FORMALIZED MATHEMATICS
Vol. 16, No. 1, Pages 7–17, 2008
DOI: 10.2478/v10037-008-0002-6

Euler's Polyhedron Formula

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Summary. Euler's polyhedron theorem states for a polyhedron p, that

$$V - E + F = 2,$$

where V, E, and F are, respectively, the number of vertices, edges, and faces of p. The formula was first stated in print by Euler in 1758 [11]. The proof given here is based on Poincaré's linear algebraic proof, stated in [17] (with a corrected proof in [18]), as adapted by Imre Lakatos in the latter's *Proofs and Refutations* [15].

As is well known, Euler's formula is not true for all polyhedra. The condition on polyhedra considered here is that of being a homology sphere, which says that the cycles (chains whose boundary is zero) are exactly the bounding chains (chains that are the boundary of a chain of one higher dimension).

The present proof actually goes beyond the three-dimensional version of the polyhedral formula given by Lakatos; it is dimension-free, in the sense that it gives a formula in which the dimension of the polyhedron is a parameter. The classical Euler relation $V - E + F = 2$ is corresponds to the case where the dimension of the polyhedron is 3.

The main theorem, expressed in the language of the present article, is

$$\mathtt{Sum\ alternating-characteristic-sequence(p)} = 0,$$

where p is a polyhedron. The alternating characteristic sequence of a polyhedron is the sequence

$$-N(-1), +N(0), -N(1), \ldots, (-1)^{\dim(p)} * N(\dim(p)),$$

where $N(k)$ is the number of polytopes of p of dimension k. The special case of $\dim(p) = 3$ yields Euler's classical relation. ($N(-1)$ and $N(3)$ will turn out to be equal, by definition, to 1.)

Two other special cases are proved: the first says that a one-dimensional "polyhedron" that is a homology sphere consists of just two vertices (and thus consists of just a single edge); the second special case asserts that a two-dimensional polyhedron that is a homology sphere (a polygon) has as many vertices as edges.

A treatment of the more general version of Euler's relation can be found in [12] and [6]. The former contains a proof of Steinitz's theorem, which shows

ISSN 1426–2630(p), 1898-9934(e)

that the abstract polyhedra treated in Poincaré's proof, which might not appear to be about polyhedra in the usual sense of the word, are in fact embeddable in $\mathbf{R}^3$ under certain conditions. It would be valuable to formalize a proof of Steinitz's theorem and relate it to the development contained here.

MML identifier: `POLYFORM`, version: `7.8.05 4.89.993`

The terminology and notation used here are introduced in the following articles: [9], [27], [28], [7], [8], [21], [10], [4], [22], [3], [5], [14], [19], [26], [23], [13], [25], [24], [16], [20], [29], [1], and [2].

1. Set-theoretical Preliminaries

The following propositions are true:

(1) For all sets X, c, d such that there exist sets a, b such that $a \neq b$ and $X = \{a, b\}$ and c, $d \in X$ and $c \neq d$ holds $X = \{c, d\}$.

(2) For every function f such that f is one-to-one holds $\overline{\overline{\operatorname{dom} f}} = \overline{\overline{\operatorname{rng} f}}$.

2. Arithmetical Preliminaries

In the sequel n denotes a natural number and k denotes an integer.

Next we state the proposition

(3) If $1 \leq k$, then k is a natural number.

Let a be an integer and let b be a natural number. Then $a \cdot b$ is an element of $\mathbb{Z}$.

One can prove the following propositions:

(4) 1 is odd.

(5) 2 is even.

(6) 3 is odd.

(7) 4 is even.

(8) If n is even, then $(-1)^n = 1$.

(9) If n is odd, then $(-1)^n = -1$.

(10) $(-1)^n$ is an integer.

Let a be an integer and let n be a natural number. Then a^n is an element of $\mathbb{Z}$.

We now state four propositions:

(11) For all finite sequences p, q, r holds $\operatorname{len}(p \frown q) \leq \operatorname{len}(p \frown (q \frown r))$.

(12) $1 < n + 2$.

(13) $(-1)^2 = 1$.

(14) For every natural number n holds $(-1)^n = (-1)^{n+2}$.

3. Preliminaries on Finite Sequences

Let f be a finite sequence of elements of $\mathbb{Z}$ and let k be a natural number. Observe that f_k is integer.

The following propositions are true:

(15) Let a, b, s be finite sequences of elements of $\mathbb{Z}$. Suppose that

(i) $\operatorname{len} s > 0$,

(ii) $\operatorname{len} a = \operatorname{len} s$,

(iii) $\operatorname{len} s = \operatorname{len} b$,

(iv) for every natural number n such that $1 \leq n \leq \operatorname{len} s$ holds $s_n = a_n + b_n$, and

(v) for every natural number k such that $1 \leq k < \operatorname{len} s$ holds $b_k = -a_{k+1}$.

Then $\sum s = a_1 + b_{\operatorname{len} s}$.

(16) For all finite sequences p, q, r holds $\operatorname{len}(p \frown q \frown r) = \operatorname{len} p + \operatorname{len} q + \operatorname{len} r$.

(17) For every set x and for all finite sequences p, q holds $(\langle x \rangle \frown p \frown q)_1 = x$.

(18) For every set x and for all finite sequences p, q holds $(p \frown q \frown \langle x \rangle)_{\operatorname{len} p + \operatorname{len} q + 1} = x$.

(19) For all finite sequences p, q, r and for every natural number k such that $\operatorname{len} p < k \leq \operatorname{len}(p \frown q)$ holds $(p \frown q \frown r)_k = q_{k - \operatorname{len} p}$.

Let a be an integer. Then $\langle a \rangle$ is a finite sequence of elements of $\mathbb{Z}$.

Let a, b be integers. Then $\langle a, b \rangle$ is a finite sequence of elements of $\mathbb{Z}$.

Let a, b, c be integers. Then $\langle a, b, c \rangle$ is a finite sequence of elements of $\mathbb{Z}$.

Let p, q be finite sequences of elements of $\mathbb{Z}$. Then $p \frown q$ is a finite sequence of elements of $\mathbb{Z}$.

We now state four propositions:

(20) For all finite sequences p, q of elements of $\mathbb{Z}$ holds $\sum p \frown q = (\sum p) + \sum q$.

(21) For every integer k and for every finite sequence p of elements of $\mathbb{Z}$ holds $\sum \langle k \rangle \frown p = k + \sum p$.

(22) For all finite sequences p, q, r of elements of $\mathbb{Z}$ holds $\sum p \frown q \frown r = (\sum p) + \sum q + \sum r$.

(23) For every element a of $\mathbf{Z}_2$ holds $\sum \langle a \rangle = a$.

4. Polyhedra and Incidence Matrices

Let X, Y be sets. An incidence matrix of X and Y is an element of $\{0_{\mathbf{Z}_2}, 1_{\mathbf{Z}_2}\}^{X \times Y}$.

We now state the proposition

(24) For all sets X, Y holds $X \times Y \longmapsto 1_{\mathbf{Z}_2}$ is an incidence matrix of X and Y.

Polyhedron is defined by the condition (Def. 1).

(Def. 1) There exists a finite sequence-yielding finite sequence F and there exists a function yielding finite sequence I such that

(i) $\operatorname{len} I = \operatorname{len} F - 1$,

(ii) for every natural number n such that $1 \leq n < \operatorname{len} F$ holds $I(n)$ is an incidence matrix of $\operatorname{rng} F(n)$ and $\operatorname{rng} F(n+1)$,

(iii) for every natural number n such that $1 \leq n \leq \operatorname{len} F$ holds $F(n)$ is non empty and $F(n)$ is one-to-one, and

(iv) it $= \langle F, I \rangle$.

In the sequel p denotes a polyhedron, k denotes an integer, and n denotes a natural number.

Let us consider p. Then $p_{\mathbf{1}}$ is a finite sequence-yielding finite sequence. Then $p_{\mathbf{2}}$ is a function yielding finite sequence.

Let p be a polyhedron. The functor $\dim(p)$ yielding an element of $\mathbb{N}$ is defined by:

(Def. 2) $\dim(p) = \operatorname{len}(p_{\mathbf{1}})$.

Let p be a polyhedron and let k be an integer. The functor $P_{k,p}$ yielding a finite set is defined by the conditions (Def. 3).

(Def. 3)(i) If $k < -1$, then $P_{k,p} = \emptyset$,

(ii) if $k = -1$, then $P_{k,p} = \{\emptyset\}$,

(iii) if $-1 < k < \dim(p)$, then $P_{k,p} = \operatorname{rng} p_{\mathbf{1}}(k+1)$,

(iv) if $k = \dim(p)$, then $P_{k,p} = \{p\}$, and

(v) if $k > \dim(p)$, then $P_{k,p} = \emptyset$.

One can prove the following two propositions:

(25) If $-1 < k < \dim(p)$, then $k+1$ is a natural number and $1 \leq k+1 \leq \dim(p)$.

(26) $P_{k,p}$ is non empty iff $-1 \leq k \leq \dim(p)$.

Let p be a polyhedron and let k be an integer. Let us assume that $-1 \leq k \leq \dim(p)$. k-polytope of p is defined by:

(Def. 4) It $\in P_{k,p}$.

Next we state the proposition

(27) If $k < \dim(p)$, then $k - 1 < \dim(p)$.

Let p be a polyhedron and let k be an integer. The functor $\eta_{p,k}$ yielding an incidence matrix of $P_{k-1,p}$ and $P_{k,p}$ is defined by the conditions (Def. 5).

(Def. 5)(i) If $k < 0$, then $\eta_{p,k} = \emptyset$,

(ii) if $k = 0$, then $\eta_{p,k} = \{\emptyset\} \times P_{0,p} \longmapsto 1_{\mathbf{Z}_2}$,

(iii) if $0 < k < \dim(p)$, then $\eta_{p,k} = p_{\mathbf{2}}(k)$,

(iv) if $k = \dim(p)$, then $\eta_{p,k} = P_{\dim(p)-1,p} \times \{p\} \longmapsto 1_{\mathbf{Z}_2}$, and

(v) if $k > \dim(p)$, then $\eta_{p,k} = \emptyset$.

Let p be a polyhedron and let k be an integer. The functor $S_{k,p}$ yielding a finite sequence is defined by the conditions (Def. 6).

(Def. 6)(i) If $k < -1$, then $S_{k,p} = \varepsilon_{\emptyset}$,

(ii) if $k = -1$, then $S_{k,p} = \langle \emptyset \rangle$,

(iii) if $-1 < k < \dim(p)$, then $S_{k,p} = p_{\mathbf{1}}(k+1)$,

(iv) if $k = \dim(p)$, then $S_{k,p} = \langle p \rangle$, and

(v) if $k > \dim(p)$, then $S_{k,p} = \varepsilon_{\emptyset}$.

Let p be a polyhedron and let k be an integer. The functor $N_{p,k}$ yielding an element of $\mathbb{N}$ is defined as follows:

(Def. 7) $N_{p,k} = \overline{\overline{P_{k,p}}}$.

Let p be a polyhedron. The functor V_p yields an element of $\mathbb{N}$ and is defined by:

(Def. 8) $V_p = N_{p,0}$.

The functor E_p yields an element of $\mathbb{N}$ and is defined by:

(Def. 9) $E_p = N_{p,1}$.

The functor F_p yielding an element of $\mathbb{N}$ is defined by:

(Def. 10) $F_p = N_{p,2}$.

Next we state several propositions:

(28) $\mathrm{dom}(S_{k,p}) = \mathrm{Seg}(N_{p,k})$.

(29) $\mathrm{len}(S_{k,p}) = N_{p,k}$.

(30) $\mathrm{rng}(S_{k,p}) = P_{k,p}$.

(31) $N_{p,-1} = 1$.

(32) $N_{p,\dim(p)} = 1$.

Let p be a polyhedron, let k be an integer, and let n be a natural number. Let us assume that $1 \leq n \leq N_{p,k}$ and $-1 \leq k \leq \dim(p)$. The functor $P^n_{p,k}$ yielding an element of $P_{k,p}$ is defined by:

(Def. 11) $P^n_{p,k} = S_{k,p}(n)$.

We now state three propositions:

(33) Suppose $-1 \leq k \leq \dim(p)$. Let x be a k-polytope of p. Then there exists a natural number n such that $x = P^n_{p,k}$ and $1 \leq n \leq N_{p,k}$.

(34) $S_{k,p}$ is one-to-one.

(35) Suppose $-1 \leq k \leq \dim(p)$. Let m, n be natural numbers. If $1 \leq n \leq N_{p,k}$ and $1 \leq m \leq N_{p,k}$ and $P^n_{p,k} = P^m_{p,k}$, then $m = n$.

Let p be a polyhedron, let k be an integer, let x be a $(k-1)$-polytope of p, and let y be a k-polytope of p. Let us assume that $0 \leq k \leq \dim(p)$. The functor $x(y)$ yields an element of $\mathbf{Z}_2$ and is defined by:

(Def. 12) $x(y) = \eta_{p,k}(x, y)$.

5. The Chain Spaces and their Subspaces. Boundary of a k-chain

Let p be a polyhedron and let k be an integer. The functor $C_{k,p}$ yielding a finite dimensional vector space over $\mathbf{Z}_2$ is defined by:

(Def. 13) $C_{k,p} = B_{P_{k,p}}$.

We now state two propositions:

(36) For every k-polytope x of p holds $0_{C_{k,p}}{}^{@}x = 0_{\mathbf{Z}_2}$.

(37) $N_{p,k} = \dim(C_{k,p})$.

Let p be a polyhedron and let k be an integer. The functor k-chains p yielding a non empty finite set is defined by:

(Def. 14) k-chains $p = 2^{P_{k,p}}$.

Let p be a polyhedron, let k be an integer, let x be a $(k-1)$-polytope of p, and let v be an element of $C_{k,p}$. The functor $v(x)$ yielding a finite sequence of elements of $\mathbf{Z}_2$ is defined by the conditions (Def. 15).

(Def. 15)(i) If $P_{k-1,p}$ is empty, then $v(x) = \varepsilon_{\emptyset}$, and

(ii) if $P_{k-1,p}$ is non empty, then $\mathrm{len}(v(x)) = N_{p,k}$ and for every natural number n such that $1 \leq n \leq N_{p,k}$ holds $v(x)(n) = (v^{@}P^n_{p,k}) \cdot x(P^n_{p,k})$.

We now state several propositions:

(38) For all elements c, d of $C_{k,p}$ and for every k-polytope x of p holds $(c + d)^{@}x = c^{@}x + d^{@}x$.

(39) For all elements c, d of $C_{k,p}$ and for every $(k-1)$-polytope x of p holds $(c + d)(x) = c(x) + d(x)$.

(40) For all elements c, d of $C_{k,p}$ and for every $(k-1)$-polytope x of p holds $\sum(c(x) + d(x)) = (\sum c(x)) + \sum d(x)$.

(41) For all elements c, d of $C_{k,p}$ and for every $(k-1)$-polytope x of p holds $\sum(c + d)(x) = (\sum c(x)) + \sum d(x)$.

(42) For every element c of $C_{k,p}$ and for every element a of $\mathbf{Z}_2$ and for every k-polytope x of p holds $(a \cdot c)^{@}x = a \cdot (c^{@}x)$.

(43) For every element c of $C_{k,p}$ and for every element a of $\mathbf{Z}_2$ and for every k-polytope x of p holds $(a \cdot c)(x) = a \cdot c(x)$.

(44) For all elements c, d of $C_{k,p}$ holds $c = d$ iff for every k-polytope x of p holds $c^{@}x = d^{@}x$.

(45) For all elements c, d of $C_{k,p}$ holds $c = d$ iff for every k-polytope x of p holds $x \in c$ iff $x \in d$.

The scheme *ChainEx* deals with a polyhedron $\mathcal{A}$, an integer $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a subset c of $P_{\mathcal{B},\mathcal{A}}$ such that for every $\mathcal{B}$-polytope x of $\mathcal{A}$ holds $x \in c$ iff $\mathcal{P}[x]$ and $x \in P_{\mathcal{B},\mathcal{A}}$

for all values of the parameters.

Let p be a polyhedron, let k be an integer, and let v be an element of $C_{k,p}$. The functor ∂v yields an element of $C_{k-1,p}$ and is defined by the conditions (Def. 16).

(Def. 16)(i) If $P_{k-1,p}$ is empty, then $\partial v = 0_{C_{k-1,p}}$, and

(ii) if $P_{k-1,p}$ is non empty, then for every $(k-1)$-polytope x of p holds $x \in \partial v$ iff $\sum v(x) = 1_{\mathbf{Z}_2}$.

One can prove the following proposition

(46) For every element c of $C_{k,p}$ and for every $(k-1)$-polytope x of p holds $\partial c^{@} x = \sum c(x)$.

Let p be a polyhedron and let k be an integer. The functor $\partial_k p$ yields a function from $C_{k,p}$ into $C_{k-1,p}$ and is defined by:

(Def. 17) For every element c of $C_{k,p}$ holds $\partial_k p(c) = \partial c$.

One can prove the following propositions:

(47) For all elements c, d of $C_{k,p}$ holds $\partial(c+d) = \partial c + \partial d$.

(48) For every element a of $\mathbf{Z}_2$ and for every element c of $C_{k,p}$ holds $\partial(a \cdot c) = a \cdot \partial c$.

(49) $\partial_k p$ is a linear transformation from $C_{k,p}$ to $C_{k-1,p}$.

Let p be a polyhedron and let k be an integer. Then $\partial_k p$ is a linear transformation from $C_{k,p}$ to $C_{k-1,p}$.

Let p be a polyhedron and let k be an integer. The functor $Z_{k,p}$ yielding a subspace of $C_{k,p}$ is defined as follows:

(Def. 18) $Z_{k,p} = \ker \partial_k p$.

Let p be a polyhedron and let k be an integer. The functor $|Z_{k,p}|$ yields a non empty subset of k-chains p and is defined by:

(Def. 19) $|Z_{k,p}| = \Omega_{Z_{k,p}}$.

Let p be a polyhedron and let k be an integer. The functor $B_{k,p}$ yields a subspace of $C_{k,p}$ and is defined as follows:

(Def. 20) $B_{k,p} = \mathrm{im}(\partial_{k+1} p)$.

Let p be a polyhedron and let k be an integer. The functor $|B_{k,p}|$ yielding a non empty subset of k-chains p is defined by:

(Def. 21) $|B_{k,p}| = \Omega_{B_{k,p}}$.

Let p be a polyhedron and let k be an integer. The functor $\mathrm{BZ}_{k,p}$ yields a subspace of $C_{k,p}$ and is defined as follows:

(Def. 22) $\mathrm{BZ}_{k,p} = B_{k,p} \cap Z_{k,p}$.

Let p be a polyhedron and let k be an integer.

The functor k-bounding-circuits p yields a non empty subset of k-chains p and is defined as follows:

(Def. 23) k-bounding-circuits $p = \Omega_{\mathrm{BZ}_{k,p}}$.

The following proposition is true

(50) $\dim(C_{k,p}) = \mathrm{rank}(\partial_k p) + \mathrm{nullity}(\partial_k p)$.

6. Simply Connected and Eulerian Polyhedra

Let p be a polyhedron. We say that p is being a homology sphere if and only if:

(Def. 24) For every integer k holds $|Z_{k,p}| = |B_{k,p}|$.

The following proposition is true

(51) p is being a homology sphere iff for every integer n holds $Z_{n,p} = B_{n,p}$.

Let p be a polyhedron. The functor $\widehat{p}$ yielding a finite sequence of elements of $\mathbb{Z}$ is defined as follows:

(Def. 25) $\mathrm{len}\,\widehat{p} = \dim(p) + 2$ and for every natural number k such that $1 \leq k \leq \dim(p) + 2$ holds $\widehat{p}(k) = (-1)^k \cdot N_{p,k-2}$.

Let p be a polyhedron. The functor $\bar{p}$ yields a finite sequence of elements of $\mathbb{Z}$ and is defined by:

(Def. 26) $\mathrm{len}\,\bar{p} = \dim(p)$ and for every natural number k such that $1 \leq k \leq \dim(p)$ holds $\bar{p}(k) = (-1)^{k+1} \cdot N_{p,k-1}$.

Let p be a polyhedron. The functor $\overline{p}$ yielding a finite sequence of elements of $\mathbb{Z}$ is defined as follows:

(Def. 27) $\mathrm{len}\,\overline{p} = \dim(p) + 1$ and for every natural number k such that $1 \leq k \leq \dim(p) + 1$ holds $\overline{p}(k) = (-1)^{k+1} \cdot N_{p,k-1}$.

One can prove the following proposition

(52) If $1 \leq n \leq \mathrm{len}\,\bar{p}$, then $\bar{p}(n) = (-1)^{n+1} \cdot \dim(B_{n-2,p}) + (-1)^{n+1} \cdot \dim(Z_{n-1,p})$.

Let p be a polyhedron. We say that p is Eulerian if and only if:

(Def. 28) $\sum \bar{p} = 1 + (-1)^{\dim(p)+1}$.

One can prove the following proposition

(53) $\overline{p} = \bar{p} \frown \langle (-1)^{\dim(p)} \rangle$.

Let p be a polyhedron. Let us observe that p is Eulerian if and only if:

(Def. 29) $\sum \overline{p} = 1$.

One can prove the following proposition

(54) $\widehat{p} = \langle -1 \rangle ^\frown \overline{p}$.

Let p be a polyhedron. Let us observe that p is Eulerian if and only if:

(Def. 30) $\sum \widehat{p} = 0$.

7. The Extremal Chain Spaces

The following propositions are true:

(55) $P_{0,p}$ is non empty.

(56) $\overline{\overline{\Omega_{C_{-1,p}}}} = 2$.

(57) $\Omega_{C_{-1,p}} = \{\emptyset, \{\emptyset\}\}$.

(58) For every k-polytope x of p and for every $(k-1)$-polytope e of p such that $k = 0$ and $e = \emptyset$ holds $e(x) = 1_{\mathbf{Z}_2}$.

(59) Let k be an integer, x be a k-polytope of p, v be an element of $C_{k,p}$, e be a $(k-1)$-polytope of p, and n be a natural number. If $k = 0$ and $v = \{x\}$ and $e = \emptyset$ and $x = P^n_{p,k}$ and $1 \leq n \leq N_{p,k}$, then $v(e)(n) = 1_{\mathbf{Z}_2}$.

(60) Let k be an integer, x be a k-polytope of p, e be a $(k-1)$-polytope of p, v be an element of $C_{k,p}$, and m, n be natural numbers. Suppose $k = 0$ and $v = \{x\}$ and $x = P^n_{p,k}$ and $1 \leq m \leq N_{p,k}$ and $1 \leq n \leq N_{p,k}$ and $m \neq n$. Then $v(e)(m) = 0_{\mathbf{Z}_2}$.

(61) Let k be an integer, x be a k-polytope of p, v be an element of $C_{k,p}$, and e be a $(k-1)$-polytope of p. If $k = 0$ and $v = \{x\}$ and $e = \emptyset$, then $\sum v(e) = 1_{\mathbf{Z}_2}$.

(62) For every 0-polytope x of p holds $\partial_0 p(\{x\}) = \{\emptyset\}$.

(63) $\dim(B_{(-1),p}) = 1$.

(64) $\overline{\overline{\Omega_{C_{\dim(p),p}}}} = 2$.

(65) $\{p\}$ is an element of $C_{\dim(p),p}$.

(66) $\{p\} \in \Omega_{C_{\dim(p),p}}$.

(67) $P_{\dim(p)-1,p}$ is non empty.

Let p be a polyhedron. Note that $P_{\dim(p)-1,p}$ is non empty.

The following propositions are true:

(68) $\Omega_{C_{\dim(p),p}} = \{0_{C_{\dim(p),p}}, \{p\}\}$.

(69) For every element x of $C_{\dim(p),p}$ holds $x = 0_{C_{\dim(p),p}}$ or $x = \{p\}$.

(70) For all elements x, y of $C_{\dim(p),p}$ such that $x \neq y$ holds $x = 0_{C_{\dim(p),p}}$ or $y = 0_{C_{\dim(p),p}}$.

(71) $S_{\dim(p),p} = \langle p \rangle$.

(72) $P^1_{p,\dim(p)} = p$.

(73) For every element c of $C_{\dim(p),p}$ and for every $\dim(p)$-polytope x of p such that $c = \{p\}$ holds $c^{@}x = 1_{\mathbf{Z}_2}$.

(74) For every $(\dim(p) - 1)$-polytope x of p and for every $\dim(p)$-polytope c of p such that $c = p$ holds $x(c) = 1_{\mathbf{Z}_2}$.

(75) For every $(\dim(p)-1)$-polytope x of p and for every element c of $C_{\dim(p),p}$ such that $c = \{p\}$ holds $c(x) = \langle 1_{\mathbf{Z}_2} \rangle$.

(76) For every $(\dim(p)-1)$-polytope x of p and for every element c of $C_{\dim(p),p}$ such that $c = \{p\}$ holds $\sum c(x) = 1_{\mathbf{Z}_2}$.

(77) $\partial_{\dim(p)} p(\{p\}) = P_{\dim(p)-1,p}$.

(78) $\partial_{\dim(p)} p$ is one-to-one.

(79) $\dim(B_{\dim(p)-1,p}) = 1$.

(80) If p is being a homology sphere, then $\dim(Z_{\dim(p)-1,p}) = 1$.

(81) If $1 < n < \dim(p) + 2$, then $\widehat{p}(n) = \bar{p}(n-1)$.

(82) $\widehat{p} = \langle -1 \rangle \frown \bar{p} \frown \langle (-1)^{\dim(p)} \rangle$.

8. A Generalized Euler Relation and its 1−, 2−, and 3−dimensional Special Cases

One can prove the following propositions:

(83) If $\dim(p)$ is odd, then $\sum \widehat{p} = (\sum \bar{p}) - 2$.

(84) If $\dim(p)$ is even, then $\sum \widehat{p} = \sum \bar{p}$.

(85) If $\dim(p) = 1$, then $\sum \bar{p} = N_{p,0}$.

(86) If $\dim(p) = 2$, then $\sum \bar{p} = N_{p,0} - N_{p,1}$.

(87) If $\dim(p) = 3$, then $\sum \bar{p} = (N_{p,0} - N_{p,1}) + N_{p,2}$.

(88) If $\dim(p) = 0$, then p is Eulerian.

(89) If p is being a homology sphere, then p is Eulerian.

(90) If p is being a homology sphere and $\dim(p) = 1$, then $V_p = 2$.

(91) If p is being a homology sphere and $\dim(p) = 2$, then $V_p = E_p$.

(92) If p is being a homology sphere and $\dim(p) = 3$, then $(V_p - E_p) + F_p = 2$.

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Received October 9, 2007