# Riemann Integral of Functions from $\mathbb{R}$ into $\mathcal{R}^{n}$ 

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#### Abstract

Summary. In this article, we define the Riemann Integral of functions from $\mathbb{R}$ into $\mathcal{R}^{n}$, and prove the linearity of this operator. The presented method is based on [21].


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The articles [22], [1], [23], [5], [6], [15], [20], [24], [7], [17], [16], [2], [4], [3], [8], [18], [9], [12], [10], [14], [13], [19], and [11] provide the notation and terminology for this paper.

## 1. Preliminaries

Let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathbb{R}$, let $S$ be a non empty Division of $A$, and let $D$ be an element of $S$. A finite sequence of elements of $\mathbb{R}$ is said to be a middle volume of $f$ and $D$ if it satisfies the conditions (Def. 1).
(Def. 1)(i) len it $=$ len $D$, and
(ii) for every natural number $i$ such that $i \in \operatorname{dom} D$ there exists an element $r$ of $\mathbb{R}$ such that $r \in \operatorname{rng}(f \upharpoonright \operatorname{divset}(D, i))$ and $\operatorname{it}(i)=r \cdot \operatorname{vol}(\operatorname{divset}(D, i))$.
Let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathbb{R}$, let $S$ be a non empty Division of $A$, let $D$ be an element of $S$, and let $F$ be a middle volume of $f$ and $D$. The functor middle_sum $(f, F)$ yielding a real number is defined as follows:
(Def. 2) middle_sum $(f, F)=\sum F$.
We now state four propositions:
(1) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}, S$ be a non empty Division of $A, D$ be an element of $S$, and $F$ be a middle volume of $f$ and $D$. If $f \upharpoonright A$ is lower bounded, then lower_sum $(f, D) \leq$ middle_sum $(f, F)$.
(2) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}, S$ be a non empty Division of $A, D$ be an element of $S$, and $F$ be a middle volume of $f$ and $D$. If $f \upharpoonright A$ is upper bounded, then middle_sum $(f, F) \leq$ upper_sum $(f, D)$.
(3) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}, S$ be a non empty Division of $A, D$ be an element of $S$, and $e$ be a real number. Suppose $f \upharpoonright A$ is lower bounded and $0<e$. Then there exists a middle volume $F$ of $f$ and $D$ such that middle_sum $(f, F) \leq \operatorname{lower} \_$_sum $(f, D)+e$.
(4) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}, S$ be a non empty Division of $A, D$ be an element of $S$, and $e$ be a real number. Suppose $f \upharpoonright A$ is upper bounded and $0<e$. Then there exists a middle volume $F$ of $f$ and $D$ such that upper_sum $(f, D)-e \leq \operatorname{middle\_ sum~}(f, F)$.
Let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathbb{R}$, and let $T$ be a DivSequence of $A$. A function from $\mathbb{N}$ into $\mathbb{R}^{*}$ is said to be a middle volume sequence of $f$ and $T$ if:
(Def. 3) For every element $k$ of $\mathbb{N}$ holds it $(k)$ is a middle volume of $f$ and $T(k)$.
Let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathbb{R}$, let $T$ be a DivSequence of $A$, let $S$ be a middle volume sequence of $f$ and $T$, and let $k$ be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of $f$ and $T(k)$.

Let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathbb{R}$, let $T$ be a DivSequence of $A$, and let $S$ be a middle volume sequence of $f$ and $T$. The functor middle_sum $(f, S)$ yields a sequence of real numbers and is defined by:
(Def. 4) For every element $i$ of $\mathbb{N}$ holds (middle_sum $(f, S))(i)=\operatorname{middle}$ _sum $(f, S(i))$.
We now state several propositions:
(5) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}, T$ be a DivSequence of $A, S$ be a middle volume sequence of $f$ and $T$, and $i$ be an element of $\mathbb{N}$. If $f \upharpoonright A$ is lower bounded, then (lower_sum $(f, T))(i) \leq$ (middle_sum $(f, S))(i)$.
(6) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}, T$ be a DivSequence of $A, S$ be a middle volume sequence of $f$ and $T$, and $i$ be an element of $\mathbb{N}$. If $f \upharpoonright A$ is upper bounded, then (middle_sum $(f, S))(i) \leq$ (upper_sum $(f, T))(i)$.
(7) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}$, $T$ be a DivSequence of $A$, and $e$ be an element of $\mathbb{R}$. Suppose $0<e$ and $f \upharpoonright A$ is lower bounded. Then there exists a middle volume sequence $S$ of
$f$ and $T$ such that for every element $i$ of $\mathbb{N}$ holds (middle_sum $(f, S))(i) \leq$ (lower_sum $(f, T))(i)+e$.
(8) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}$, $T$ be a DivSequence of $A$, and $e$ be an element of $\mathbb{R}$. Suppose $0<e$ and $f \upharpoonright A$ is upper bounded. Then there exists a middle volume sequence $S$ of $f$ and $T$ such that for every element $i$ of $\mathbb{N}$ holds (upper_sum $(f, T))(i)-e \leq$ (middle_sum $(f, S))(i)$.
(9) Let $A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathbb{R}, T$ be a DivSequence of $A$, and $S$ be a middle volume sequence of $f$ and $T$. Suppose $f$ is bounded and $f$ is integrable on $A$ and $\delta_{T}$ is convergent and $\lim \left(\delta_{T}\right)=0$. Then middle_sum $(f, S)$ is convergent and $\lim$ middle_sum $(f, S)=$ integral $f$.
(10) Let $A$ be a closed-interval subset of $\mathbb{R}$ and $f$ be a function from $A$ into $\mathbb{R}$. Suppose $f$ is bounded. Then $f$ is integrable on $A$ if and only if there exists a real number $I$ such that for every DivSequence $T$ of $A$ and for every middle volume sequence $S$ of $f$ and $T$ such that $\delta_{T}$ is convergent and $\lim \left(\delta_{T}\right)=0$ holds middle_sum $(f, S)$ is convergent and lim middle_sum $(f, S)=I$.
Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathcal{R}^{n}$, let $S$ be a non empty Division of $A$, and let $D$ be an element of $S$. A finite sequence of elements of $\mathcal{R}^{n}$ is said to be a middle volume of $f$ and $D$ if it satisfies the conditions (Def. 5).
(Def. 5)(i) $\quad$ len it $=\operatorname{len} D$, and
(ii) for every natural number $i$ such that $i \in \operatorname{dom} D$ there exists an element $r$ of $\mathcal{R}^{n}$ such that $r \in \operatorname{rng}(f \upharpoonright \operatorname{divset}(D, i))$ and $\operatorname{it}(i)=\operatorname{vol}(\operatorname{divset}(D, i)) \cdot r$.
Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathcal{R}^{n}$, let $S$ be a non empty Division of $A$, let $D$ be an element of $S$, and let $F$ be a middle volume of $f$ and $D$. The functor middle_sum $(f, F)$ yielding an element of $\mathcal{R}^{n}$ is defined by the condition (Def. 6).
(Def. 6) Let $i$ be an element of $\mathbb{N}$. Suppose $i \in \operatorname{Seg} n$. Then there exists a finite sequence $F_{1}$ of elements of $\mathbb{R}$ such that $F_{1}=\operatorname{proj}(i, n) \cdot F$ and (middle_sum $(f, F))(i)=\sum F_{1}$.
Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathcal{R}^{n}$, and let $T$ be a DivSequence of $A$. A function from $\mathbb{N}$ into $\left(\mathcal{R}^{n}\right)^{*}$ is said to be a middle volume sequence of $f$ and $T$ if:
(Def. 7) For every element $k$ of $\mathbb{N}$ holds it $(k)$ is a middle volume of $f$ and $T(k)$.
Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a function from $A$ into $\mathcal{R}^{n}$, let $T$ be a DivSequence of $A$, let $S$ be a middle volume sequence of $f$ and $T$, and let $k$ be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of $f$ and $T(k)$.

Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, let $f$ be a
function from $A$ into $\mathcal{R}^{n}$, let $T$ be a DivSequence of $A$, and let $S$ be a middle volume sequence of $f$ and $T$. The functor middle_sum $(f, S)$ yields a sequence of $\left\langle\mathcal{E}^{n},\|\cdot\|\right\rangle$ and is defined as follows:
(Def. 8) For every element $i$ of $\mathbb{N}$ holds (middle_sum $(f, S))(i)=$ middle_sum $(f, S(i))$.
Let $n$ be an element of $\mathbb{N}$, let $Z$ be a non empty set, and let $f, g$ be partial functions from $Z$ to $\mathcal{R}^{n}$. The functor $f+g$ yielding a partial function from $Z$ to $\mathcal{R}^{n}$ is defined by:
(Def. 9) $\operatorname{dom}(f+g)=\operatorname{dom} f \cap \operatorname{dom} g$ and for every element $c$ of $Z$ such that $c \in \operatorname{dom}(f+g)$ holds $(f+g)_{c}=f_{c}+g_{c}$.
The functor $f-g$ yielding a partial function from $Z$ to $\mathcal{R}^{n}$ is defined as follows:
(Def. 10) $\operatorname{dom}(f-g)=\operatorname{dom} f \cap \operatorname{dom} g$ and for every element $c$ of $Z$ such that $c \in \operatorname{dom}(f-g)$ holds $(f-g)_{c}=f_{c}-g_{c}$.
Let $n$ be an element of $\mathbb{N}$, let $r$ be a real number, let $Z$ be a non empty set, and let $f$ be a partial function from $Z$ to $\mathcal{R}^{n}$. The functor $r f$ yielding a partial function from $Z$ to $\mathcal{R}^{n}$ is defined as follows:
(Def. 11) $\operatorname{dom}(r f)=\operatorname{dom} f$ and for every element $c$ of $Z$ such that $c \in \operatorname{dom}(r f)$ holds $(r f)_{c}=r \cdot f_{c}$.

## 2. Definition of Riemann Integral of Functions from $\mathbb{R}$ into $\mathcal{R}^{n}$

Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, and let $f$ be a function from $A$ into $\mathcal{R}^{n}$. We say that $f$ is bounded if and only if:
(Def. 12) For every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{Seg} n \operatorname{holds} \operatorname{proj}(i, n) \cdot f$ is bounded.
Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, and let $f$ be a function from $A$ into $\mathcal{R}^{n}$. We say that $f$ is integrable if and only if:
(Def. 13) For every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{Seg} n \operatorname{holds} \operatorname{proj}(i, n) \cdot f$ is integrable on $A$.
Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, and let $f$ be a function from $A$ into $\mathcal{R}^{n}$. The functor integral $f$ yielding an element of $\mathcal{R}^{n}$ is defined by:
(Def. 14) dom integral $f=\operatorname{Seg} n$ and for every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{Seg} n$ holds (integral $f)(i)=$ integral $\operatorname{proj}(i, n) \cdot f$.
One can prove the following two propositions:
(11) Let $n$ be an element of $\mathbb{N}, A$ be a closed-interval subset of $\mathbb{R}, f$ be a function from $A$ into $\mathcal{R}^{n}, T$ be a DivSequence of $A$, and $S$ be a middle volume sequence of $f$ and $T$. Suppose $f$ is bounded and integrable and $\delta_{T}$ is convergent and $\lim \left(\delta_{T}\right)=0$. Then middle_sum $(f, S)$ is convergent and $\lim$ middle_sum $(f, S)=\operatorname{integral} f$.
(12) Let $n$ be an element of $\mathbb{N}, A$ be a closed-interval subset of $\mathbb{R}$, and $f$ be a function from $A$ into $\mathcal{R}^{n}$. Suppose $f$ is bounded. Then $f$ is integrable if and only if there exists an element $I$ of $\mathcal{R}^{n}$ such that for every DivSequence $T$ of $A$ and for every middle volume sequence $S$ of $f$ and $T$ such that $\delta_{T}$ is convergent and $\lim \left(\delta_{T}\right)=0$ holds middle_sum $(f, S)$ is convergent and $\lim$ middle_sum $(f, S)=I$.
Let $n$ be an element of $\mathbb{N}$ and let $f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}$. We say that $f$ is bounded if and only if:
(Def. 15) For every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{Seg} n \operatorname{holds} \operatorname{proj}(i, n) \cdot f$ is bounded.
Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, and let $f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}$. We say that $f$ is integrable on $A$ if and only if:
(Def. 16) For every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{Seg} n \operatorname{holds} \operatorname{proj}(i, n) \cdot f$ is integrable on $A$.
Let $n$ be an element of $\mathbb{N}$, let $A$ be a closed-interval subset of $\mathbb{R}$, and let $f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}$. The functor $\int_{A} f(x) d x$ yields an element of $\mathcal{R}^{n}$ and is defined by:
(Def. 17) $\operatorname{dom} \int_{A} f(x) d x=\operatorname{Seg} n$ and for every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{Seg} n$ $\operatorname{holds}\left(\int_{A} f(x) d x\right)(i)=\int_{A}(\operatorname{proj}(i, n) \cdot f)(x) d x$.
The following two propositions are true:
(13) Let $n$ be an element of $\mathbb{N}, A$ be a closed-interval subset of $\mathbb{R}, f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}$, and $g$ be a function from $A$ into $\mathcal{R}^{n}$. Suppose $f \upharpoonright A=g$. Then $f$ is integrable on $A$ if and only if $g$ is integrable.
(14) Let $n$ be an element of $\mathbb{N}, A$ be a closed-interval subset of $\mathbb{R}, f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}$, and $g$ be a function from $A$ into $\mathcal{R}^{n}$. If $f \upharpoonright A=g$, then $\int_{A} f(x) d x=$ integral $g$.
Let $a, b$ be real numbers, let $n$ be an element of $\mathbb{N}$, and let $f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}$. The functor $\int_{a}^{b} f(x) d x$ yielding an element of $\mathcal{R}^{n}$ is defined as follows:
(Def. 18) $\operatorname{dom} \int_{a}^{b} f(x) d x=\operatorname{Seg} n$ and for every element $i$ of $\mathbb{N}$ such that $i \in \operatorname{Seg} n$ holds $\left(\int_{a}^{b} f(x) d x\right)(i)=\int_{a}^{b}(\operatorname{proj}(i, n) \cdot f)(x) d x$.

## 3. Linearity of Integration Operator

We now state several propositions:
(15) Let $n$ be an element of $\mathbb{N}, f_{1}, f_{2}$ be partial functions from $\mathbb{R}$ to $\mathcal{R}^{n}$, and $i$ be an element of $\mathbb{N}$. If $i \in \operatorname{Seg} n$, then $\operatorname{proj}(i, n) \cdot\left(f_{1}+f_{2}\right)=\operatorname{proj}(i, n)$. $f_{1}+\operatorname{proj}(i, n) \cdot f_{2}$ and $\operatorname{proj}(i, n) \cdot\left(f_{1}-f_{2}\right)=\operatorname{proj}(i, n) \cdot f_{1}-\operatorname{proj}(i, n) \cdot f_{2}$.
(16) Let $n$ be an element of $\mathbb{N}, r$ be a real number, $f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}$, and $i$ be an element of $\mathbb{N}$. If $i \in \operatorname{Seg} n$, then $\operatorname{proj}(i, n) \cdot(r f)=$ $r(\operatorname{proj}(i, n) \cdot f)$.
(17) Let $n$ be an element of $\mathbb{N}, A$ be a closed-interval subset of $\mathbb{R}$, and $f_{1}$, $f_{2}$ be partial functions from $\mathbb{R}$ to $\mathcal{R}^{n}$. Suppose $f_{1}$ is integrable on $A$ and $f_{2}$ is integrable on $A$ and $A \subseteq \operatorname{dom} f_{1}$ and $A \subseteq \operatorname{dom} f_{2}$ and $f_{1}\lceil A$ is bounded and $f_{2} \upharpoonright A$ is bounded. Then $f_{1}+f_{2}$ is integrable on $A$ and $f_{1}-f_{2}$ is integrable on $A$ and $\int_{A}\left(f_{1}+f_{2}\right)(x) d x=\int_{A} f_{1}(x) d x+\int_{A} f_{2}(x) d x$ and $\int_{A}\left(f_{1}-f_{2}\right)(x) d x=\int_{A} f_{1}(x) d x-\int_{A} f_{2}(x) d x$.
(18) Let $n$ be an element of $\mathbb{N}, r$ be a real number, $A$ be a closed-interval subset of $\mathbb{R}$, and $f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}$. Suppose $A \subseteq \operatorname{dom} f$ and $f$ is integrable on $A$ and $f \upharpoonright A$ is bounded. Then $r f$ is integrable on $A$ and $\int_{A}(r f)(x) d x=r \cdot \int_{A} f(x) d x$.
(19) Let $n$ be an element of $\mathbb{N}, f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}, A$ be a closed-interval subset of $\mathbb{R}$, and $a, b$ be real numbers. If $A=[a, b]$, then $\int_{A} f(x) d x=\int_{a}^{b} f(x) d x$.
(20) Let $n$ be an element of $\mathbb{N}, f$ be a partial function from $\mathbb{R}$ to $\mathcal{R}^{n}, A$ be a closed-interval subset of $\mathbb{R}$, and $a, b$ be real numbers. If $A=[b, a]$, then $-\int_{A} f(x) d x=\int_{a}^{b} f(x) d x$.

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