

Equivalence of Deterministic and Nondeterministic Epsilon Automata

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Summary. Based on concepts introduced in [14], semiautomata and left-languages, automata and right-languages, and languages accepted by automata are defined. The powerset construction is defined for transition systems, semiautomata and automata. Finally, the equivalence of deterministic and nondeterministic epsilon automata is shown.

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The terminology and notation used in this paper have been introduced in the following articles: [1], [8], [2], [11], [6], [18], [7], [9], [17], [16], [15], [4], [10], [13], [3], [12], [5], and [14].

1. PRELIMINARIES

For simplicity, we adopt the following convention: x, y, X denote sets, E denotes a non empty set, e denotes an element of E , u, u_1, v, v_1, v_2, w denote elements of E^ω , F denotes a subset of E^ω , i, k, l denote natural numbers, \mathfrak{T} denotes a non empty transition-system over F , and S, T denote subsets of \mathfrak{T} .

One can prove the following propositions:

- (1) If $i \geq k + l$, then $i \geq k$.
- (2) For all finite sequences a, b such that $a \frown b = a$ or $b \frown a = a$ holds $b = \emptyset$.
- (3) For all finite sequences p, q such that $k \in \text{dom } p$ and $\text{len } p + 1 = \text{len } q$ holds $k + 1 \in \text{dom } q$.
- (4) If $\text{len } u = 1$, then there exists e such that $\langle e \rangle = u$ and $e = u(0)$.

- (5) If $k \neq 0$ and $\text{len } u \leq k + 1$, then there exist v_1, v_2 such that $\text{len } v_1 \leq k$ and $\text{len } v_2 \leq k$ and $u = v_1 \hat{\ } v_2$.
- (6) For all finite 0-sequences p, q such that $\langle x \rangle \hat{\ } p = \langle y \rangle \hat{\ } q$ holds $x = y$ and $p = q$.
- (7) If $\text{len } u > 0$, then there exist e, u_1 such that $u = \langle e \rangle \hat{\ } u_1$.

Let us consider E . One can verify that $\text{Lex } E$ is non empty.

Next we state three propositions:

- (8) $\langle \rangle_E \notin \text{Lex } E$.
- (9) $u \in \text{Lex } E$ iff $\text{len } u = 1$.
- (10) If $u \neq v$ and $u, v \in \text{Lex } E$, then it is not true that there exists w such that $u \hat{\ } w = v$ or $w \hat{\ } u = v$.

2. TRANSITION SYSTEMS OVER $\text{Lex } E$

The following propositions are true:

- (11) For every transition-system \mathfrak{T} over $\text{Lex } E$ holds $\langle \rangle_E \notin \text{rng dom}$ (the transition of \mathfrak{T}).
- (12) For every transition-system \mathfrak{T} over $\text{Lex } E$ such that the transition of \mathfrak{T} is a function holds \mathfrak{T} is deterministic.

3. POWERSSET CONSTRUCTION FOR TRANSITION SYSTEMS

Let us consider E, F, \mathfrak{T} . The functor $\text{bool } \mathfrak{T}$ yielding a strict transition-system over $\text{Lex } E$ is defined by the conditions (Def. 1).

- (Def. 1)(i) The carrier of $\text{bool } \mathfrak{T} = 2^{\text{the carrier of } \mathfrak{T}}$, and
- (ii) for all S, w, T holds $\langle \langle S, w \rangle, T \rangle \in \text{the transition of } \text{bool } \mathfrak{T}$ iff $\text{len } w = 1$ and $T = w\text{-succ}_{\mathfrak{T}}(S)$.

Let us consider E, F, \mathfrak{T} . Note that $\text{bool } \mathfrak{T}$ is non empty and deterministic.

Let us consider E, F and let \mathfrak{T} be a finite non empty transition-system over F . One can check that $\text{bool } \mathfrak{T}$ is finite.

The following two propositions are true:

- (13) If $x, \langle e \rangle \Rightarrow_{\text{bool } \mathfrak{T}}^* y, \langle \rangle_E$, then $x, \langle e \rangle \Rightarrow_{\text{bool } \mathfrak{T}} y, \langle \rangle_E$.
- (14) If $\text{len } w = 1$, then $X = w\text{-succ}_{\mathfrak{T}}(S)$ iff $S, w \Rightarrow_{\text{bool } \mathfrak{T}}^* X$.

4. SEMIAUTOMATA

Let us consider E, F . We consider semiautomata over F as extensions of transition-system over F as systems

\langle a carrier, a transition, an initial state \rangle ,

where the carrier is a set, the transition is a relation between the carrier $\times F$ and the carrier, and the initial state is a subset of the carrier.

Let us consider E, F and let \mathfrak{S} be a semiautomaton over F . We say that \mathfrak{S} is deterministic if and only if:

(Def. 2) The transition-system of \mathfrak{S} is deterministic and $\text{Card}(\text{the initial state of } \mathfrak{S}) = 1$.

Let us consider E, F . One can check that there exists a semiautomaton over F which is strict, non empty, finite, and deterministic.

In the sequel \mathfrak{S} is a non empty semiautomaton over F .

Let us consider E, F, \mathfrak{S} . Observe that the transition-system of \mathfrak{S} is non empty.

Let us consider E, F, \mathfrak{S} . The functor $\text{bool } \mathfrak{S}$ yields a strict semiautomaton over $\text{Lex } E$ and is defined by the conditions (Def. 3).

(Def. 3)(i) The transition-system of $\text{bool } \mathfrak{S} = \text{bool}(\text{the transition-system of } \mathfrak{S})$,
and
(ii) the initial state of $\text{bool } \mathfrak{S} = \{\langle \rangle_E\text{-succ}_{\mathfrak{S}}(\text{the initial state of } \mathfrak{S})\}$.

Let us consider E, F, \mathfrak{S} . Observe that $\text{bool } \mathfrak{S}$ is non empty and deterministic.

The following proposition is true

(15) The carrier of $\text{bool } \mathfrak{S} = 2^{\text{the carrier of } \mathfrak{S}}$.

Let us consider E, F and let \mathfrak{S} be a finite non empty semiautomaton over F . Observe that $\text{bool } \mathfrak{S}$ is finite.

5. LEFT-LANGUAGES

Let us consider E, F, \mathfrak{S} and let Q be a subset of \mathfrak{S} . The functor $\text{left-Lang } Q$ yields a subset of E^ω and is defined as follows:

(Def. 4) $\text{left-Lang } Q = \{w : Q \text{ meets } w\text{-succ}_{\mathfrak{S}}(\text{the initial state of } \mathfrak{S})\}$.

Next we state the proposition

(16) For every subset Q of \mathfrak{S} holds $w \in \text{left-Lang } Q$ iff Q meets $w\text{-succ}_{\mathfrak{S}}(\text{the initial state of } \mathfrak{S})$.

6. AUTOMATA

Let us consider E, F . We consider automata over F as extensions of semiautomaton over F as systems

\langle a carrier, a transition, an initial state, final states \rangle ,

where the carrier is a set, the transition is a relation between the carrier $\times F$ and the carrier, the initial state is a subset of the carrier, and the final states constitute a subset of the carrier.

Let us consider E, F and let \mathfrak{A} be a automaton over F . We say that \mathfrak{A} is deterministic if and only if:

(Def. 5) The semiautomaton of \mathfrak{A} is deterministic.

Let us consider E, F . Observe that there exists a automaton over F which is strict, non empty, finite, and deterministic.

In the sequel \mathfrak{A} denotes a non empty automaton over F and p, q denote elements of \mathfrak{A} .

Let us consider E, F, \mathfrak{A} . One can check that the transition-system of \mathfrak{A} is non empty and the semiautomaton of \mathfrak{A} is non empty.

Let us consider E, F, \mathfrak{A} . The functor $\text{bool}\mathfrak{A}$ yields a strict automaton over $\text{Lex } E$ and is defined by the conditions (Def. 6).

(Def. 6)(i) The semiautomaton of $\text{bool}\mathfrak{A} = \text{bool}$ (the semiautomaton of \mathfrak{A}), and
(ii) the final states of $\text{bool}\mathfrak{A} = \{Q; Q \text{ ranges over elements of } \text{bool}\mathfrak{A} : Q \text{ meets the final states of } \mathfrak{A}\}$.

Let us consider E, F, \mathfrak{A} . One can check that $\text{bool}\mathfrak{A}$ is non empty and deterministic.

The following proposition is true

(17) The carrier of $\text{bool}\mathfrak{A} = 2^{\text{the carrier of } \mathfrak{A}}$.

Let us consider E, F and let \mathfrak{A} be a finite non empty automaton over F . Note that $\text{bool}\mathfrak{A}$ is finite.

7. RIGHT-LANGUAGES

Let us consider E, F, \mathfrak{A} and let Q be a subset of \mathfrak{A} . The functor $\text{right-Lang } Q$ yields a subset of E^ω and is defined as follows:

(Def. 7) $\text{right-Lang } Q = \{w : w\text{-succ}_{\mathfrak{A}}(Q) \text{ meets the final states of } \mathfrak{A}\}$.

The following proposition is true

(18) For every subset Q of \mathfrak{A} holds $w \in \text{right-Lang } Q$ iff $w\text{-succ}_{\mathfrak{A}}(Q)$ meets the final states of \mathfrak{A} .

8. LANGUAGES ACCEPTED BY AUTOMATA

Let us consider E, F, \mathfrak{A} . The language generated by \mathfrak{A} yielding a subset of E^ω is defined by the condition (Def. 8).

(Def. 8) The language generated by $\mathfrak{A} = \{u : \bigvee_{p,q} (p \in \text{the initial state of } \mathfrak{A} \wedge q \in \text{the final states of } \mathfrak{A} \wedge p, u \Rightarrow_{\mathfrak{A}}^* q)\}$.

The following propositions are true:

- (19) $w \in \text{the language generated by } \mathfrak{A}$ if and only if there exist p, q such that $p \in \text{the initial state of } \mathfrak{A}$ and $q \in \text{the final states of } \mathfrak{A}$ and $p, w \Rightarrow_{\mathfrak{A}}^* q$.
- (20) $w \in \text{the language generated by } \mathfrak{A}$ if and only if $w\text{-succ}_{\mathfrak{A}}(\text{the initial state of } \mathfrak{A})$ meets the final states of \mathfrak{A} .
- (21) The language generated by $\mathfrak{A} = \text{left-Lang}(\text{the final states of } \mathfrak{A})$.
- (22) The language generated by $\mathfrak{A} = \text{right-Lang}(\text{the initial state of } \mathfrak{A})$.

9. EQUIVALENCE OF DETERMINISTIC AND NONDETERMINISTIC EPSILON AUTOMATA

In the sequel \mathfrak{T} denotes a non empty transition-system over $\text{Lex } E \cup \{\langle \rangle_E\}$.

One can prove the following three propositions:

- (23) For every reduction sequence R w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $R(1)_2 = \langle e \rangle \wedge u$ and $R(\text{len } R)_2 = \langle \rangle_E$ holds $R(2)_2 = \langle e \rangle \wedge u$ or $R(2)_2 = u$.
- (24) For every reduction sequence R w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $R(1)_2 = u$ and $R(\text{len } R)_2 = \langle \rangle_E$ holds $\text{len } R > \text{len } u$.
- (25) For every reduction sequence R w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $R(1)_2 = u \wedge v$ and $R(\text{len } R)_2 = \langle \rangle_E$ there exists l such that $l \in \text{dom } R$ and $R(l)_2 = v$.

Let us consider E, u, v . The functor $\text{chop}(u, v)$ yielding an element of E^ω is defined by:

- (Def. 9)(i) For every w such that $w \wedge v = u$ holds $\text{chop}(u, v) = w$ if there exists w such that $w \wedge v = u$,
- (ii) $\text{chop}(u, v) = u$, otherwise.

The following propositions are true:

- (26) Let p be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. Suppose $p(1) = \langle x, u \wedge w \rangle$ and $p(\text{len } p) = \langle y, v \wedge w \rangle$. Then there exists a reduction sequence q w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $q(1) = \langle x, u \rangle$ and $q(\text{len } q) = \langle y, v \rangle$.
- (27) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, u \wedge w \rangle$ to $\langle y, v \wedge w \rangle$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, u \rangle$ to $\langle y, v \rangle$.
- (28) If $x, u \wedge w \Rightarrow_{\mathfrak{T}}^* y, v \wedge w$, then $x, u \Rightarrow_{\mathfrak{T}}^* y, v$.
- (29) For all elements p, q of \mathfrak{T} such that $p, u \wedge v \Rightarrow_{\mathfrak{T}}^* q$ there exists an element r of \mathfrak{T} such that $p, u \Rightarrow_{\mathfrak{T}}^* r$ and $r, v \Rightarrow_{\mathfrak{T}}^* q$.

$$(30) \quad w \cap v\text{-succ}_{\mathfrak{T}}(X) = v\text{-succ}_{\mathfrak{T}}(w\text{-succ}_{\mathfrak{T}}(X)).$$

$$(31) \quad \text{bool } \mathfrak{T} \text{ is a non empty transition-system over } \text{Lex } E \cup \{\langle \rangle_E\}.$$

$$(32) \quad w\text{-succ}_{\text{bool } \mathfrak{T}}(\{v\text{-succ}_{\mathfrak{T}}(X)\}) = \{v \cap w\text{-succ}_{\mathfrak{T}}(X)\}.$$

In the sequel \mathfrak{S} denotes a non empty semiautomaton over $\text{Lex } E \cup \{\langle \rangle_E\}$.

One can prove the following proposition

$$(33) \quad w\text{-succ}_{\text{bool } \mathfrak{S}}(\{\langle \rangle_E\text{-succ}_{\mathfrak{S}}(X)\}) = \{w\text{-succ}_{\mathfrak{S}}(X)\}.$$

In the sequel \mathfrak{A} denotes a non empty automaton over $\text{Lex } E \cup \{\langle \rangle_E\}$ and P denotes a subset of \mathfrak{A} .

Next we state several propositions:

$$(34) \quad \text{If } x \in \text{the final states of } \mathfrak{A} \text{ and } x \in P, \text{ then } P \in \text{the final states of } \text{bool } \mathfrak{A}.$$

$$(35) \quad \text{If } X \in \text{the final states of } \text{bool } \mathfrak{A}, \text{ then } X \text{ meets the final states of } \mathfrak{A}.$$

$$(36) \quad \text{The initial state of } \text{bool } \mathfrak{A} = \{\langle \rangle_E\text{-succ}_{\mathfrak{A}}(\text{the initial state of } \mathfrak{A})\}.$$

$$(37) \quad w\text{-succ}_{\text{bool } \mathfrak{A}}(\{\langle \rangle_E\text{-succ}_{\mathfrak{A}}(X)\}) = \{w\text{-succ}_{\mathfrak{A}}(X)\}.$$

$$(38) \quad w\text{-succ}_{\text{bool } \mathfrak{A}}(\text{the initial state of } \text{bool } \mathfrak{A}) = \{w\text{-succ}_{\mathfrak{A}}(\text{the initial state of } \mathfrak{A})\}.$$

$$(39) \quad \text{The language generated by } \mathfrak{A} = \text{the language generated by } \text{bool } \mathfrak{A}.$$

$$(40) \quad \text{Let } \mathfrak{A} \text{ be a non empty automaton over } \text{Lex } E \cup \{\langle \rangle_E\}. \text{ Then there exists a non empty deterministic automaton } \mathfrak{A}_1 \text{ over } \text{Lex } E \text{ such that the language generated by } \mathfrak{A} = \text{the language generated by } \mathfrak{A}_1.$$

$$(41) \quad \text{Let } \mathfrak{F} \text{ be a non empty finite automaton over } \text{Lex } E \cup \{\langle \rangle_E\}. \text{ Then there exists a non empty deterministic finite automaton } \mathfrak{A}_2 \text{ over } \text{Lex } E \text{ such that the language generated by } \mathfrak{F} = \text{the language generated by } \mathfrak{A}_2.$$

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