

Abstract Simplicial Complexes

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Summary. In this article we define the notion of abstract simplicial complexes and operations on them. We introduce the following basic notions: simplex, face, vertex, degree, skeleton, subdivision and substructure, and prove some of their properties.

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The articles [2], [5], [6], [10], [8], [14], [1], [7], [3], [4], [11], [13], [16], [12], [15], and [9] provide the notation and terminology for this paper.

1. PRELIMINARIES

For simplicity, we adopt the following convention: x, y, X, Y, Z are sets, D is a non empty set, n, k are natural numbers, and i, i_1, i_2 are integers.

Let us consider X . We introduce X has empty element as an antonym of X has non empty elements.

Note that there exists a set which is empty and finite-membered and every set which is empty is also finite-membered. Let X be a finite set. Note that $\{X\}$ is finite-membered and 2^X is finite-membered. Let Y be a finite set. Observe that $\{X, Y\}$ is finite-membered.

Let X be a finite-membered set. Observe that every subset of X is finite-membered. Let Y be a finite-membered set. One can check that $X \cup Y$ is finite-membered.

Let X be a finite finite-membered set. Note that $\bigcup X$ is finite.

One can verify the following observations:

- * every set which is empty is also subset-closed,

- * every set which has empty element is also non empty,
- * every set which is non empty and subset-closed has also empty element,
and
- * there exists a set which has empty element.

Let us consider X . Observe that $\text{SubFin}(X)$ is finite-membered and there exists a family of subsets of X which is subset-closed, finite, and finite-membered.

Let X be a subset-closed set. One can check that $\text{SubFin}(X)$ is subset-closed.

Next we state the proposition

- (1) Y is subset-closed iff for every X such that $X \in Y$ holds $2^X \subseteq Y$.

Let A, B be subset-closed sets. Note that $A \cup B$ is subset-closed and $A \cap B$ is subset-closed.

Let us consider X . The subset-closure of X yields a subset-closed set and is defined by the conditions (Def. 1).

- (Def. 1)(i) $X \subseteq$ the subset-closure of X , and
(ii) for every Y such that $X \subseteq Y$ and Y is subset-closed holds the subset-closure of $X \subseteq Y$.

The following proposition is true

- (2) $x \in$ the subset-closure of X iff there exists y such that $x \subseteq y$ and $y \in X$.

Let us consider X and let F be a family of subsets of X . Then the subset-closure of F is a subset-closed family of subsets of X .

Observe that the subset-closure of \emptyset is empty. Let X be a non empty set. Note that the subset-closure of X is non empty.

Let X be a set with a non-empty element. One can check that the subset-closure of X has a non-empty element.

Let X be a finite-membered set. Note that the subset-closure of X is finite-membered.

The following propositions are true:

- (3) If $X \subseteq Y$ and Y is subset-closed, then the subset-closure of $X \subseteq Y$.
(4) The subset-closure of $\{X\} = 2^X$.
(5) The subset-closure of $X \cup Y = (\text{the subset-closure of } X) \cup (\text{the subset-closure of } Y)$.
(6) X is finer than Y iff the subset-closure of $X \subseteq$ the subset-closure of Y .
(7) If X is subset-closed, then the subset-closure of $X = X$.
(8) If the subset-closure of $X \subseteq X$, then X is subset-closed.

Let us consider Y, X and let n be a set. The subsets of X and Y with cardinality limited by n yields a family of subsets of Y and is defined by the condition (Def. 2).

- (Def. 2) Let A be a subset of Y . Then $A \in$ the subsets of X and Y with cardinality limited by n if and only if $A \in X$ and $\text{Card } A \subseteq \text{Card } n$.

Let us consider D . One can verify that there exists a family of subsets of D which is finite, subset-closed, and finite-membered and has a non-empty element.

Let us consider Y , X and let n be a finite set. One can check that the subsets of X and Y with cardinality limited by n is finite-membered.

Let us consider Y , let X be a subset-closed set, and let n be a set. Note that the subsets of X and Y with cardinality limited by n is subset-closed.

Let us consider Y , let X be a set with empty element, and let n be a set. One can check that the subsets of X and Y with cardinality limited by n has empty element.

Let us consider D , let X be a subset-closed family of subsets of D with a non-empty element, and let n be a non empty set. Note that the subsets of X and D with cardinality limited by n has a non-empty element.

Let us consider X , let Y be a family of subsets of X , and let n be a set. We introduce the subsets of Y with cardinality limited by n as a synonym of the subsets of Y and X with cardinality limited by n .

Let us observe that every set which is empty is also \subseteq -linear and there exists a set which is empty and \subseteq -linear.

Let X be a \subseteq -linear set. Note that every subset of X is \subseteq -linear.

The following propositions are true:

- (9) If X is non empty, finite, and \subseteq -linear, then $\bigcup X \in X$.
- (10) For every finite \subseteq -linear set X such that X has non empty elements holds $\text{Card } X \subseteq \text{Card } \bigcup X$.
- (11) If X is \subseteq -linear and $\bigcup X$ misses x , then $X \cup \{\bigcup X \cup x\}$ is \subseteq -linear.
- (12) Let X be a non empty set. Then there exists a family Y of subsets of X such that
 - (i) Y is \subseteq -linear and has non empty elements,
 - (ii) $X \in Y$,
 - (iii) $\text{Card } X = \text{Card } Y$, and
 - (iv) for every Z such that $Z \in Y$ and $\text{Card } Z \neq 1$ there exists x such that $x \in Z$ and $Z \setminus \{x\} \in Y$.
- (13) Let Y be a family of subsets of X . Suppose Y is finite and \subseteq -linear and has non empty elements and $X \in Y$. Then there exists a family Y' of subsets of X such that
 - (i) $Y \subseteq Y'$,
 - (ii) Y' is \subseteq -linear and has non empty elements,
 - (iii) $\text{Card } X = \text{Card } Y'$, and
 - (iv) for every Z such that $Z \in Y'$ and $\text{Card } Z \neq 1$ there exists x such that $x \in Z$ and $Z \setminus \{x\} \in Y'$.

2. SIMPLICIAL COMPLEXES

A simplicial complex structure is a topological structure.

In the sequel K denotes a simplicial complex structure.

Let us consider K and let A be a subset of K . We introduce A is simplex-like as a synonym of A is open.

Let us consider K and let S be a family of subsets of K . We introduce S is simplex-like as a synonym of S is open.

Let us consider K . One can check that there exists a family of subsets of K which is empty and simplex-like.

The following proposition is true

- (14) For every family S of subsets of K holds S is simplex-like iff $S \subseteq$ the topology of K .

Let us consider K and let v be an element of K . We say that v is vertex-like if and only if:

- (Def. 3) There exists a subset S of K such that S is simplex-like and $v \in S$.

Let us consider K . The functor $\text{Vertices } K$ yielding a subset of K is defined by:

- (Def. 4) For every element v of K holds $v \in \text{Vertices } K$ iff v is vertex-like.

Let K be a simplicial complex structure. A vertex of K is an element of $\text{Vertices } K$.

Let K be a simplicial complex structure. We say that K is finite-vertices if and only if:

- (Def. 5) $\text{Vertices } K$ is finite.

Let us consider K . We say that K is locally-finite if and only if:

- (Def. 6) For every vertex v of K holds $\{S \subseteq K: S \text{ is simplex-like} \wedge v \in S\}$ is finite.

Let us consider K . We say that K is empty-membered if and only if:

- (Def. 7) The topology of K is empty-membered.

We say that K has non empty elements if and only if:

- (Def. 8) The topology of K has non empty elements.

Let us consider K . We introduce K has a non-empty element as an antonym of K is empty-membered. We introduce K has empty element as an antonym of K has non empty elements.

Let us consider X . A simplicial complex structure is said to be a simplicial complex structure of X if:

- (Def. 9) $\Omega_{\text{it}} \subseteq X$.

Let us consider X and let K_1 be a simplicial complex structure of X . We say that K_1 is total if and only if:

(Def. 10) $\Omega_{(K_1)} = X$.

One can check the following observations:

- * every simplicial complex structure which has empty element is also non void,
- * every simplicial complex structure which has a non-empty element is also non void,
- * every simplicial complex structure which is non void and empty-membered has also empty element,
- * every simplicial complex structure which is non void and subset-closed has also empty element,
- * every simplicial complex structure which is empty-membered is also subset-closed and finite-vertices,
- * every simplicial complex structure which is finite-vertices is also locally-finite and finite-degree, and
- * every simplicial complex structure which is locally-finite and subset-closed is also finite-membered.

Let us consider X . Observe that there exists a simplicial complex structure of X which is empty, void, empty-membered, and strict.

Let us consider D . Note that there exists a simplicial complex structure of D which is non empty, non void, total, empty-membered, and strict and there exists a simplicial complex structure of D which is non empty, total, finite-vertices, subset-closed, and strict and has empty element and a non-empty element.

Let us observe that there exists a simplicial complex structure which is non empty, finite-vertices, subset-closed, and strict and has empty element and a non-empty element.

Let K be a simplicial complex structure with a non-empty element. Observe that Vertices K is non empty.

Let K be a finite-vertices simplicial complex structure. Note that every family of subsets of K which is simplex-like is also finite.

Let K be a finite-membered simplicial complex structure. Note that every family of subsets of K which is simplex-like is also finite-membered.

Next we state several propositions:

- (15) Vertices K is empty iff K is empty-membered.
- (16) Vertices $K = \bigcup (\text{the topology of } K)$.
- (17) For every subset S of K such that S is simplex-like holds $S \subseteq \text{Vertices } K$.
- (18) If K is finite-vertices, then the topology of K is finite.
- (19) If the topology of K is finite and K is non finite-vertices, then K is non finite-membered.
- (20) If K is subset-closed and the topology of K is finite, then K is finite-vertices.

3. THE SIMPLICIAL COMPLEX GENERATED ON THE SET

Let us consider X and let Y be a family of subsets of X . The complex of Y yielding a strict simplicial complex structure of X is defined as follows:

(Def. 11) The complex of $Y = \langle X, \text{the subset-closure of } Y \rangle$.

Let us consider X and let Y be a family of subsets of X . One can verify that the complex of Y is total and subset-closed.

Let us consider X and let Y be a non empty family of subsets of X . Note that the complex of Y has empty element.

Let us consider X and let Y be a finite-membered family of subsets of X . Note that the complex of Y is finite-membered.

Let us consider X and let Y be a finite finite-membered family of subsets of X . Observe that the complex of Y is finite-vertices.

One can prove the following proposition

(21) If K is subset-closed, then the topological structure of $K = \text{the complex of the topology of } K$.

Let us consider X . A simplicial complex of X is a finite-membered subset-closed simplicial complex structure of X .

Let K be a non void simplicial complex structure. A simplex of K is a simplex-like subset of K .

Let K be a simplicial complex structure with empty element. One can check that every subset of K which is empty is also simplex-like and there exists a simplex of K which is empty.

Let K be a non void finite-membered simplicial complex structure. Note that there exists a simplex of K which is finite.

4. THE DEGREE OF SIMPLICIAL COMPLEXES

Let us consider K . The functor $\text{degree}(K)$ yields an extended real number and is defined as follows:

- (Def. 12)(i) For every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq \text{degree}(K) + 1$ and there exists a subset S of K such that S is simplex-like and $\text{Card } S = \text{degree}(K) + 1$ if K is non void and finite-degree,
- (ii) $\text{degree}(K) = -1$ if K is void,
- (iii) $\text{degree}(K) = +\infty$, otherwise.

Let K be a finite-degree simplicial complex structure. Note that $\text{degree}(K) + 1$ is natural and $\text{degree}(K)$ is integer.

The following propositions are true:

(22) $\text{degree}(K) = -1$ iff K is empty-membered.

(23) $-1 \leq \text{degree}(K)$.

- (24) For every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq \text{degree}(K) + 1$.
- (25) Suppose K is non void or $i \geq -1$. Then $\text{degree}(K) \leq i$ if and only if the following conditions are satisfied:
 - (i) K is finite-membered, and
 - (ii) for every finite subset S of K such that S is simplex-like holds $\overline{\overline{S}} \leq i + 1$.
- (26) For every finite subset A of X holds $\text{degree}(\text{the complex of } \{A\}) = \overline{\overline{A}} - 1$.

5. SUBCOMPLEXES

Let us consider X and let K_1 be a simplicial complex structure of X . A simplicial complex of X is said to be a subsimplicial complex of K_1 if:

(Def. 13) $\Omega_{\text{it}} \subseteq \Omega_{(K_1)}$ and the topology of it \subseteq the topology of K_1 .

In the sequel K_1 denotes a simplicial complex structure of X and S_1 denotes a subsimplicial complex of K_1 .

Let us consider X , K_1 . One can check that there exists a subsimplicial complex of K_1 which is empty, void, and strict.

Let us consider X and let K_1 be a void simplicial complex structure of X . Observe that every subsimplicial complex of K_1 is void.

Let us consider D and let K_2 be a non void subset-closed simplicial complex structure of D . Note that there exists a subsimplicial complex of K_2 which is non void.

Let us consider X and let K_1 be a finite-vertices simplicial complex structure of X . One can check that every subsimplicial complex of K_1 is finite-vertices.

Let us consider X and let K_1 be a finite-degree simplicial complex structure of X . Note that every subsimplicial complex of K_1 is finite-degree.

Next we state several propositions:

- (27) Every subsimplicial complex of S_1 is a subsimplicial complex of K_1 .
- (28) Let A be a subset of K_1 and S be a finite-membered family of subsets of A . Suppose the subset-closure of $S \subseteq$ the topology of K_1 . Then the complex of S is a strict subsimplicial complex of K_1 .
- (29) Let K_1 be a subset-closed simplicial complex structure of X , A be a subset of K_1 , and S be a finite-membered family of subsets of A . Suppose $S \subseteq$ the topology of K_1 . Then the complex of S is a strict subsimplicial complex of K_1 .
- (30) Let Y_1, Y_2 be families of subsets of X . Suppose Y_1 is finite-membered and finer than Y_2 . Then the complex of Y_1 is a subsimplicial complex of the complex of Y_2 .
- (31) Vertices $S_1 \subseteq$ Vertices K_1 .
- (32) $\text{degree}(S_1) \leq \text{degree}(K_1)$.

Let us consider X, K_1, S_1 . We say that S_1 is maximal if and only if:

(Def. 14) For every subset A of S_1 such that $A \in$ the topology of K_1 holds A is simplex-like.

We now state the proposition

(33) S_1 is maximal iff $2^{\Omega(S_1)} \cap$ the topology of $K_1 \subseteq$ the topology of S_1 .

Let us consider X, K_1 . Note that there exists a subsimplicial complex of K_1 which is maximal and strict.

We now state three propositions:

(34) Let S_2 be a subsimplicial complex of S_1 . Suppose S_1 is maximal and S_2 is maximal. Then S_2 is a maximal subsimplicial complex of K_1 .

(35) Let S_2 be a subsimplicial complex of S_1 . If S_2 is a maximal subsimplicial complex of K_1 , then S_2 is maximal.

(36) Let K_3, K_4 be maximal subsimplicial complexes of K_1 .

Suppose $\Omega_{(K_3)} = \Omega_{(K_4)}$. Then the topological structure of $K_3 =$ the topological structure of K_4 .

Let us consider X , let K_1 be a subset-closed simplicial complex structure of X , and let A be a subset of K_1 . Let us assume that $2^A \cap$ the topology of K_1 is finite-membered. The functor $K_1 \upharpoonright A$ yields a maximal strict subsimplicial complex of K_1 and is defined as follows:

(Def. 15) $\Omega_{K_1 \upharpoonright A} = A$.

In the sequel S_3 denotes a simplicial complex of X .

Let us consider X, S_3 and let A be a subset of S_3 . Then $S_3 \upharpoonright A$ is a maximal strict subsimplicial complex of S_3 and it can be characterized by the condition:

(Def. 16) $\Omega_{S_3 \upharpoonright A} = A$.

The following four propositions are true:

(37) For every subset A of S_3 holds the topology of $S_3 \upharpoonright A = 2^A \cap$ the topology of S_3 .

(38) For all subsets A, B of S_3 and for every subset B' of $S_3 \upharpoonright A$ such that $B' = B$ holds $S_3 \upharpoonright A \upharpoonright B' = S_3 \upharpoonright B$.

(39) $S_3 \upharpoonright \Omega_{(S_3)} =$ the topological structure of S_3 .

(40) For all subsets A, B of S_3 such that $A \subseteq B$ holds $S_3 \upharpoonright A$ is a subsimplicial complex of $S_3 \upharpoonright B$.

Let us observe that every integer is finite.

6. THE SKELETON OF A SIMPLICIAL COMPLEX

Let us consider X, K_1 and let i be a real number. The skeleton of K_1 and i yielding a simplicial complex structure of X is defined by the condition (Def. 17).

(Def. 17) The skeleton of K_1 and i is the complex of the subsets of the topology of K_1 with cardinality limited by $i + 1$.

Let us consider X , K_1 . Observe that the skeleton of K_1 and -1 is empty-membered. Let us consider i . Note that the skeleton of K_1 and i is finite-degree.

Let us consider X , let K_1 be an empty-membered simplicial complex structure of X , and let us consider i . One can check that the skeleton of K_1 and i is empty-membered.

Let us consider D , let K_2 be a non void subset-closed simplicial complex structure of D , and let us consider i . One can check that the skeleton of K_2 and i is non void.

One can prove the following proposition

(41) If $-1 \leq i_1 \leq i_2$, then the skeleton of K_1 and i_1 is a subsimplicial complex of the skeleton of K_1 and i_2 .

Let us consider X , let K_1 be a subset-closed simplicial complex structure of X , and let us consider i . Then the skeleton of K_1 and i is a subsimplicial complex of K_1 .

We now state several propositions:

(42) If K_1 is subset-closed and the skeleton of K_1 and i is empty-membered, then K_1 is empty-membered or $i = -1$.

(43) $\text{degree}(\text{the skeleton of } K_1 \text{ and } i) \leq \text{degree}(K_1)$.

(44) If $-1 \leq i$, then $\text{degree}(\text{the skeleton of } K_1 \text{ and } i) \leq i$.

(45) If $-1 \leq i$ and the skeleton of K_1 and i is the topological structure of K_1 , then $\text{degree}(K_1) \leq i$.

(46) If K_1 is subset-closed and $\text{degree}(K_1) \leq i$, then the skeleton of K_1 and i is the topological structure of K_1 .

In the sequel K is a non void subset-closed simplicial complex structure.

Let us consider K and let i be a real number. Let us assume that i is integer.

A finite simplex of K is said to be a simplex of i and K if:

(Def. 18)(i) $\overline{\text{it}} = i + 1$ if $-1 \leq i \leq \text{degree}(K)$,

(ii) it is empty, otherwise.

Let us consider K . Note that every simplex of -1 and K is empty.

The following three propositions are true:

(47) For every simplex S of i and K such that S is non empty holds i is natural.

(48) Every finite simplex S of K is a simplex of $\overline{S} - 1$ and K .

(49) Let K be a non void subset-closed simplicial complex structure of D , S be a non void subsimplicial complex of K , i be an integer, and A be a simplex of i and S . If A is non empty or $i \leq \text{degree}(S)$ or $\text{degree}(S) = \text{degree}(K)$, then A is a simplex of i and K .

Let us consider K and let i be a real number. Let us assume that i is integer and $i \leq \text{degree}(K)$. Let S be a simplex of i and K . A simplex of $\max(i-1, -1)$ and K is said to be a face of S if:

(Def. 19) It $\subseteq S$.

One can prove the following proposition

- (50) Let S be a simplex of n and K . Suppose $n \leq \text{degree}(K)$. Then X is a face of S if and only if there exists x such that $x \in S$ and $S \setminus \{x\} = X$.

7. THE SUBDIVISION OF A SIMPLICIAL COMPLEX

In the sequel P is a function.

Let us consider X, K_1, P . The functor $\text{subdivision}(P, K_1)$ yields a strict simplicial complex structure of X and is defined by the conditions (Def. 20).

- (Def. 20)(i) $\Omega_{\text{subdivision}(P, K_1)} = \Omega_{(K_1)}$, and
(ii) for every subset A of $\text{subdivision}(P, K_1)$ holds A is simplex-like iff there exists a \subseteq -linear finite simplex-like family S of subsets of K_1 such that $A = P^\circ S$.

Let us consider X, K_1, P . One can verify that $\text{subdivision}(P, K_1)$ is subset-closed and finite-membered and has empty element.

Let us consider X , let K_1 be a void simplicial complex structure of X , and let us consider P . Observe that $\text{subdivision}(P, K_1)$ is empty-membered.

The following propositions are true:

- (51) $\text{degree}(\text{subdivision}(P, K_1)) \leq \text{degree}(K_1) + 1$.
(52) If $\text{dom } P$ has non empty elements, then $\text{degree}(\text{subdivision}(P, K_1)) \leq \text{degree}(K_1)$.

Let us consider X , let K_1 be a finite-degree simplicial complex structure of X , and let us consider P . Note that $\text{subdivision}(P, K_1)$ is finite-degree.

Let us consider X , let K_1 be a finite-vertices simplicial complex structure of X , and let us consider P . One can check that $\text{subdivision}(P, K_1)$ is finite-vertices.

One can prove the following propositions:

- (53) Let K_1 be a subset-closed simplicial complex structure of X and given P . Suppose that
(i) $\text{dom } P$ has non empty elements, and
(ii) for every n such that $n \leq \text{degree}(K_1)$ there exists a subset S of K_1 such that S is simplex-like and $\text{Card } S = n + 1$ and $2_+^S \subseteq \text{dom } P$ and $P^\circ 2_+^S$ is a subset of K_1 and $P|2_+^S$ is one-to-one.
Then $\text{degree}(\text{subdivision}(P, K_1)) = \text{degree}(K_1)$.
(54) If $Y \subseteq Z$, then $\text{subdivision}(P|Y, K_1)$ is a subsimplicial complex of $\text{subdivision}(P|Z, K_1)$.

- (55) If $\text{dom } P \cap \text{the topology of } K_1 \subseteq Y$, then $\text{subdivision}(P|Y, K_1) = \text{subdivision}(P, K_1)$.
- (56) If $Y \subseteq Z$, then $\text{subdivision}(Y|P, K_1)$ is a subsimplicial complex of $\text{subdivision}(Z|P, K_1)$.
- (57) If $P^\circ(\text{the topology of } K_1) \subseteq Y$, then $\text{subdivision}(Y|P, K_1) = \text{subdivision}(P, K_1)$.
- (58) $\text{subdivision}(P, S_1)$ is a subsimplicial complex of $\text{subdivision}(P, K_1)$.
- (59) For every subset A of $\text{subdivision}(P, K_1)$ such that $\text{dom } P \subseteq \text{the topology of } S_1$ and $A = \Omega_{(S_1)}$ holds $\text{subdivision}(P, S_1) = \text{subdivision}(P, K_1)|A$.
- (60) Let K_3, K_4 be simplicial complex structures of X . Suppose the topological structure of $K_3 = \text{the topological structure of } K_4$. Then $\text{subdivision}(P, K_3) = \text{subdivision}(P, K_4)$.

Let us consider X, K_1, P, n . The functor $\text{subdivision}(n, P, K_1)$ yielding a simplicial complex structure of X is defined by the condition (Def. 21).

- (Def. 21) There exists a function F such that
- (i) $F(0) = K_1$,
 - (ii) $F(n) = \text{subdivision}(n, P, K_1)$,
 - (iii) $\text{dom } F = \mathbb{N}$, and
 - (iv) for every k and for every simplicial complex structure K'_1 of X such that $K'_1 = F(k)$ holds $F(k+1) = \text{subdivision}(P, K'_1)$.

Next we state several propositions:

- (61) $\text{subdivision}(0, P, K_1) = K_1$.
- (62) $\text{subdivision}(1, P, K_1) = \text{subdivision}(P, K_1)$.
- (63) For every natural number n_1 such that $n_1 = n + k$ holds $\text{subdivision}(n_1, P, K_1) = \text{subdivision}(n, P, \text{subdivision}(k, P, K_1))$.
- (64) $\Omega_{\text{subdivision}(n, P, K_1)} = \Omega_{(K_1)}$.
- (65) $\text{subdivision}(n, P, S_1)$ is a subsimplicial complex of $\text{subdivision}(n, P, K_1)$.

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