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Stability of n-Bit Generalized Full Adder Circuits (GFAs). Part II

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Summary. We continue to formalize the concept of the Generalized Full Addition and Subtraction circuits (GFAs), define the structures of calculation units for the Redundant Signed Digit (RSD) operations, then prove its stability of the calculations. Generally, one-bit binary full adder assumes positive weights to all of its three binary inputs and two outputs. We define the circuit structure of two-types n-bit GFAs using the recursive construction to use the RSD arithmetic logical units that we generalize full adder to have both positive and negative weights to inputs and outputs. The motivation for this research is to establish a technique based on formalized mathematics and its applications for calculation circuits with high reliability.

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The notation and terminology used in this paper have been introduced in the following articles: [15], [2], [12], [17], [1], [7], [8], [3], [6], [13], [16], [14], [11], [10], [9], [4], [5], and [18]. For simplicity the following abbreviations are introduced

$$\eta_0 = \mathit{Boolean}^0 \longmapsto \mathit{false}$$
$$\eta_1 = \mathit{Boolean}^0 \longmapsto \mathit{true}$$
$$\Sigma_0 = \text{1GateCircStr}(\varepsilon, \eta_0)$$
$$\Sigma_1 = \text{1GateCircStr}(\varepsilon, \eta_1)$$
$$\mathfrak{C}_0 = \text{1GateCircuit}(\varepsilon, \eta_0)$$
$$\mathfrak{C}_1 = \text{1GateCircuit}(\varepsilon, \eta_1)$$

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1. n-Bit Generalized Full Adder Circuit (TYPE-0)

Let n be a natural number and let x, y be finite sequences. The functor n-BitGFA0Str(x, y) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by the condition (Def. 1).

(Def. 1) There exist many sorted sets f, h indexed by $\mathbb{N}$ such that

(i) n-BitGFA0Str$(x, y) = f(n)$,

(ii) $f(0) = \Sigma_0$,

(iii) $h(0) = \langle \varepsilon, \eta_0 \rangle$, and

(iv) for every element n of $\mathbb{N}$ and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitGFA0Str$(x(n+1), y(n+1), z)$ and $h(n+1) =$ GFA0CarryOutput$(x(n+1), y(n+1), z)$.

Let n be an element of $\mathbb{N}$ and let x, y be finite sequences. The functor n-BitGFA0Circ(x, y) yields a Boolean strict circuit of n-BitGFA0Str(x, y) with denotation held in gates and is defined by the condition (Def. 2).

(Def. 2) There exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that

(i) n-BitGFA0Str$(x, y) = f(n)$,

(ii) n-BitGFA0Circ$(x, y) = g(n)$,

(iii) $f(0) = \Sigma_0$,

(iv) $g(0) = \mathfrak{C}_0$,

(v) $h(0) = \langle \varepsilon, \eta_0 \rangle$, and

(vi) for every element n of $\mathbb{N}$ and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitGFA0Str$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitGFA0Circ$(x(n+1), y(n+1), z)$ and $h(n+1) =$ GFA0CarryOutput$(x(n+1), y(n+1), z)$.

Let n be an element of $\mathbb{N}$ and let x, y be finite sequences. The functor n-BitGFA0CarryOutput(x, y) yields an element of InnerVertices(n-BitGFA0Str(x, y)) and is defined by the condition (Def. 3).

(Def. 3) There exists a many sorted set h indexed by $\mathbb{N}$ such that n-BitGFA0CarryOutput$(x, y) = h(n)$ and $h(0) = \langle \varepsilon, \eta_0 \rangle$ and for every element n of $\mathbb{N}$ holds $h(n+1) =$ GFA0CarryOutput$(x(n+1), y(n+1), h(n))$.

The following propositions are true:

(1) Let x, y be finite sequences and f, g, h be many sorted sets indexed by $\mathbb{N}$. Suppose that

(i) $f(0) = \Sigma_0$,

(ii) $g(0) = \mathfrak{C}_0$,

(iii) $h(0) = \langle\varepsilon, \eta_0\rangle$, and

(iv) for every element n of $\mathbb{N}$ and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot \mathrm{BitGFA0Str}(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot \mathrm{BitGFA0Circ}(x(n+1), y(n+1), z)$ and $h(n+1) = \mathrm{GFA0CarryOutput}(x(n+1), y(n+1), z)$.

Let n be an element of $\mathbb{N}$. Then n-$\mathrm{BitGFA0Str}(x, y) = f(n)$ and n-$\mathrm{BitGFA0Circ}(x, y) = g(n)$ and n-$\mathrm{BitGFA0CarryOutput}(x, y) = h(n)$.

(2) For all finite sequences a, b holds 0-$\mathrm{BitGFA0Str}(a, b) = \Sigma_0$ and 0-$\mathrm{BitGFA0Circ}(a, b) = \mathfrak{C}_0$ and 0-$\mathrm{BitGFA0CarryOutput}(a, b) = \langle\varepsilon, \eta_0\rangle$.

(3) Let a, b be finite sequences and c be a set. Suppose $c = \langle\varepsilon, \eta_0\rangle$. Then 1-$\mathrm{BitGFA0Str}(a, b) = \Sigma_0 +\cdot \mathrm{BitGFA0Str}(a(1), b(1), c)$ and 1-$\mathrm{BitGFA0Circ}(a, b) = \mathfrak{C}_0 +\cdot \mathrm{BitGFA0Circ}(a(1), b(1), c)$ and 1-$\mathrm{BitGFA0CarryOutput}(a, b) = \mathrm{GFA0CarryOutput}(a(1), b(1), c)$.

(4) For all sets a, b, c such that $c = \langle\varepsilon, \eta_0\rangle$ holds 1-$\mathrm{BitGFA0Str}(\langle a\rangle, \langle b\rangle) = \Sigma_0 +\cdot \mathrm{BitGFA0Str}(a, b, c)$ and 1-$\mathrm{BitGFA0Circ}(\langle a\rangle, \langle b\rangle) = \mathfrak{C}_0 +\cdot \mathrm{BitGFA0Circ}(a, b, c)$ and 1-$\mathrm{BitGFA0CarryOutput}(\langle a\rangle, \langle b\rangle) = \mathrm{GFA0CarryOutput}(a, b, c)$.

(5) Let n be an element of $\mathbb{N}$, p, q be finite sequences with length n, and p_1, p_2, q_1, q_2 be finite sequences. Then n-$\mathrm{BitGFA0Str}(p ^\frown p_1, q ^\frown q_1) = n$-$\mathrm{BitGFA0Str}(p ^\frown p_2, q ^\frown q_2)$ and n-$\mathrm{BitGFA0Circ}(p ^\frown p_1, q ^\frown q_1) = n$-$\mathrm{BitGFA0Circ}(p ^\frown p_2, q ^\frown q_2)$ and n-$\mathrm{BitGFA0CarryOutput}(p ^\frown p_1, q ^\frown q_1) = n$-$\mathrm{BitGFA0CarryOutput}(p ^\frown p_2, q ^\frown q_2)$.

(6) Let n be an element of $\mathbb{N}$, x, y be finite sequences with length n, and a, b be sets. Then $(n+1)$-$\mathrm{BitGFA0Str}(x ^\frown \langle a\rangle, y ^\frown \langle b\rangle) = (n$-$\mathrm{BitGFA0Str}(x, y)) +\cdot \mathrm{BitGFA0Str}(a, b, n$-$\mathrm{BitGFA0CarryOutput}(x, y))$ and $(n+1)$-$\mathrm{BitGFA0Circ}(x ^\frown \langle a\rangle, y ^\frown \langle b\rangle) = (n$-$\mathrm{BitGFA0Circ}(x, y)) +\cdot \mathrm{BitGFA0Circ}(a, b, n$-$\mathrm{BitGFA0CarryOutput}(x, y))$ and $(n+1)$-$\mathrm{BitGFA0CarryOutput}(x ^\frown \langle a\rangle, y ^\frown \langle b\rangle) = \mathrm{GFA0CarryOutput}(a, b, n$-$\mathrm{BitGFA0CarryOutput}(x, y))$.

(7) Let n be an element of $\mathbb{N}$ and x, y be finite sequences. Then $(n+1)$-$\mathrm{BitGFA0Str}(x, y) = (n$-$\mathrm{BitGFA0Str}(x, y)) +\cdot \mathrm{BitGFA0Str}(x(n+1), y(n+1), n$-$\mathrm{BitGFA0CarryOutput}(x, y))$ and $(n+1)$-$\mathrm{BitGFA0Circ}(x, y) = (n$-$\mathrm{BitGFA0Circ}(x, y)) +\cdot \mathrm{BitGFA0Circ}(x(n+1), y(n+1), n$-$\mathrm{BitGFA0CarryOutput}(x, y))$ and $(n+1)$-$\mathrm{BitGFA0CarryOutput}(x, y) = \mathrm{GFA0CarryOutput}(x(n+1), y(n+1), n$-$\mathrm{BitGFA0CarryOutput}(x, y))$.

(8) For all elements n, m of $\mathbb{N}$ such that $n \leq m$ and for all finite sequences x, y holds $\mathrm{InnerVertices}(n$-$\mathrm{BitGFA0Str}(x, y)) \subseteq \mathrm{InnerVertices}(m$-$\mathrm{BitGFA0Str}(x, y))$.

(9) For every element n of $\mathbb{N}$ and for all finite sequences x, y holds

InnerVertices$((n+1)$-BitGFA0Str$(x,\ y))$ = InnerVertices(n-BitGFA0Str$(x,\ y))\cup$InnerVertices(BitGFA0Str$(x(n+1),\ y(n+1)$, n-BitGFA0CarryOutput$(x,y)))$.

Let k, n be elements of $\mathbb{N}$. Let us assume that $k \geq 1$ and $k \leq n$. Let x, y be finite sequences. The functor $(k,\ n)$-BitGFA0AdderOutput$(x,\ y)$ yielding an element of InnerVertices(n-BitGFA0Str$(x,\ y)$) is defined as follows:

(Def. 4) There exists an element i of $\mathbb{N}$ such that $k = i+1$ and $(k,\ n)$-BitGFA0AdderOutput$(x,\ y)$ = GFA0AdderOutput$(x(k),\ y(k)$, i-BitGFA0CarryOutput$(x,\ y))$.

Next we state two propositions:

(10) For all elements n, k of $\mathbb{N}$ such that $k < n$ and for all finite sequences x, y holds $(k+1,\ n)$-BitGFA0AdderOutput$(x,\ y)$ = GFA0AdderOutput$(x(k+1),\ y(k+1)$, k-BitGFA0CarryOutput$(x,\ y))$.

(11) For every element n of $\mathbb{N}$ and for all finite sequences x, y holds InnerVertices(n-BitGFA0Str$(x,\ y)$) is a binary relation.

Let n be an element of $\mathbb{N}$ and let x, y be finite sequences. Observe that n-BitGFA0CarryOutput$(x,\ y)$ is pair.

One can prove the following three propositions:

(12) Let f, g be nonpair yielding finite sequences and n be an element of $\mathbb{N}$. Then InputVertices$((n+1)$-BitGFA0Str$(f,\ g))$ = InputVertices(n-BitGFA0Str$(f,\ g))\cup$(InputVertices(BitGFA0Str$(f(n+1),\ g(n+1)$, n-BitGFA0CarryOutput$(f,\ g)))\setminus\{n$-BitGFA0CarryOutput$(f,\ g)\})$ and InnerVertices(n-BitGFA0Str$(f,\ g)$) is a binary relation and InputVertices(n-BitGFA0Str$(f,\ g)$) has no pairs.

(13) For every element n of $\mathbb{N}$ and for all nonpair yielding finite sequences x, y with length n holds InputVertices(n-BitGFA0Str$(x,\ y)$) $= \operatorname{rng} x \cup \operatorname{rng} y$.

(14) Let n be an element of $\mathbb{N}$, x, y be nonpair yielding finite sequences with length n, and s be a state of n-BitGFA0Circ$(x,\ y)$. Then Following$(s, 1+2\cdot n)$ is stable.

2. n-Bit Generalized Full Adder Circuit (TYPE-1)

Let n be a natural number and let x, y be finite sequences. The functor n-BitGFA1Str$(x,\ y)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by the condition (Def. 5).

(Def. 5) There exist many sorted sets f, h indexed by $\mathbb{N}$ such that

(i) n-BitGFA1Str$(x,\ y) = f(n)$,

(ii) $f(0) = \Sigma_1$,

(iii) $h(0) = \langle\varepsilon,\ \eta_1\rangle$, and

(iv) for every element n of $\mathbb{N}$ and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitGFA1Str($x(n+1)$, $y(n+1)$, z) and $h(n+1) =$ GFA1CarryOutput($x(n+1)$, $y(n+1)$, z).

Let n be an element of $\mathbb{N}$ and let x, y be finite sequences. The functor n-BitGFA1Circ(x, y) yielding a Boolean strict circuit of n-BitGFA1Str(x, y) with denotation held in gates is defined by the condition (Def. 6).

(Def. 6) There exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that

(i) n-BitGFA1Str(x, y) $= f(n)$,

(ii) n-BitGFA1Circ(x, y) $= g(n)$,

(iii) $f(0) = \Sigma_1$,

(iv) $g(0) = \mathfrak{C}_1$,

(v) $h(0) = \langle \varepsilon, \eta_1 \rangle$, and

(vi) for every element n of $\mathbb{N}$ and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitGFA1Str($x(n+1)$, $y(n+1)$, z) and $g(n+1) = A +\cdot$ BitGFA1Circ($x(n+1)$, $y(n+1)$, z) and $h(n+1) =$ GFA1CarryOutput($x(n+1)$, $y(n+1)$, z).

Let n be an element of $\mathbb{N}$ and let x, y be finite sequences. The functor n-BitGFA1CarryOutput(x, y) yields an element of InnerVertices(n-BitGFA1Str(x, y)) and is defined by the condition (Def. 7).

(Def. 7) There exists a many sorted set h indexed by $\mathbb{N}$ such that n-BitGFA1CarryOutput(x, y) $= h(n)$ and $h(0) = \langle \varepsilon, \eta_1 \rangle$ and for every element n of $\mathbb{N}$ holds $h(n+1) =$ GFA1CarryOutput($x(n+1)$, $y(n+1)$, $h(n)$).

One can prove the following propositions:

(15) Let x, y be finite sequences and f, g, h be many sorted sets indexed by $\mathbb{N}$. Suppose that

(i) $f(0) = \Sigma_1$,

(ii) $g(0) = \mathfrak{C}_1$,

(iii) $h(0) = \langle \varepsilon, \eta_1 \rangle$, and

(iv) for every element n of $\mathbb{N}$ and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitGFA1Str($x(n+1)$, $y(n+1)$, z) and $g(n+1) = A +\cdot$ BitGFA1Circ($x(n+1)$, $y(n+1)$, z) and $h(n+1) =$ GFA1CarryOutput($x(n+1)$, $y(n+1)$, z).

Let n be an element of $\mathbb{N}$. Then n-BitGFA1Str(x, y) $= f(n)$ and n-BitGFA1Circ(x, y) $= g(n)$ and n-BitGFA1CarryOutput(x, y) $= h(n)$.

(16) For all finite sequences a, b holds 0-BitGFA1Str(a, b) $= \Sigma_1$ and 0-BitGFA1Circ(a, b) $= \mathfrak{C}_1$ and 0-BitGFA1CarryOutput(a, b) $= \langle\varepsilon, \eta_1\rangle$.

(17) Let a, b be finite sequences and c be a set. Suppose $c = \langle\varepsilon, \eta_1\rangle$. Then 1-BitGFA1Str($a$, b) $= \Sigma_1 +\cdot$ BitGFA1Str($a(1)$, $b(1)$, c) and 1-BitGFA1Circ(a, b) $= \mathfrak{C}_1 +\cdot$ BitGFA1Circ($a(1)$, $b(1)$, c) and 1-BitGFA1CarryOutput(a, b) = GFA1CarryOutput($a(1)$, $b(1)$, c).

(18) For all sets a, b, c such that $c = \langle\varepsilon, \eta_1\rangle$ holds 1-BitGFA1Str($\langle a\rangle$, $\langle b\rangle$) $= \Sigma_1 +\cdot$ BitGFA1Str(a, b, c) and 1-BitGFA1Circ($\langle a\rangle$, $\langle b\rangle$) $= \mathfrak{C}_1 +\cdot$ BitGFA1Circ(a, b, c) and 1-BitGFA1CarryOutput($\langle a\rangle$, $\langle b\rangle$) = GFA1CarryOutput(a, b, c).

(19) Let n be an element of $\mathbb{N}$, p, q be finite sequences with length n, and p_1, p_2, q_1, q_2 be finite sequences. Then n-BitGFA1Str($p \frown p_1$, $q \frown q_1$) = n-BitGFA1Str($p \frown p_2$, $q \frown q_2$) and n-BitGFA1Circ($p \frown p_1$, $q \frown q_1$) = n-BitGFA1Circ($p \frown p_2$, $q \frown q_2$) and n-BitGFA1CarryOutput($p \frown p_1$, $q \frown q_1$) = n-BitGFA1CarryOutput($p \frown p_2$, $q \frown q_2$).

(20) Let n be an element of $\mathbb{N}$, x, y be finite sequences with length n, and a, b be sets. Then $(n+1)$-BitGFA1Str($x \frown \langle a\rangle$, $y \frown \langle b\rangle$) = ($n$-BitGFA1Str($x$, y))$+\cdot$ BitGFA1Str(a, b, n-BitGFA1CarryOutput(x, y)) and $(n + 1)$-BitGFA1Circ($x \frown \langle a\rangle$, $y \frown \langle b\rangle$) = ($n$-BitGFA1Circ($x$, y))$+\cdot$ BitGFA1Circ(a, b, n-BitGFA1CarryOutput(x, y)) and $(n+1)$-BitGFA1CarryOutput($x \frown \langle a\rangle$, $y \frown \langle b\rangle$) = GFA1CarryOutput($a$, b, n-BitGFA1CarryOutput(x, y)).

(21) Let n be an element of $\mathbb{N}$ and x, y be finite sequences. Then $(n+1)$-BitGFA1Str(x, y) = (n-BitGFA1Str(x, y))$+\cdot$ BitGFA1Str($x(n+1)$, $y(n+1)$, n-BitGFA1CarryOutput(x, y)) and $(n+1)$-BitGFA1Circ(x, y) = (n-BitGFA1Circ(x, y))$+\cdot$ BitGFA1Circ($x(n+1)$, $y(n+1)$, n-BitGFA1CarryOutput(x, y)) and $(n+1)$-BitGFA1CarryOutput(x, y) = GFA1CarryOutput($x(n+1)$, $y(n+1)$, n-BitGFA1CarryOutput(x, y)).

(22) For all elements n, m of $\mathbb{N}$ such that $n \leq m$ and for all finite sequences x, y holds InnerVertices(n-BitGFA1Str(x, y)) $\subseteq$ InnerVertices(m-BitGFA1Str(x, y)).

(23) For every element n of $\mathbb{N}$ and for all finite sequences x, y holds InnerVertices($(n+1)$-BitGFA1Str(x, y)) = InnerVertices(n-BitGFA1Str(x, y))$\cup$InnerVertices(BitGFA1Str($x(n+1)$, $y(n+1)$, n-BitGFA1CarryOutput (x, y))).

Let k, n be elements of $\mathbb{N}$. Let us assume that $k \geq 1$ and $k \leq n$. Let x, y be finite sequences. The functor (k, n)-BitGFA1AdderOutput(x, y) yielding an element of InnerVertices(n-BitGFA1Str(x, y)) is defined by:

(Def. 8) There exists an element i of $\mathbb{N}$ such that $k = i + 1$ and (k, n)-BitGFA1AdderOutput(x, y) = GFA1AdderOutput($x(k)$, $y(k)$,

i-BitGFA1CarryOutput(x, y)).

Next we state two propositions:

(24) For all elements n, k of $\mathbb{N}$ such that $k < n$ and for all finite sequences x, y holds $(k+1, n)$-BitGFA1AdderOutput(x, y) = GFA1AdderOutput$(x(k+1), y(k+1), k$-BitGFA1CarryOutput$(x, y))$.

(25) For every element n of $\mathbb{N}$ and for all finite sequences x, y holds InnerVertices(n-BitGFA1Str(x, y)) is a binary relation.

Let n be an element of $\mathbb{N}$ and let x, y be finite sequences. One can check that n-BitGFA1CarryOutput(x, y) is pair.

We now state three propositions:

(26) Let f, g be nonpair yielding finite sequences and n be an element of $\mathbb{N}$. Then InputVertices(($n+1$)-BitGFA1Str(f, g)) = InputVertices(n-BitGFA1Str(f, g))$\cup$(InputVertices(BitGFA1Str$(f(n+1), g(n+1), n$-BitGFA1CarryOutput(f, g))) $\setminus$ $\{n$-BitGFA1CarryOutput$(f, g)\}$) and InnerVertices(n-BitGFA1Str(f, g)) is a binary relation and InputVertices(n-BitGFA1Str(f, g)) has no pairs.

(27) For every element n of $\mathbb{N}$ and for all nonpair yielding finite sequences x, y with length n holds InputVertices(n-BitGFA1Str(x, y)) = $\operatorname{rng} x \cup \operatorname{rng} y$.

(28) Let n be an element of $\mathbb{N}$, x, y be nonpair yielding finite sequences with length n, and s be a state of n-BitGFA1Circ(x, y). Then Following$(s, 1 + 2 \cdot n)$ is stable.

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