Labelled State Transition Systems

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Summary. This article introduces labelled state transition systems, where transitions may be labelled by words from a given alphabet. Reduction relations from [4] are used to define transitions between states, acceptance of words, and reachable states. Deterministic transition systems are also defined.

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The articles [1], [8], [2], [11], [6], [17], [7], [9], [16], [15], [14], [4], [10], [13], [3], [12], and [5] provide the notation and terminology for this paper.

1. PRELIMINARIES

For simplicity, we adopt the following convention: $x, x_1, x_2, y, y_1, y_2, z, z_1, z_2, X, X_1, X_2$ are sets, E is a non empty set, e is an element of E , $u, v, v_1, v_2, w, w_1, w_2$ are elements of E^ω , F, F_1, F_2 are subsets of E^ω , and k, l are natural numbers.

Next we state a number of propositions:

- (1) For every finite sequence p such that $k \in \text{dom } p$ holds $(\langle x \rangle \frown p)(k+1) = p(k)$.
- (2) For every finite sequence p such that $p \neq \emptyset$ there exists a finite sequence q and there exists x such that $p = q \frown \langle x \rangle$ and $\text{len } p = \text{len } q + 1$.
- (3) For every finite sequence p such that $k \in \text{dom } p$ and $k+1 \notin \text{dom } p$ holds $\text{len } p = k$.
- (4) Let R be a binary relation, P be a reduction sequence w.r.t. R , and q_1, q_2 be finite sequences. Suppose $P = q_1 \frown q_2$ and $\text{len } q_1 > 0$ and $\text{len } q_2 > 0$. Then q_1 is a reduction sequence w.r.t. R and q_2 is a reduction sequence w.r.t. R .

- (5) Let R be a binary relation and P be a reduction sequence w.r.t. R . Suppose $\text{len } P > 1$. Then there exists a reduction sequence Q w.r.t. R such that $\langle P(1) \rangle \cap Q = P$ and $\text{len } Q + 1 = \text{len } P$.
- (6) Let R be a binary relation and P be a reduction sequence w.r.t. R . Suppose $\text{len } P > 1$. Then there exists a reduction sequence Q w.r.t. R such that $Q \cap \langle P(\text{len } P) \rangle = P$ and $\text{len } Q + 1 = \text{len } P$.
- (7) Let R be a binary relation and P be a reduction sequence w.r.t. R . Suppose $\text{len } P > 1$. Then there exists a reduction sequence Q w.r.t. R such that $\text{len } Q + 1 = \text{len } P$ and for every k such that $k \in \text{dom } Q$ holds $Q(k) = P(k + 1)$.
- (8) For every binary relation R such that $\langle x, y \rangle$ is a reduction sequence w.r.t. R holds $\langle x, y \rangle \in R$.
- (9) If $w = u \cap v$, then $\text{len } u \leq \text{len } w$ and $\text{len } v \leq \text{len } w$.
- (10) If $w = u \cap v$ and $u \neq \langle \rangle_E$ and $v \neq \langle \rangle_E$, then $\text{len } u < \text{len } w$ and $\text{len } v < \text{len } w$.
- (11) If $w_1 \cap v_1 = w_2 \cap v_2$ and if $\text{len } w_1 = \text{len } w_2$ or $\text{len } v_1 = \text{len } v_2$, then $w_1 = w_2$ and $v_1 = v_2$.
- (12) If $w_1 \cap v_1 = w_2 \cap v_2$ and if $\text{len } w_1 \leq \text{len } w_2$ or $\text{len } v_1 \geq \text{len } v_2$, then there exists u such that $w_1 \cap u = w_2$ and $v_1 = u \cap v_2$.
- (13) If $w_1 \cap v_1 = w_2 \cap v_2$, then there exists u such that $w_1 \cap u = w_2$ and $v_1 = u \cap v_2$ or there exists u such that $w_2 \cap u = w_1$ and $v_2 = u \cap v_1$.

Let us consider X . We introduce transition-systems over X which are extensions of 1-sorted structure and are systems

$\langle \text{a carrier, a transition} \rangle$,

where the carrier is a set and the transition is a relation between the carrier $\times$ X and the carrier.

2. TRANSITION SYSTEMS OVER SUBSETS OF E^ω

Let us consider E, F and let $\mathfrak{T}$ be a transition-system over F . We say that $\mathfrak{T}$ is deterministic if and only if the conditions (Def. 1) are satisfied.

- (Def. 1)(i) The transition of $\mathfrak{T}$ is a function,
- (ii) $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$, and
 - (iii) for every element s of $\mathfrak{T}$ and for all u, v such that $u \neq v$ and $\langle s, u \rangle \in \text{dom (the transition of } \mathfrak{T})$ and $\langle s, v \rangle \in \text{dom (the transition of } \mathfrak{T})$ it is not true that there exists w such that $u \cap w = v$ or $v \cap w = u$.

We now state the proposition

- (14) For every transition-system $\mathfrak{T}$ over F such that $\text{dom (the transition of } \mathfrak{T}) = \emptyset$ holds $\mathfrak{T}$ is deterministic.

Let us consider E, F . Observe that there exists a transition-system over F which is strict, non empty, finite, and deterministic.

3. PRODUCTIONS

Let us consider X , let $\mathfrak{T}$ be a transition-system over X , and let us consider x, y, z . The predicate $x, y \rightarrow_{\mathfrak{T}} z$ is defined by:

(Def. 2) $\langle \langle x, y \rangle, z \rangle \in$ the transition of $\mathfrak{T}$.

We now state several propositions:

- (15) Let $\mathfrak{T}$ be a transition-system over X . Suppose $x, y \rightarrow_{\mathfrak{T}} z$. Then
 - (i) $x \in \mathfrak{T}$,
 - (ii) $y \in X$,
 - (iii) $z \in \mathfrak{T}$,
 - (iv) $x \in \text{dom dom}(\text{the transition of } \mathfrak{T})$,
 - (v) $y \in \text{rng dom}(\text{the transition of } \mathfrak{T})$, and
 - (vi) $z \in \text{rng}(\text{the transition of } \mathfrak{T})$.
- (16) Let $\mathfrak{T}_1$ be a transition-system over X_1 and $\mathfrak{T}_2$ be a transition-system over X_2 . Suppose the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$. If $x, y \rightarrow_{\mathfrak{T}_1} z$, then $x, y \rightarrow_{\mathfrak{T}_2} z$.
- (17) Let $\mathfrak{T}$ be a transition-system over F . Suppose the transition of $\mathfrak{T}$ is a function. If $x, y \rightarrow_{\mathfrak{T}} z_1$ and $x, y \rightarrow_{\mathfrak{T}} z_2$, then $z_1 = z_2$.
- (18) For every deterministic transition-system $\mathfrak{T}$ over F such that $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$ holds $x, \langle \rangle_E \not\rightarrow_{\mathfrak{T}} y$.
- (19) Let $\mathfrak{T}$ be a deterministic transition-system over F . If $u \neq v$ and $x, u \rightarrow_{\mathfrak{T}} z_1$ and $x, v \rightarrow_{\mathfrak{T}} z_2$, then it is not true that there exists w such that $u \wedge w = v$ or $v \wedge w = u$.

4. DIRECT TRANSITIONS

Let us consider E, F , let $\mathfrak{T}$ be a transition-system over F , and let us consider x_1, x_2, y_1, y_2 . The predicate $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$ is defined as follows:

(Def. 3) There exist v, w such that $v = y_2$ and $x_1, w \rightarrow_{\mathfrak{T}} y_1$ and $x_2 = w \wedge v$.

The following propositions are true:

- (20) Let $\mathfrak{T}$ be a transition-system over F . Suppose $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$. Then $x_1, y_1 \in \mathfrak{T}$ and $x_2, y_2 \in E^\omega$ and $x_1 \in \text{dom dom}(\text{the transition of } \mathfrak{T})$ and $y_1 \in \text{rng}(\text{the transition of } \mathfrak{T})$.
- (21) Let $\mathfrak{T}_1$ be a transition-system over F_1 and $\mathfrak{T}_2$ be a transition-system over F_2 . Suppose the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$ and $x_1, x_2 \Rightarrow_{\mathfrak{T}_1} y_1, y_2$. Then $x_1, x_2 \Rightarrow_{\mathfrak{T}_2} y_1, y_2$.

- (22) For every transition-system $\mathfrak{T}$ over F such that $x, u \Rightarrow_{\mathfrak{T}} y, v$ there exists w such that $x, w \rightarrow_{\mathfrak{T}} y$ and $u = w \wedge v$.
- (23) For every transition-system $\mathfrak{T}$ over F holds $x, y \rightarrow_{\mathfrak{T}} z$ iff $x, y \Rightarrow_{\mathfrak{T}} z, \langle \rangle_E$.
- (24) For every transition-system $\mathfrak{T}$ over F holds $x, v \rightarrow_{\mathfrak{T}} y$ iff $x, v \wedge w \Rightarrow_{\mathfrak{T}} y, w$.
- (25) For every transition-system $\mathfrak{T}$ over F such that $x, u \Rightarrow_{\mathfrak{T}} y, v$ holds $x, u \wedge w \Rightarrow_{\mathfrak{T}} y, v \wedge w$.
- (26) For every transition-system $\mathfrak{T}$ over F such that $x, u \Rightarrow_{\mathfrak{T}} y, v$ holds $\text{len } u \geq \text{len } v$.
- (27) Let $\mathfrak{T}$ be a transition-system over F . Suppose the transition of $\mathfrak{T}$ is a function. If $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, z$ and $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_2, z$, then $y_1 = y_2$.
- (28) For every transition-system $\mathfrak{T}$ over F such that $\langle \rangle_E \notin \text{rng dom}$ (the transition of $\mathfrak{T}$) holds $x, z \not\Rightarrow_{\mathfrak{T}} y, z$.
- (29) For every transition-system $\mathfrak{T}$ over F such that $\langle \rangle_E \notin \text{rng dom}$ (the transition of $\mathfrak{T}$) holds if $x, u \Rightarrow_{\mathfrak{T}} y, v$, then $\text{len } u > \text{len } v$.
- (30) For every deterministic transition-system $\mathfrak{T}$ over F such that $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, z_1$ and $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_2, z_2$ holds $y_1 = y_2$ and $z_1 = z_2$.

5. REDUCTION RELATION

In the sequel $\mathfrak{T}$ is a non empty transition-system over F , s, t are elements of $\mathfrak{T}$, and S is a subset of $\mathfrak{T}$.

Let us consider $E, F, \mathfrak{T}$. The functor $\Rightarrow_{\mathfrak{T}}$ yielding a binary relation on (the carrier of $\mathfrak{T}$) $\times E^{\omega}$ is defined as follows:

(Def. 4) $\langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$ iff $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$.

The following propositions are true:

- (31) If $\langle x, y \rangle \in \Rightarrow_{\mathfrak{T}}$, then there exist s, v, t, w such that $x = \langle s, v \rangle$ and $y = \langle t, w \rangle$.
- (32) Suppose $\langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$. Then $x_1, y_1 \in \mathfrak{T}$ and $x_2, y_2 \in E^{\omega}$ and $x_1 \in \text{dom dom}$ (the transition of $\mathfrak{T}$) and $y_1 \in \text{rng}$ (the transition of $\mathfrak{T}$).
- (33) If $x \in \Rightarrow_{\mathfrak{T}}$, then there exist s, t, v, w such that $x = \langle \langle s, v \rangle, \langle t, w \rangle \rangle$.
- (34) Let $\mathfrak{T}_1$ be a non empty transition-system over F_1 and $\mathfrak{T}_2$ be a non empty transition-system over F_2 . Suppose the carrier of $\mathfrak{T}_1 =$ the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$. Then $\Rightarrow_{\mathfrak{T}_1} = \Rightarrow_{\mathfrak{T}_2}$.
- (35) If $\langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$, then there exist v, w such that $v = y_2$ and $x_1, w \rightarrow_{\mathfrak{T}} y_1$ and $x_2 = w \wedge v$.
- (36) If $\langle \langle x, u \rangle, \langle y, v \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$, then there exists w such that $x, w \rightarrow_{\mathfrak{T}} y$ and $u = w \wedge v$.
- (37) $x, y \rightarrow_{\mathfrak{T}} z$ iff $\langle \langle x, y \rangle, \langle z, \langle \rangle_E \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$.
- (38) $x, v \rightarrow_{\mathfrak{T}} y$ iff $\langle \langle x, v \wedge w \rangle, \langle y, w \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$.

- (39) If $\langle \langle x, u \rangle, \langle y, v \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$, then $\langle \langle x, u \wedge w \rangle, \langle y, v \wedge w \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$.
- (40) If $\langle \langle x, u \rangle, \langle y, v \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$, then $\text{len } u \geq \text{len } v$.
- (41) If the transition of $\mathfrak{T}$ is a function, then if $\langle x, \langle y_1, z \rangle \rangle, \langle x, \langle y_2, z \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$, then $y_1 = y_2$.
- (42) If $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$, then if $\langle \langle x, u \rangle, \langle y, v \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$, then $\text{len } u > \text{len } v$.
- (43) If $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$, then $\langle \langle x, z \rangle, \langle y, z \rangle \rangle \notin \Rightarrow_{\mathfrak{T}}$.
- (44) If $\mathfrak{T}$ is deterministic, then if $\langle x, y_1 \rangle, \langle x, y_2 \rangle \in \Rightarrow_{\mathfrak{T}}$, then $y_1 = y_2$.
- (45) If $\mathfrak{T}$ is deterministic, then if $\langle x, \langle y_1, z_1 \rangle \rangle, \langle x, \langle y_2, z_2 \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$, then $y_1 = y_2$ and $z_1 = z_2$.
- (46) If $\mathfrak{T}$ is deterministic, then $\Rightarrow_{\mathfrak{T}}$ is function-like.

6. REDUCTION SEQUENCES

Let us consider x, E . The functor $\text{dim}_2(x, E)$ yields an element of E^ω and is defined as follows:

$$\text{(Def. 5)} \quad \text{dim}_2(x, E) = \begin{cases} x_2, & \text{if there exist } y, u \text{ such that } x = \langle y, u \rangle, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next we state a number of propositions:

- (47) Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given k . If $k, k+1 \in \text{dom } P$, then there exist s, v, t, w such that $P(k) = \langle s, v \rangle$ and $P(k+1) = \langle t, w \rangle$.
- (48) Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given k . If $k, k+1 \in \text{dom } P$, then $P(k) = \langle P(k)_1, P(k)_2 \rangle$ and $P(k+1) = \langle P(k+1)_1, P(k+1)_2 \rangle$.
- (49) Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given k . Suppose $k, k+1 \in \text{dom } P$. Then
 - (i) $P(k)_1 \in \mathfrak{T}$,
 - (ii) $P(k)_2 \in E^\omega$,
 - (iii) $P(k+1)_1 \in \mathfrak{T}$,
 - (iv) $P(k+1)_2 \in E^\omega$,
 - (v) $P(k)_1 \in \text{dom dom (the transition of } \mathfrak{T})$, and
 - (vi) $P(k+1)_1 \in \text{rng (the transition of } \mathfrak{T})$.
- (50) Let $\mathfrak{T}_1$ be a non empty transition-system over F_1 and $\mathfrak{T}_2$ be a non empty transition-system over F_2 . Suppose the carrier of $\mathfrak{T}_1 =$ the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$. Then every reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}_1}$ is a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}_2}$.
- (51) Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If there exist x, u such that $P(1) = \langle x, u \rangle$, then for every k such that $k \in \text{dom } P$ holds $\text{dim}_2(P(k), E) = P(k)_2$.
- (52) Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(\text{len } P) = \langle y, w \rangle$, then for every k such that $k \in \text{dom } P$ there exists u such that $P(k)_2 = u \wedge w$.

- (53) For every reduction sequence P w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $P(1) = \langle x, v \rangle$ and $P(\text{len } P) = \langle y, w \rangle$ there exists u such that $v = u \wedge w$.
- (54) Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, u \rangle$ and $P(\text{len } P) = \langle y, u \rangle$, then for every k such that $k \in \text{dom } P$ holds $P(k)_2 = u$.
- (55) Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given k . Suppose $k, k+1 \in \text{dom } P$. Then there exist v, w such that $v = P(k+1)_2$ and $P(k)_1, w \rightarrow_{\mathfrak{T}} P(k+1)_1$ and $P(k)_2 = w \wedge v$.
- (56) Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given k . Suppose $k, k+1 \in \text{dom } P$ and $P(k) = \langle x, u \rangle$ and $P(k+1) = \langle y, v \rangle$. Then there exists w such that $x, w \rightarrow_{\mathfrak{T}} y$ and $u = w \wedge v$.
- (57) $x, y \rightarrow_{\mathfrak{T}} z$ iff $\langle \langle x, y \rangle, \langle z, \langle \rangle_E \rangle \rangle$ is a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$.
- (58) $x, v \rightarrow_{\mathfrak{T}} y$ iff $\langle \langle x, v \wedge w \rangle, \langle y, w \rangle \rangle$ is a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$.
- (59) For every reduction sequence P w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $P(1) = \langle x, v \rangle$ and $P(\text{len } P) = \langle y, w \rangle$ holds $\text{len } v \geq \text{len } w$.
- (60) Suppose $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$. Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, u \rangle$ and $P(\text{len } P) = \langle y, u \rangle$, then $\text{len } P = 1$ and $x = y$.
- (61) Suppose $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$. Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1)_2 = P(\text{len } P)_2$, then $\text{len } P = 1$.
- (62) Suppose $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$. Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, u \rangle$ and $P(\text{len } P) = \langle y, \langle \rangle_E \rangle$, then $\text{len } P \leq \text{len } u + 1$.
- (63) Suppose $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$. Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, \langle e \rangle \rangle$ and $P(\text{len } P) = \langle y, \langle \rangle_E \rangle$, then $\text{len } P = 2$.
- (64) Suppose $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$. Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = \langle x, v \rangle$ and $P(\text{len } P) = \langle y, w \rangle$, then $\text{len } v > \text{len } w$ or $\text{len } P = 1$ and $x = y$ and $v = w$.
- (65) Suppose $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$. Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given k . If $k, k+1 \in \text{dom } P$, then $P(k)_2 \neq P(k+1)_2$.
- (66) Suppose $\langle \rangle_E \notin \text{rng dom (the transition of } \mathfrak{T})$. Let P be a reduction sequence w.r.t. $\Rightarrow_{\mathfrak{T}}$ and given k, l . If $k, l \in \text{dom } P$ and $k < l$, then $P(k)_2 \neq P(l)_2$.
- (67) Suppose $\mathfrak{T}$ is deterministic. Let P, Q be reduction sequences w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = Q(1)$, then for every k such that $k \in \text{dom } P$ and $k \in \text{dom } Q$ holds $P(k) = Q(k)$.
- (68) If $\mathfrak{T}$ is deterministic, then for all reduction sequences P, Q w.r.t. $\Rightarrow_{\mathfrak{T}}$ such that $P(1) = Q(1)$ and $\text{len } P = \text{len } Q$ holds $P = Q$.
- (69) Suppose $\mathfrak{T}$ is deterministic. Let P, Q be reduction sequences w.r.t. $\Rightarrow_{\mathfrak{T}}$. If $P(1) = Q(1)$ and $P(\text{len } P)_2 = Q(\text{len } Q)_2$, then $P = Q$.

7. REDUCTIONS

The following propositions are true:

- (70) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, v \rangle$ to $\langle y, w \rangle$, then there exists u such that $v = u \cap w$.
- (71) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, u \rangle$ to $\langle y, v \rangle$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, u \cap w \rangle$ to $\langle y, v \cap w \rangle$.
- (72) If $x, y \rightarrow_{\mathfrak{T}} z$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, y \rangle$ to $\langle z, \langle \rangle_E \rangle$.
- (73) If $x, v \rightarrow_{\mathfrak{T}} y$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, v \cap w \rangle$ to $\langle y, w \rangle$.
- (74) If $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$, then $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x_1, x_2 \rangle$ to $\langle y_1, y_2 \rangle$.
- (75) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, v \rangle$ to $\langle y, w \rangle$, then $\text{len } v \geq \text{len } w$.
- (76) If $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, w \rangle$ to $\langle y, v \cap w \rangle$, then $v = \langle \rangle_E$.
- (77) If $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$, then if $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, v \rangle$ to $\langle y, w \rangle$, then $\text{len } v > \text{len } w$ or $x = y$ and $v = w$.
- (78) If $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$, then if $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, u \rangle$ to $\langle y, u \rangle$, then $x = y$.
- (79) If $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$, then if $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x, \langle e \rangle \rangle$ to $\langle y, \langle \rangle_E \rangle$, then $\langle \langle x, \langle e \rangle \rangle, \langle y, \langle \rangle_E \rangle \rangle \in \Rightarrow_{\mathfrak{T}}$.
- (80) If $\mathfrak{T}$ is deterministic, then if $\Rightarrow_{\mathfrak{T}}$ reduces x to $\langle y_1, z \rangle$ and $\Rightarrow_{\mathfrak{T}}$ reduces x to $\langle y_2, z \rangle$, then $y_1 = y_2$.

8. TRANSITIONS

Let us consider $E, F, \mathfrak{T}, x_1, x_2, y_1, y_2$. The predicate $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1, y_2$ is defined as follows:

(Def. 6) $\Rightarrow_{\mathfrak{T}}$ reduces $\langle x_1, x_2 \rangle$ to $\langle y_1, y_2 \rangle$.

We now state a number of propositions:

- (81) Let $\mathfrak{T}_1$ be a non empty transition-system over F_1 and $\mathfrak{T}_2$ be a non empty transition-system over F_2 . Suppose the carrier of $\mathfrak{T}_1 =$ the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$. If $x_1, x_2 \Rightarrow_{\mathfrak{T}_1}^* y_1, y_2$, then $x_1, x_2 \Rightarrow_{\mathfrak{T}_2}^* y_1, y_2$.
- (82) $x, y \Rightarrow_{\mathfrak{T}}^* x, y$.
- (83) If $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1, y_2$ and $y_1, y_2 \Rightarrow_{\mathfrak{T}}^* z_1, z_2$, then $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* z_1, z_2$.
- (84) If $x, y \rightarrow_{\mathfrak{T}} z$, then $x, y \Rightarrow_{\mathfrak{T}}^* z, \langle \rangle_E$.
- (85) If $x, v \rightarrow_{\mathfrak{T}} y$, then $x, v \cap w \Rightarrow_{\mathfrak{T}}^* y, w$.
- (86) If $x, u \Rightarrow_{\mathfrak{T}}^* y, v$, then $x, u \cap w \Rightarrow_{\mathfrak{T}}^* y, v \cap w$.
- (87) If $x_1, x_2 \Rightarrow_{\mathfrak{T}} y_1, y_2$, then $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1, y_2$.
- (88) If $x, v \Rightarrow_{\mathfrak{T}}^* y, w$, then there exists u such that $v = u \cap w$.
- (89) If $x, v \Rightarrow_{\mathfrak{T}}^* y, w$, then $\text{len } w \leq \text{len } v$.
- (90) If $x, w \Rightarrow_{\mathfrak{T}}^* y, v \cap w$, then $v = \langle \rangle_E$.

- (91) If $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$, then $x, u \Rightarrow_{\mathfrak{T}}^* y, u$ iff $x = y$.
- (92) If $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$, then if $x, \langle e \rangle \Rightarrow_{\mathfrak{T}}^* y, \langle \rangle_E$, then $x, \langle e \rangle \Rightarrow_{\mathfrak{T}} y, \langle \rangle_E$.
- (93) If $\mathfrak{T}$ is deterministic, then if $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1, z$ and $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_2, z$, then $y_1 = y_2$.

9. ACCEPTANCE OF WORDS

Let us consider $E, F, \mathfrak{T}, x_1, x_2, y$. The predicate $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y$ is defined as follows:

(Def. 7) $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y, \langle \rangle_E$.

We now state several propositions:

- (94) Let $\mathfrak{T}_1$ be a non empty transition-system over F_1 and $\mathfrak{T}_2$ be a non empty transition-system over F_2 . Suppose the carrier of $\mathfrak{T}_1 =$ the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$. If $x, y \Rightarrow_{\mathfrak{T}_1}^* z$, then $x, y \Rightarrow_{\mathfrak{T}_2}^* z$.
- (95) $x, \langle \rangle_E \Rightarrow_{\mathfrak{T}}^* x$.
- (96) If $x, u \Rightarrow_{\mathfrak{T}}^* y$, then $x, u \wedge v \Rightarrow_{\mathfrak{T}}^* y, v$.
- (97) If $x, y \rightarrow_{\mathfrak{T}} z$, then $x, y \Rightarrow_{\mathfrak{T}}^* z$.
- (98) If $x_1, x_2 \Rightarrow_{\mathfrak{T}} y, \langle \rangle_E$, then $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y$.
- (99) If $x, u \Rightarrow_{\mathfrak{T}}^* y$ and $y, v \Rightarrow_{\mathfrak{T}}^* z$, then $x, u \wedge v \Rightarrow_{\mathfrak{T}}^* z$.
- (100) If $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$, then $x, \langle \rangle_E \Rightarrow_{\mathfrak{T}}^* y$ iff $x = y$.
- (101) If $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$, then if $x, \langle e \rangle \Rightarrow_{\mathfrak{T}}^* y$, then $x, \langle e \rangle \Rightarrow_{\mathfrak{T}} y, \langle \rangle_E$.
- (102) If $\mathfrak{T}$ is deterministic, then if $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_1$ and $x_1, x_2 \Rightarrow_{\mathfrak{T}}^* y_2$, then $y_1 = y_2$.

10. REACHABLE STATES

Let us consider $E, F, \mathfrak{T}, x, X$. The functor $x\text{-succ}_{\mathfrak{T}}(X)$ yields a subset of $\mathfrak{T}$ and is defined as follows:

(Def. 8) $x\text{-succ}_{\mathfrak{T}}(X) = \{s : \bigvee_t (t \in X \wedge t, x \Rightarrow_{\mathfrak{T}}^* s)\}$.

The following propositions are true:

- (103) $s \in x\text{-succ}_{\mathfrak{T}}(X)$ iff there exists t such that $t \in X$ and $t, x \Rightarrow_{\mathfrak{T}}^* s$.
- (104) If $\langle \rangle_E \notin \text{rng dom}(\text{the transition of } \mathfrak{T})$, then $\langle \rangle_E\text{-succ}_{\mathfrak{T}}(S) = S$.
- (105) Let $\mathfrak{T}_1$ be a non empty transition-system over F_1 and $\mathfrak{T}_2$ be a non empty transition-system over F_2 . Suppose the carrier of $\mathfrak{T}_1 =$ the carrier of $\mathfrak{T}_2$ and the transition of $\mathfrak{T}_1 =$ the transition of $\mathfrak{T}_2$. Then $x\text{-succ}_{\mathfrak{T}_1}(X) = x\text{-succ}_{\mathfrak{T}_2}(X)$.

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