $L^p(F_1(n), M, k)$ and there exists a real number r such that $r = \int |F_1(n)|^k \,\mathrm{d}M$ and $\|S_1(n)\| = r^{\frac{1}{k}}$.

(64) Let S_1 be a sequence of L^p-Space(M, k). Then there exists a sequence F_1 of partial functions from X into $\mathbb{R}$ with the same dom such that for every element n of $\mathbb{N}$ holds
$F_1(n) \in L^p$ functions(M, k) and $F_1(n) \in S_1(n)$ and $S_1(n) =$ a.e-eq-class $L^p(F_1(n), M, k)$ and there exists a real number r such that $0 \leq r = \int |F_1(n)|^k \,\mathrm{d}M$ and $\|S_1(n)\| = r^{\frac{1}{k}}$.

(65) Let X be a real normed space, S_1 be a sequence of X, and S_0 be a point of X. If $\|S_1 - S_0\|$ is convergent and $\lim \|S_1 - S_0\| = 0$, then S_1 is convergent and $\lim S_1 = S_0$.

(66) Let X be a real normed space and S_1 be a sequence of X. Suppose S_1 is Cauchy sequence by norm. Then there exists an increasing function N from $\mathbb{N}$ into $\mathbb{N}$ such that for all elements i, j of $\mathbb{N}$ if $j \geq N(i)$, then $\|S_1(j) - S_1(N(i))\| < 2^{-i}$.

(67) Let F be a sequence of partial functions from X into $\mathbb{R}$. Suppose that for every natural number m holds $F(m) \in L^p$ functions(M, k). Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \in L^p$ functions(M, k).

(68) Let F be a sequence of partial functions from X into $\mathbb{R}$. Suppose that for every natural number m holds $F(m)$ is non-negative. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is non-negative.

(69) Let F be a sequence of partial functions from X into $\mathbb{R}$, x be an element of X, and n, m be natural numbers. Suppose F has the same dom and $x \in \operatorname{dom} F(0)$ and for every natural number k holds $F(k)$ is non-negative and $n \leq m$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)(x)$.

(70) For every sequence F of partial functions from X into $\mathbb{R}$ such that F has the same dom holds $|F|$ has the same dom.

(71) Let k be a great or equal to 1 real number and S_1 be a sequence of L^p-Space(M, k). If S_1 is Cauchy sequence by norm, then S_1 is convergent.

Let us consider X, S, M and let k be a great or equal to 1 real number. Observe that L^p-Space(M, k) is complete.

6. Relations between L^1 Space and L^p Space

One can prove the following propositions:

(72) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{CosetSet}\, M = \mathrm{CosetSet}(M, 1)$.

(73) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{addCoset}\, M = \mathrm{addCoset}(M, 1)$.

(74) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{zeroCoset}\, M = \mathrm{zeroCoset}(M, 1)$.

(75) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{lmultCoset}\, M = \mathrm{lmultCoset}(M, 1)$.

(76) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\text{pre-}L\text{-Space}\, M = \text{Pre-}\, L^p\text{-Space}(M, 1)$.

(77) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $L^1\text{-Norm}(M) = L^p\text{-Norm}(M, 1)$.

(78) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $L^1\text{-Space}(M) = L^p\text{-Space}(M, 1)$.

References

[1] Jonathan Backer, Piotr Rudnicki, and Christoph Schwarzweller. Ring ideals. *Formalized Mathematics*, 9(**3**):565–582, 2001.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[4] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.

[5] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.

[6] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.

[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[8] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[12] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[13] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006, doi:10.2478/v10037-006-0008-x.

[14] Noboru Endou, Yasunari Shidama, and Keiko Narita. Egoroff's theorem. *Formalized Mathematics*, 16(**1**):57–63, 2008, doi:10.2478/v10037-008-0009-z.

[15] P. R. Halmos. *Measure Theory.* Springer-Verlag, 1987.

[16] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[17] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[18] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[19] Keiko Narita, Noboru Endou, and Yasunari Shidama. Integral of complex-valued measurable function. *Formalized Mathematics*, 16(**4**):319–324, 2008, doi:10.2478/v10037-008-0039-6.
[20] Keiko Narita, Noboru Endou, and Yasunari Shidama. Lebesgue's convergence theorem of complex-valued function. *Formalized Mathematics*, 17(**2**):137–145, 2009, doi: 10.2478/v10037-009-0015-9.
[21] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[22] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[23] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(**1**):17–21, 1992.
[24] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[25] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.
[26] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.
[27] Walter Rudin. *Real and Complex Analysis*. Mc Graw-Hill, Inc., 1974.
[28] Yasunari Shidama and Noboru Endou. Integral of real-valued measurable function. *Formalized Mathematics*, 14(**4**):143–152, 2006, doi:10.2478/v10037-006-0018-8.
[29] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[30] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[31] Yasushige Watase, Noboru Endou, and Yasunari Shidama. On L^1 space formed by real-valued partial functions. *Formalized Mathematics*, 16(**4**):361–369, 2008, doi:10.2478/v10037-008-0044-9.
[32] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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