Stability of the 4-2 Binary Addition Circuit Cells. Part I

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Summary. To evaluate our formal verification method on a real-size calculation circuit, in this article, we continue to formalize the concept of the 4-2 Binary Addition Cell primitives (FTAs) to define the structures of calculation units for a very fast multiplication algorithm for VLSI implementation [11]. We define the circuit structure of four-types FTAs, TYPE-0 to TYPE-3, using the series constructions of the Generalized Full Adder Circuits (GFAs) that generalized adder to have for each positive and negative weights to inputs and outputs [15]. We then successfully prove its circuit stability of the calculation outputs after four-steps. The motivation for this research is to establish a technique based on formalized mathematics and its applications for calculation circuits with high reliability.

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The terminology and notation used in this paper are introduced in the following papers: [8], [10], [14], [3], [13], [1], [7], [9], [6], [5], [4], [2], [12], and [15]. For simplicity the following abbreviations are introduced

$$\begin{aligned} \text{BitGFA}i\text{Str} &\longmapsto \Sigma_i \\ \text{BitGFA}i\text{Circ} &\longmapsto \mathfrak{C}_i \\ \text{GFA}i\text{AdderOutput} &\longmapsto \mathfrak{a}_i \\ \text{GFA}i\text{CarryOutput} &\longmapsto \mathfrak{c}_i \\ \text{InnerVertices} &\longmapsto \mathcal{IV} \end{aligned}$$

1. STABILITY OF 4-2 BINARY ADDITION CIRCUIT CELL (TYPE-0)

Let a_1, b_1, c_1, d_1, c_2 be sets. The functor $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 1) $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2) = \Sigma_0(a_1, b_1, c_1) + \cdot \Sigma_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1)$.

Let a_1, b_1, c_1, d_1, c_2 be sets. The functor $\text{BitFTA0Circ}(a_1, b_1, c_1, d_1, c_2)$ yields a strict Boolean circuit of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ with denotation held in gates and is defined as follows:

(Def. 2) $\text{BitFTA0Circ}(a_1, b_1, c_1, d_1, c_2) = \mathfrak{C}_0(a_1, b_1, c_1) + \cdot \mathfrak{C}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1)$.

One can prove the following propositions:

- (1) Let a_1, b_1, c_1, d_1, c_2 be sets. Then $\mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)) = \{\langle\langle a_1, b_1 \rangle, \text{xor}_2 \rangle, \mathbf{a}_0(a_1, b_1, c_1)\} \cup \{\langle\langle a_1, b_1 \rangle, \text{and}_2 \rangle, \langle\langle b_1, c_1 \rangle, \text{and}_2 \rangle, \langle\langle c_1, a_1 \rangle, \text{and}_2 \rangle, \mathfrak{c}_0(a_1, b_1, c_1)\} \cup \{\langle\langle \mathbf{a}_0(a_1, b_1, c_1), c_2 \rangle, \text{xor}_2 \rangle, \mathbf{a}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1)\} \cup \{\langle\langle \mathbf{a}_0(a_1, b_1, c_1), c_2 \rangle, \text{and}_2 \rangle, \langle\langle c_2, d_1 \rangle, \text{and}_2 \rangle, \langle\langle d_1, \mathbf{a}_0(a_1, b_1, c_1) \rangle, \text{and}_2 \rangle, \mathfrak{c}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1)\}$.
- (2) For all sets a_1, b_1, c_1, d_1, c_2 holds $\mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ is a binary relation.
- (3) For all non pair sets a_1, b_1, c_1, d_1 and for every set c_2 such that $c_2 \neq \langle\langle d_1, \mathbf{a}_0(a_1, b_1, c_1) \rangle, \text{and}_2 \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_0(a_1, b_1, c_1))$ holds $\text{InputVertices}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)) = \{a_1, b_1, c_1, d_1, c_2\}$.
- (4) Let a_1, b_1, c_1, d_1, c_2 be sets. Then $a_1 \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $b_1 \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $c_1 \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $d_1 \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $c_2 \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\langle\langle a_1, b_1 \rangle, \text{xor}_2 \rangle \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\mathbf{a}_0(a_1, b_1, c_1) \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\langle\langle a_1, b_1 \rangle, \text{and}_2 \rangle \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\langle\langle b_1, c_1 \rangle, \text{and}_2 \rangle \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\langle\langle c_1, a_1 \rangle, \text{and}_2 \rangle \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\mathfrak{c}_0(a_1, b_1, c_1) \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\langle\langle \mathbf{a}_0(a_1, b_1, c_1), c_2 \rangle, \text{xor}_2 \rangle \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\mathbf{a}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1) \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\langle\langle \mathbf{a}_0(a_1, b_1, c_1), c_2 \rangle, \text{and}_2 \rangle \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\langle\langle c_2, d_1 \rangle, \text{and}_2 \rangle \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\langle\langle d_1, \mathbf{a}_0(a_1, b_1, c_1) \rangle, \text{and}_2 \rangle \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$ and $\mathfrak{c}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1) \in$ the carrier of $\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2)$.
- (5) Let a_1, b_1, c_1, d_1, c_2 be sets. Then $\langle\langle a_1, b_1 \rangle, \text{xor}_2 \rangle \in \mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and $\mathbf{a}_0(a_1, b_1, c_1) \in \mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and

$\langle\langle a_1, b_1 \rangle, \text{and}_2 \rangle, \langle\langle b_1, c_1 \rangle, \text{and}_2 \rangle, \langle\langle c_1, a_1 \rangle, \text{and}_2 \rangle \in \mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and $\mathbf{c}_0(a_1, b_1, c_1) \in \mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and $\langle\langle \mathbf{a}_0(a_1, b_1, c_1), c_2 \rangle, \text{xor}_2 \rangle, \mathbf{a}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1), \langle\langle \mathbf{a}_0(a_1, b_1, c_1), c_2 \rangle, \text{and}_2 \rangle, \langle\langle c_2, d_1 \rangle, \text{and}_2 \rangle, \langle\langle d_1, \mathbf{a}_0(a_1, b_1, c_1) \rangle, \text{and}_2 \rangle, \mathbf{c}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1) \in \mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$.

- (6) Let a_1, b_1, c_1, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathbf{a}_0(a_1, b_1, c_1) \rangle, \text{and}_2 \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_0(a_1, b_1, c_1))$. Then $a_1, b_1, c_1, d_1, c_2 \in \text{InputVertices}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$.

Let a_1, b_1, c_1, d_1, c_2 be sets. The functor $\text{BitFTA0CarryOutput}(a_1, b_1, c_1, d_1, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and is defined as follows:

(Def. 3) $\text{BitFTA0CarryOutput}(a_1, b_1, c_1, d_1, c_2) = \mathbf{c}_0(a_1, b_1, c_1)$.

The functor $\text{BitFTA0AdderOutputI}(a_1, b_1, c_1, d_1, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and is defined as follows:

(Def. 4) $\text{BitFTA0AdderOutputI}(a_1, b_1, c_1, d_1, c_2) = \mathbf{a}_0(a_1, b_1, c_1)$.

The functor $\text{BitFTA0AdderOutputP}(a_1, b_1, c_1, d_1, c_2)$ yielding an element of $\mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ is defined by:

(Def. 5) $\text{BitFTA0AdderOutputP}(a_1, b_1, c_1, d_1, c_2) = \mathbf{c}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1)$.

The functor $\text{BitFTA0AdderOutputQ}(a_1, b_1, c_1, d_1, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA0Str}(a_1, b_1, c_1, d_1, c_2))$ and is defined by:

(Def. 6) $\text{BitFTA0AdderOutputQ}(a_1, b_1, c_1, d_1, c_2) = \mathbf{a}_0(\mathbf{a}_0(a_1, b_1, c_1), c_2, d_1)$.

The following propositions are true:

- (7) Let a_1, b_1, c_1 be non pair sets, d_1, c_2 be sets, s be a state of $\text{BitFTA0Circ}(a_1, b_1, c_1, d_1, c_2)$, and a_2, a_3, a_4 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_1)$ and $a_4 = s(c_1)$. Then $(\text{Following}(s, 2))(\text{BitFTA0CarryOutput}(a_1, b_1, c_1, d_1, c_2)) = a_2 \wedge a_3 \vee a_3 \wedge a_4 \vee a_4 \wedge a_2$ and $(\text{Following}(s, 2))(\text{BitFTA0AdderOutputI}(a_1, b_1, c_1, d_1, c_2)) = a_2 \oplus a_3 \oplus a_4$.
- (8) Let a_1, b_1, c_1, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathbf{a}_0(a_1, b_1, c_1) \rangle, \text{and}_2 \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_0(a_1, b_1, c_1))$. Let s be a state of $\text{BitFTA0Circ}(a_1, b_1, c_1, d_1, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_1)$ and $a_4 = s(c_1)$ and $a_5 = s(d_1)$ and $a_6 = s(c_2)$. Then $(\text{Following}(s, 2))(\mathbf{a}_0(a_1, b_1, c_1)) = a_2 \oplus a_3 \oplus a_4$ and $(\text{Following}(s, 2))(a_1) = a_2$ and $(\text{Following}(s, 2))(b_1) = a_3$ and $(\text{Following}(s, 2))(c_1) = a_4$ and $(\text{Following}(s, 2))(d_1) = a_5$ and $(\text{Following}(s, 2))(c_2) = a_6$.
- (9) Let a_1, b_1, c_1, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathbf{a}_0(a_1, b_1, c_1) \rangle, \text{and}_2 \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_0(a_1, b_1, c_1))$. Let s be a state of $\text{BitFTA0Circ}(a_1, b_1, c_1, d_1, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_1)$ and $a_4 = s(c_1)$ and $a_5 = s(d_1)$

and $a_6 = s(c_2)$. Then $(\text{Following}(s, 4))(\text{BitFTA0AdderOutputP}(a_1, b_1, c_1, d_1, c_2)) = (a_2 \oplus a_3 \oplus a_4) \wedge a_6 \vee a_6 \wedge a_5 \vee a_5 \wedge (a_2 \oplus a_3 \oplus a_4)$ and $(\text{Following}(s, 4))(\text{BitFTA0AdderOutputQ}(a_1, b_1, c_1, d_1, c_2)) = a_2 \oplus a_3 \oplus a_4 \oplus a_5 \oplus a_6$.

- (10) Let a_1, b_1, c_1, d_1 be non pair sets and c_2 be a set. If $c_2 \neq \langle \langle d_1, \mathbf{a}_0(a_1, b_1, c_1) \rangle, \text{and}_2 \rangle$, then for every state s of $\text{BitFTA0Circ}(a_1, b_1, c_1, d_1, c_2)$ holds $\text{Following}(s, 4)$ is stable.

2. STABILITY OF 4-2 BINARY ADDITION CIRCUIT CELL (TYPE-1)

Let a_1, b_2, c_1, d_2, c_2 be sets. The functor $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

- (Def. 7) $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2) = \Sigma_1(a_1, b_2, c_1) + \Sigma_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2)$.

Let a_1, b_2, c_1, d_2, c_2 be sets. The functor $\text{BitFTA1Circ}(a_1, b_2, c_1, d_2, c_2)$ yields a strict Boolean circuit of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ with denotation held in gates and is defined by:

- (Def. 8) $\text{BitFTA1Circ}(a_1, b_2, c_1, d_2, c_2) = \mathfrak{C}_1(a_1, b_2, c_1) + \mathfrak{C}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2)$.

Next we state several propositions:

- (11) Let a_1, b_2, c_1, d_2, c_2 be sets. Then $\mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)) = \{ \langle \langle a_1, b_2 \rangle, \text{xor}2c \rangle, \mathbf{a}_1(a_1, b_2, c_1) \} \cup \{ \langle \langle a_1, b_2 \rangle, \text{and}2c \rangle, \langle \langle b_2, c_1 \rangle, \text{and}2a \rangle, \langle \langle c_1, a_1 \rangle, \text{and}2 \rangle, \mathbf{c}_1(a_1, b_2, c_1) \} \cup \{ \langle \langle \mathbf{a}_1(a_1, b_2, c_1), c_2 \rangle, \text{xor}2c \rangle, \mathbf{a}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2) \} \cup \{ \langle \langle \mathbf{a}_1(a_1, b_2, c_1), c_2 \rangle, \text{and}2a \rangle, \langle \langle c_2, d_2 \rangle, \text{and}2c \rangle, \langle \langle d_2, \mathbf{a}_1(a_1, b_2, c_1) \rangle, \text{and}2b \rangle, \mathbf{c}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2) \}$.
- (12) For all sets a_1, b_2, c_1, d_2, c_2 holds $\mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ is a binary relation.
- (13) For all non pair sets a_1, b_2, c_1, d_2 and for every set c_2 such that $c_2 \neq \langle \langle d_2, \mathbf{a}_1(a_1, b_2, c_1) \rangle, \text{and}2b \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_1(a_1, b_2, c_1))$ holds $\text{InputVertices}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)) = \{a_1, b_2, c_1, d_2, c_2\}$.
- (14) Let a_1, b_2, c_1, d_2, c_2 be sets. Then $a_1 \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $b_2 \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $c_1 \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $d_2 \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $c_2 \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\langle \langle a_1, b_2 \rangle, \text{xor}2c \rangle \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\mathbf{a}_1(a_1, b_2, c_1) \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\langle \langle a_1, b_2 \rangle, \text{and}2c \rangle \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\langle \langle b_2, c_1 \rangle, \text{and}2a \rangle \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\langle \langle c_1, a_1 \rangle, \text{and}2 \rangle \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\mathbf{c}_1(a_1, b_2,$

- $c_1) \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\langle \langle \mathbf{a}_1(a_1, b_2, c_1), c_2 \rangle, \text{xor2c} \rangle \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\mathbf{a}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2) \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\langle \langle \mathbf{a}_1(a_1, b_2, c_1), c_2 \rangle, \text{and}_{2a} \rangle \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\langle \langle c_2, d_2 \rangle, \text{and}_{2c} \rangle \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\langle \langle d_2, \mathbf{a}_1(a_1, b_2, c_1) \rangle, \text{and}_{2b} \rangle \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$ and $\mathbf{c}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2) \in$ the carrier of $\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2)$.
- (15) Let a_1, b_2, c_1, d_2, c_2 be sets. Then $\langle \langle a_1, b_2 \rangle, \text{xor2c} \rangle \in \mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ and $\mathbf{a}_1(a_1, b_2, c_1) \in \mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ and $\langle \langle a_1, b_2 \rangle, \text{and}_{2c} \rangle, \langle \langle b_2, c_1 \rangle, \text{and}_{2a} \rangle, \langle \langle c_1, a_1 \rangle, \text{and}_2 \rangle \in \mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ and $\mathbf{c}_1(a_1, b_2, c_1) \in \mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ and $\langle \langle \mathbf{a}_1(a_1, b_2, c_1), c_2 \rangle, \text{xor2c} \rangle, \mathbf{a}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2), \langle \langle \mathbf{a}_1(a_1, b_2, c_1), c_2 \rangle, \text{and}_{2a} \rangle, \langle \langle c_2, d_2 \rangle, \text{and}_{2c} \rangle, \langle \langle d_2, \mathbf{a}_1(a_1, b_2, c_1) \rangle, \text{and}_{2b} \rangle, \mathbf{c}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2) \in \mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$.
- (16) Let a_1, b_2, c_1, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle \langle d_2, \mathbf{a}_1(a_1, b_2, c_1) \rangle, \text{and}_{2b} \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_1(a_1, b_2, c_1))$. Then $a_1, b_2, c_1, d_2, c_2 \in \text{InputVertices}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$.

Let a_1, b_2, c_1, d_2, c_2 be sets. The functor $\text{BitFTA1CarryOutput}(a_1, b_2, c_1, d_2, c_2)$ yielding an element of $\mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ is defined as follows:

(Def. 9) $\text{BitFTA1CarryOutput}(a_1, b_2, c_1, d_2, c_2) = \mathbf{c}_1(a_1, b_2, c_1)$.

The functor $\text{BitFTA1AdderOutputI}(a_1, b_2, c_1, d_2, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ and is defined by:

(Def. 10) $\text{BitFTA1AdderOutputI}(a_1, b_2, c_1, d_2, c_2) = \mathbf{a}_1(a_1, b_2, c_1)$.

The functor $\text{BitFTA1AdderOutputP}(a_1, b_2, c_1, d_2, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ and is defined as follows:

(Def. 11) $\text{BitFTA1AdderOutputP}(a_1, b_2, c_1, d_2, c_2) = \mathbf{c}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2)$.

The functor $\text{BitFTA1AdderOutputQ}(a_1, b_2, c_1, d_2, c_2)$ yielding an element of $\mathcal{IV}(\text{BitFTA1Str}(a_1, b_2, c_1, d_2, c_2))$ is defined as follows:

(Def. 12) $\text{BitFTA1AdderOutputQ}(a_1, b_2, c_1, d_2, c_2) = \mathbf{a}_2(\mathbf{a}_1(a_1, b_2, c_1), c_2, d_2)$.

The following four propositions are true:

- (17) Let a_1, b_2, c_1 be non pair sets, d_2, c_2 be sets, s be a state of $\text{BitFTA1Circ}(a_1, b_2, c_1, d_2, c_2)$, and a_2, a_3, a_4 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_2)$ and $a_4 = s(c_1)$. Then $(\text{Following}(s, 2))(\text{BitFTA1CarryOutput}(a_1, b_2, c_1, d_2, c_2)) = a_2 \wedge \neg a_3 \vee \neg a_3 \wedge a_4 \vee a_4 \wedge a_2$ and $(\text{Following}(s, 2))(\text{BitFTA1AdderOutputI}(a_1, b_2, c_1, d_2, c_2)) = \neg(a_2 \oplus \neg a_3 \oplus a_4)$.
- (18) Let a_1, b_2, c_1, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle \langle d_2, \mathbf{a}_1(a_1, b_2, c_1) \rangle, \text{and}_{2b} \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_1(a_1, b_2, c_1))$. Let s be a state of $\text{BitFTA1Circ}(a_1, b_2, c_1, d_2, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of

Boolean. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_2)$ and $a_4 = s(c_1)$ and $a_5 = s(d_2)$ and $a_6 = s(c_2)$. Then $(\text{Following}(s, 2))(\mathbf{a}_1(a_1, b_2, c_1)) = \neg(a_2 \oplus \neg a_3 \oplus a_4)$ and $(\text{Following}(s, 2))(a_1) = a_2$ and $(\text{Following}(s, 2))(b_2) = a_3$ and $(\text{Following}(s, 2))(c_1) = a_4$ and $(\text{Following}(s, 2))(d_2) = a_5$ and $(\text{Following}(s, 2))(c_2) = a_6$.

- (19) Let a_1, b_2, c_1, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle \langle d_2, \mathbf{a}_1(a_1, b_2, c_1) \rangle, \text{and}_{2b} \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_1(a_1, b_2, c_1))$. Let s be a state of $\text{BitFTA1Circ}(a_1, b_2, c_1, d_2, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_1)$ and $a_3 = s(b_2)$ and $a_4 = s(c_1)$ and $a_5 = s(d_2)$ and $a_6 = s(c_2)$. Then $(\text{Following}(s, 4))(\text{BitFTA1AdderOutputP}(a_1, b_2, c_1, d_2, c_2)) = \neg((a_2 \oplus \neg a_3 \oplus a_4) \wedge a_6 \vee a_6 \wedge \neg a_5 \vee \neg a_5 \wedge (a_2 \oplus \neg a_3 \oplus a_4))$ and $(\text{Following}(s, 4))(\text{BitFTA1AdderOutputQ}(a_1, b_2, c_1, d_2, c_2)) = a_2 \oplus \neg a_3 \oplus a_4 \oplus \neg a_5 \oplus a_6$.
- (20) Let a_1, b_2, c_1, d_2 be non pair sets and c_2 be a set. If $c_2 \neq \langle \langle d_2, \mathbf{a}_1(a_1, b_2, c_1) \rangle, \text{and}_{2b} \rangle$, then for every state s of $\text{BitFTA1Circ}(a_1, b_2, c_1, d_2, c_2)$ holds $\text{Following}(s, 4)$ is stable.

3. STABILITY OF 4-2 BINARY ADDITION CIRCUIT CELL (TYPE-2)

Let a_7, b_1, c_3, d_1, c_2 be sets. The functor $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

- (Def. 13) $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2) = \Sigma_2(a_7, b_1, c_3) + \cdot \Sigma_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1)$.

Let a_7, b_1, c_3, d_1, c_2 be sets. The functor $\text{BitFTA2Circ}(a_7, b_1, c_3, d_1, c_2)$ yielding a strict Boolean circuit of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ with denotation held in gates is defined by:

- (Def. 14) $\text{BitFTA2Circ}(a_7, b_1, c_3, d_1, c_2) = \mathfrak{C}_2(a_7, b_1, c_3) + \cdot \mathfrak{C}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1)$.

Next we state several propositions:

- (21) Let a_7, b_1, c_3, d_1, c_2 be sets. Then $\mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)) = \{ \langle \langle a_7, b_1 \rangle, \text{xor}_{2c} \rangle, \mathbf{a}_2(a_7, b_1, c_3) \} \cup \{ \langle \langle a_7, b_1 \rangle, \text{and}_{2a} \rangle, \langle \langle b_1, c_3 \rangle, \text{and}_{2c} \rangle, \langle \langle c_3, a_7 \rangle, \text{and}_{2b} \rangle, \mathbf{c}_2(a_7, b_1, c_3) \} \cup \{ \langle \langle \mathbf{a}_2(a_7, b_1, c_3), c_2 \rangle, \text{xor}_{2c} \rangle, \mathbf{a}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1) \} \cup \{ \langle \langle \mathbf{a}_2(a_7, b_1, c_3), c_2 \rangle, \text{and}_{2c} \rangle, \langle \langle c_2, d_1 \rangle, \text{and}_{2a} \rangle, \langle \langle d_1, \mathbf{a}_2(a_7, b_1, c_3) \rangle, \text{and}_2 \rangle, \mathbf{c}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1) \}$.
- (22) For all sets a_7, b_1, c_3, d_1, c_2 holds $\mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ is a binary relation.
- (23) For all non pair sets a_7, b_1, c_3, d_1 and for every set c_2 such that $c_2 \neq \langle \langle d_1, \mathbf{a}_2(a_7, b_1, c_3) \rangle, \text{and}_2 \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_2(a_7, b_1, c_3))$ holds $\text{InputVertices}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)) = \{a_7, b_1, c_3, d_1, c_2\}$.

- (24) Let a_7, b_1, c_3, d_1, c_2 be sets. Then $a_7 \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $b_1 \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $c_3 \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $d_1 \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $c_2 \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\langle\langle a_7, b_1 \rangle, \text{xor2c} \rangle \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\mathbf{a}_2(a_7, b_1, c_3) \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\langle\langle a_7, b_1 \rangle, \text{and}_{2a} \rangle \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\langle\langle b_1, c_3 \rangle, \text{and}_{2c} \rangle \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\langle\langle c_3, a_7 \rangle, \text{and}_{2b} \rangle \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\mathbf{c}_2(a_7, b_1, c_3) \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\langle\langle \mathbf{a}_2(a_7, b_1, c_3), c_2 \rangle, \text{xor2c} \rangle \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\mathbf{a}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1) \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\langle\langle \mathbf{a}_2(a_7, b_1, c_3), c_2 \rangle, \text{and}_{2c} \rangle \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\langle\langle c_2, d_1 \rangle, \text{and}_{2a} \rangle \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\langle\langle d_1, \mathbf{a}_2(a_7, b_1, c_3) \rangle, \text{and}_2 \rangle \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$ and $\mathbf{c}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1) \in$ the carrier of $\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2)$.
- (25) Let a_7, b_1, c_3, d_1, c_2 be sets. Then $\langle\langle a_7, b_1 \rangle, \text{xor2c} \rangle \in \mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ and $\mathbf{a}_2(a_7, b_1, c_3) \in \mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ and $\langle\langle a_7, b_1 \rangle, \text{and}_{2a} \rangle, \langle\langle b_1, c_3 \rangle, \text{and}_{2c} \rangle, \langle\langle c_3, a_7 \rangle, \text{and}_{2b} \rangle \in \mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ and $\mathbf{c}_2(a_7, b_1, c_3) \in \mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ and $\langle\langle \mathbf{a}_2(a_7, b_1, c_3), c_2 \rangle, \text{xor2c} \rangle, \mathbf{a}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1), \langle\langle \mathbf{a}_2(a_7, b_1, c_3), c_2 \rangle, \text{and}_{2c} \rangle, \langle\langle c_2, d_1 \rangle, \text{and}_{2a} \rangle, \langle\langle d_1, \mathbf{a}_2(a_7, b_1, c_3) \rangle, \text{and}_2 \rangle, \mathbf{c}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1) \in \mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$.
- (26) Let a_7, b_1, c_3, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_1, \mathbf{a}_2(a_7, b_1, c_3) \rangle, \text{and}_2 \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_2(a_7, b_1, c_3))$. Then $a_7, b_1, c_3, d_1, c_2 \in \text{InputVertices}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$.

Let a_7, b_1, c_3, d_1, c_2 be sets. The functor $\text{BitFTA2CarryOutput}(a_7, b_1, c_3, d_1, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ and is defined as follows:

(Def. 15) $\text{BitFTA2CarryOutput}(a_7, b_1, c_3, d_1, c_2) = \mathbf{c}_2(a_7, b_1, c_3)$.

The functor $\text{BitFTA2AdderOutputI}(a_7, b_1, c_3, d_1, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ and is defined as follows:

(Def. 16) $\text{BitFTA2AdderOutputI}(a_7, b_1, c_3, d_1, c_2) = \mathbf{a}_2(a_7, b_1, c_3)$.

The functor $\text{BitFTA2AdderOutputP}(a_7, b_1, c_3, d_1, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ and is defined by:

(Def. 17) $\text{BitFTA2AdderOutputP}(a_7, b_1, c_3, d_1, c_2) = \mathbf{c}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1)$.

The functor $\text{BitFTA2AdderOutputQ}(a_7, b_1, c_3, d_1, c_2)$ yielding an element of $\mathcal{IV}(\text{BitFTA2Str}(a_7, b_1, c_3, d_1, c_2))$ is defined as follows:

(Def. 18) $\text{BitFTA2AdderOutputQ}(a_7, b_1, c_3, d_1, c_2) = \mathbf{a}_1(\mathbf{a}_2(a_7, b_1, c_3), c_2, d_1)$.

One can prove the following propositions:

- (27) Let a_7, b_1, c_3 be non pair sets, d_1, c_2 be sets, s be a state of $\text{BitFTA2Circ}(a_7, b_1, c_3, d_1, c_2)$, and a_2, a_3, a_4 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_1)$ and $a_4 = s(c_3)$. Then $(\text{Following}(s, 2))(\text{BitFTA2CarryOutput}(a_7, b_1, c_3, d_1, c_2)) = \neg(\neg a_2 \wedge a_3 \vee a_3 \wedge \neg a_4 \vee \neg a_4 \wedge \neg a_2)$ and $(\text{Following}(s, 2))(\text{BitFTA2AdderOutputI}(a_7, b_1, c_3, d_1, c_2)) = \neg a_2 \oplus a_3 \oplus \neg a_4$.
- (28) Let a_7, b_1, c_3, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle \langle d_1, \mathbf{a}_2(a_7, b_1, c_3) \rangle, \text{and}_2 \rangle$ and $c_2 \notin \mathcal{TV}(\Sigma_2(a_7, b_1, c_3))$. Let s be a state of $\text{BitFTA2Circ}(a_7, b_1, c_3, d_1, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_1)$ and $a_4 = s(c_3)$ and $a_5 = s(d_1)$ and $a_6 = s(c_2)$. Then $(\text{Following}(s, 2))(\mathbf{a}_2(a_7, b_1, c_3)) = \neg a_2 \oplus a_3 \oplus \neg a_4$ and $(\text{Following}(s, 2))(a_7) = a_2$ and $(\text{Following}(s, 2))(b_1) = a_3$ and $(\text{Following}(s, 2))(c_3) = a_4$ and $(\text{Following}(s, 2))(d_1) = a_5$ and $(\text{Following}(s, 2))(c_2) = a_6$.
- (29) Let a_7, b_1, c_3, d_1 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle \langle d_1, \mathbf{a}_2(a_7, b_1, c_3) \rangle, \text{and}_2 \rangle$ and $c_2 \notin \mathcal{TV}(\Sigma_2(a_7, b_1, c_3))$. Let s be a state of $\text{BitFTA2Circ}(a_7, b_1, c_3, d_1, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_1)$ and $a_4 = s(c_3)$ and $a_5 = s(d_1)$ and $a_6 = s(c_2)$. Then $(\text{Following}(s, 4))(\text{BitFTA2AdderOutputP}(a_7, b_1, c_3, d_1, c_2)) = (\neg a_2 \oplus a_3 \oplus \neg a_4) \wedge \neg a_6 \vee \neg a_6 \wedge a_5 \vee a_5 \wedge (\neg a_2 \oplus a_3 \oplus \neg a_4)$ and $(\text{Following}(s, 4))(\text{BitFTA2AdderOutputQ}(a_7, b_1, c_3, d_1, c_2)) = \neg(\neg a_2 \oplus a_3 \oplus \neg a_4 \oplus a_5 \oplus \neg a_6)$.
- (30) Let a_7, b_1, c_3, d_1 be non pair sets and c_2 be a set. If $c_2 \neq \langle \langle d_1, \mathbf{a}_2(a_7, b_1, c_3) \rangle, \text{and}_2 \rangle$, then for every state s of $\text{BitFTA2Circ}(a_7, b_1, c_3, d_1, c_2)$ holds $\text{Following}(s, 4)$ is stable.

4. STABILITY OF 4-2 BINARY ADDITION CIRCUIT CELL (TYPE-3)

Let a_7, b_2, c_3, d_2, c_2 be sets. The functor $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

$$(\text{Def. 19}) \quad \text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2) = \Sigma_3(a_7, b_2, c_3) + \cdot \Sigma_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2).$$

Let a_7, b_2, c_3, d_2, c_2 be sets. The functor $\text{BitFTA3Circ}(a_7, b_2, c_3, d_2, c_2)$ yielding a strict Boolean circuit of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ with denotation held in gates is defined by:

$$(\text{Def. 20}) \quad \text{BitFTA3Circ}(a_7, b_2, c_3, d_2, c_2) = \mathfrak{C}_3(a_7, b_2, c_3) + \cdot \mathfrak{C}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2).$$

We now state several propositions:

- (31) Let a_7, b_2, c_3, d_2, c_2 be sets. Then $\mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)) = \{\langle\langle a_7, b_2 \rangle, \text{xor}_2 \rangle, \mathbf{a}_3(a_7, b_2, c_3)\} \cup \{\langle\langle a_7, b_2 \rangle, \text{and}_{2b} \rangle, \langle\langle b_2, c_3 \rangle, \text{and}_{2b} \rangle, \langle\langle c_3, a_7 \rangle, \text{and}_{2b} \rangle, \mathbf{c}_3(a_7, b_2, c_3)\} \cup \{\langle\langle \mathbf{a}_3(a_7, b_2, c_3), c_2 \rangle, \text{xor}_2 \rangle, \mathbf{a}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2)\} \cup \{\langle\langle \mathbf{a}_3(a_7, b_2, c_3), c_2 \rangle, \text{and}_{2b} \rangle, \langle\langle c_2, d_2 \rangle, \text{and}_{2b} \rangle, \langle\langle d_2, \mathbf{a}_3(a_7, b_2, c_3) \rangle, \text{and}_{2b} \rangle, \mathbf{c}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2)\}.$
- (32) For all sets a_7, b_2, c_3, d_2, c_2 holds $\mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ is a binary relation.
- (33) For all non pair sets a_7, b_2, c_3, d_2 and for every set c_2 such that $c_2 \neq \langle\langle d_2, \mathbf{a}_3(a_7, b_2, c_3) \rangle, \text{and}_{2b} \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_3(a_7, b_2, c_3))$ holds $\text{InputVertices}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)) = \{a_7, b_2, c_3, d_2, c_2\}$.
- (34) Let a_7, b_2, c_3, d_2, c_2 be sets. Then $a_7 \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $b_2 \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $c_3 \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $d_2 \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $c_2 \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\langle\langle a_7, b_2 \rangle, \text{xor}_2 \rangle \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\mathbf{a}_3(a_7, b_2, c_3) \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\langle\langle a_7, b_2 \rangle, \text{and}_{2b} \rangle \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\langle\langle b_2, c_3 \rangle, \text{and}_{2b} \rangle \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\langle\langle c_3, a_7 \rangle, \text{and}_{2b} \rangle \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\mathbf{c}_3(a_7, b_2, c_3) \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\langle\langle \mathbf{a}_3(a_7, b_2, c_3), c_2 \rangle, \text{xor}_2 \rangle \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\mathbf{a}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2) \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\langle\langle \mathbf{a}_3(a_7, b_2, c_3), c_2 \rangle, \text{and}_{2b} \rangle \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\langle\langle c_2, d_2 \rangle, \text{and}_{2b} \rangle \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\langle\langle d_2, \mathbf{a}_3(a_7, b_2, c_3) \rangle, \text{and}_{2b} \rangle \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$ and $\mathbf{c}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2) \in$ the carrier of $\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2)$.
- (35) Let a_7, b_2, c_3, d_2, c_2 be sets. Then $\langle\langle a_7, b_2 \rangle, \text{xor}_2 \rangle \in \mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ and $\mathbf{a}_3(a_7, b_2, c_3) \in \mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ and $\langle\langle a_7, b_2 \rangle, \text{and}_{2b} \rangle, \langle\langle b_2, c_3 \rangle, \text{and}_{2b} \rangle, \langle\langle c_3, a_7 \rangle, \text{and}_{2b} \rangle \in \mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ and $\mathbf{c}_3(a_7, b_2, c_3) \in \mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ and $\langle\langle \mathbf{a}_3(a_7, b_2, c_3), c_2 \rangle, \text{xor}_2 \rangle, \mathbf{a}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2), \langle\langle \mathbf{a}_3(a_7, b_2, c_3), c_2 \rangle, \text{and}_{2b} \rangle, \langle\langle c_2, d_2 \rangle, \text{and}_{2b} \rangle, \langle\langle d_2, \mathbf{a}_3(a_7, b_2, c_3) \rangle, \text{and}_{2b} \rangle, \mathbf{c}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2) \in \mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$.
- (36) Let a_7, b_2, c_3, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle\langle d_2, \mathbf{a}_3(a_7, b_2, c_3) \rangle, \text{and}_{2b} \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_3(a_7, b_2, c_3))$. Then $a_7, b_2, c_3, d_2, c_2 \in \text{InputVertices}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$.

Let a_7, b_2, c_3, d_2, c_2 be sets. The functor $\text{BitFTA3CarryOutput}(a_7, b_2, c_3, d_2, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ and is defined by:

(Def. 21) $\text{BitFTA3CarryOutput}(a_7, b_2, c_3, d_2, c_2) = \mathbf{c}_3(a_7, b_2, c_3)$.

The functor $\text{BitFTA3AdderOutputI}(a_7, b_2, c_3, d_2, c_2)$ yields an element of

$\mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ and is defined by:

(Def. 22) $\text{BitFTA3AdderOutputI}(a_7, b_2, c_3, d_2, c_2) = \mathbf{a}_3(a_7, b_2, c_3)$.

The functor $\text{BitFTA3AdderOutputP}(a_7, b_2, c_3, d_2, c_2)$ yields an element of $\mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ and is defined by:

(Def. 23) $\text{BitFTA3AdderOutputP}(a_7, b_2, c_3, d_2, c_2) = \mathbf{c}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2)$.

The functor $\text{BitFTA3AdderOutputQ}(a_7, b_2, c_3, d_2, c_2)$ yielding an element of $\mathcal{IV}(\text{BitFTA3Str}(a_7, b_2, c_3, d_2, c_2))$ is defined by:

(Def. 24) $\text{BitFTA3AdderOutputQ}(a_7, b_2, c_3, d_2, c_2) = \mathbf{a}_3(\mathbf{a}_3(a_7, b_2, c_3), c_2, d_2)$.

One can prove the following propositions:

- (37) Let a_7, b_2, c_3 be non pair sets, d_2, c_2 be sets, s be a state of $\text{BitFTA3Circ}(a_7, b_2, c_3, d_2, c_2)$, and a_2, a_3, a_4 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_2)$ and $a_4 = s(c_3)$. Then $(\text{Following}(s, 2))(\text{BitFTA3CarryOutput}(a_7, b_2, c_3, d_2, c_2)) = \neg(\neg a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_4 \vee \neg a_4 \wedge \neg a_2)$ and $(\text{Following}(s, 2))(\text{BitFTA3AdderOutputI}(a_7, b_2, c_3, d_2, c_2)) = \neg(\neg a_2 \oplus \neg a_3 \oplus \neg a_4)$.
- (38) Let a_7, b_2, c_3, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle \langle d_2, \mathbf{a}_3(a_7, b_2, c_3) \rangle, \text{and}_{2b} \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_3(a_7, b_2, c_3))$. Let s be a state of $\text{BitFTA3Circ}(a_7, b_2, c_3, d_2, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_2)$ and $a_4 = s(c_3)$ and $a_5 = s(d_2)$ and $a_6 = s(c_2)$. Then $(\text{Following}(s, 2))(\mathbf{a}_3(a_7, b_2, c_3)) = \neg(\neg a_2 \oplus \neg a_3 \oplus \neg a_4)$ and $(\text{Following}(s, 2))(a_7) = a_2$ and $(\text{Following}(s, 2))(b_2) = a_3$ and $(\text{Following}(s, 2))(c_3) = a_4$ and $(\text{Following}(s, 2))(d_2) = a_5$ and $(\text{Following}(s, 2))(c_2) = a_6$.
- (39) Let a_7, b_2, c_3, d_2 be non pair sets and c_2 be a set. Suppose $c_2 \neq \langle \langle d_2, \mathbf{a}_3(a_7, b_2, c_3) \rangle, \text{and}_{2b} \rangle$ and $c_2 \notin \mathcal{IV}(\Sigma_3(a_7, b_2, c_3))$. Let s be a state of $\text{BitFTA3Circ}(a_7, b_2, c_3, d_2, c_2)$ and a_2, a_3, a_4, a_5, a_6 be elements of *Boolean*. Suppose $a_2 = s(a_7)$ and $a_3 = s(b_2)$ and $a_4 = s(c_3)$ and $a_5 = s(d_2)$ and $a_6 = s(c_2)$. Then $(\text{Following}(s, 4))(\text{BitFTA3AdderOutputP}(a_7, b_2, c_3, d_2, c_2)) = \neg((\neg a_2 \oplus \neg a_3 \oplus \neg a_4) \wedge \neg a_6 \vee \neg a_6 \wedge \neg a_5 \vee \neg a_5 \wedge (\neg a_2 \oplus \neg a_3 \oplus \neg a_4))$ and $(\text{Following}(s, 4))(\text{BitFTA3AdderOutputQ}(a_7, b_2, c_3, d_2, c_2)) = \neg(\neg a_2 \oplus \neg a_3 \oplus \neg a_4 \oplus \neg a_5 \oplus \neg a_6)$.
- (40) Let a_7, b_2, c_3, d_2 be non pair sets and c_2 be a set. If $c_2 \neq \langle \langle d_2, \mathbf{a}_3(a_7, b_2, c_3) \rangle, \text{and}_{2b} \rangle$, then for every state s of $\text{BitFTA3Circ}(a_7, b_2, c_3, d_2, c_2)$ holds $\text{Following}(s, 4)$ is stable.

REFERENCES

- [1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [2] Grzegorz Bancerek and Yatsuka Nakamura. Full adder circuit. Part I. *Formalized Mathematics*, 5(3):367–380, 1996.

- [3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [4] Yatsuka Nakamura and Grzegorz Bancerek. Combining of circuits. *Formalized Mathematics*, 5(2):283–295, 1996.
- [5] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, II. *Formalized Mathematics*, 5(2):273–278, 1996.
- [6] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Formalized Mathematics*, 5(2):215–220, 1996.
- [7] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(1):83–86, 1993.
- [8] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(1):25–34, 1990.
- [9] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(1):37–42, 1996.
- [10] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [11] E. Jean Vuillemin. A very fast multiplication algorithm for VLSI implementation, integration. *The VLSI Journal*, 1(1):39–52, 1983.
- [12] Katsumi Wasaki and Pauline N. Kawamoto. 2's complement circuit. *Formalized Mathematics*, 6(2):189–197, 1997.
- [13] Edmund Woronowicz. Many–argument relations. *Formalized Mathematics*, 1(4):733–737, 1990.
- [14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [15] Shin'nosuke Yamaguchi, Katsumi Wasaki, and Nobuhiro Shimo. Generalized full adder circuits (GFAs). Part I. *Formalized Mathematics*, 13(4):549–571, 2005.

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