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Complete Spaces

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Summary. This paper is a continuation of [12]. First some definitions needed to formulate Cantor's theorem on complete spaces and show several facts about them are introduced. Next section contains the proof of Cantor's theorem and some properties of complete spaces resulting from this theorem. Moreover, countable compact spaces and proofs of auxiliary facts about them is defined. I also show the important condition that every metric space is compact if and only if it is countably compact. Then I prove that every metric space is compact if and only if it is a complete and totally bounded space. I also introduce the definition of the metric space with the well metric. This article is based on [13].

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The articles [29], [3], [11], [10], [18], [26], [1], [7], [16], [22], [24], [23], [9], [8], [27], [5], [20], [12], [28], [6], [17], [4], [19], [14], [21], [2], [15], and [25] provide the terminology and notation for this paper.

1. Preliminaries

We follow the rules: i, n, m denote natural numbers, x, X, Y denote sets, and r denotes a real number.

Let M be a non empty metric structure and let S be a sequence of subsets of M. We say that S is bounded if and only if:

(Def. 1) For every i holds $S(i)$ is bounded.

Let M be a non empty reflexive metric structure. Observe that there exists a sequence of subsets of M which is bounded and non-empty.

Let M be a reflexive non empty metric structure and let S be a sequence of subsets of M. The functor $\varnothing S$ yielding a sequence of real numbers is defined by:

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(Def. 2) For every i holds $(\varnothing S)(i) = \varnothing S(i)$.

We now state several propositions:

(1) Let M be a reflexive non empty metric structure and S be a bounded sequence of subsets of M. Then $\varnothing S$ is lower bounded.

(2) Let M be a reflexive non empty metric structure and S be a bounded sequence of subsets of M. If S is descending, then $\varnothing S$ is upper bounded and $\varnothing S$ is non-increasing.

(3) Let M be a reflexive non empty metric structure and S be a bounded sequence of subsets of M. If S is ascending, then $\varnothing S$ is non-decreasing.

(4) Let M be a non empty reflexive metric structure and S be a bounded sequence of subsets of M. Suppose S is descending and $\lim \varnothing S = 0$. Let F be a sequence of M. If for every i holds $F(i) \in S(i)$, then F is Cauchy.

(5) Let M be a reflexive symmetric triangle non empty metric structure and p be a point of M. If $0 \leq r$, then $\varnothing\, \overline{\mathrm{Ball}}(p, r) \leq 2 \cdot r$.

Let M be a metric structure and let U be a subset of M. We say that U is open if and only if:

(Def. 3) $U \in$ the open set family of M.

Let M be a metric structure and let A be a subset of M. We say that A is closed if and only if:

(Def. 4) A^{c} is open.

Let M be a metric structure. Note that there exists a subset of M which is open and empty and there exists a subset of M which is closed and empty.

Let M be a non empty metric structure. One can verify that there exists a subset of M which is open and non empty and there exists a subset of M which is closed and non empty.

One can prove the following proposition

(6) Let M be a metric structure, A be a subset of M, and A' be a subset of M_{top} such that $A' = A$. Then

(i) A is open iff A' is open, and

(ii) A is closed iff A' is closed.

Let T be a topological structure and let S be a sequence of subsets of T. We say that S is open if and only if:

(Def. 5) For every i holds $S(i)$ is open.

We say that S is closed if and only if:

(Def. 6) For every i holds $S(i)$ is closed.

Let T be a topological space. Observe that there exists a sequence of subsets of T which is open and there exists a sequence of subsets of T which is closed.

Let T be a non empty topological space. One can verify that there exists a sequence of subsets of T which is open and non-empty and there exists a sequence of subsets of T which is closed and non-empty.

Let M be a metric structure and let S be a sequence of subsets of M. We say that S is open if and only if:

(Def. 7) For every i holds $S(i)$ is open.

We say that S is closed if and only if:

(Def. 8) For every i holds $S(i)$ is closed.

Let M be a non empty metric space. Note that there exists a sequence of subsets of M which is non-empty, bounded, and open and there exists a sequence of subsets of M which is non-empty, bounded, and closed.

The following propositions are true:

(7) Let M be a metric structure, S be a sequence of subsets of M, and S' be a sequence of subsets of M_{top} such that $S' = S$. Then
 (i) S is open iff S' is open, and
 (ii) S is closed iff S' is closed.

(8) Let M be a reflexive symmetric triangle non empty metric structure and S, C_1 be subsets of M. Suppose S is bounded. Let S' be a subset of M_{top}. If $S = S'$ and $C_1 = \overline{S'}$, then C_1 is bounded and $\varnothing S = \varnothing C_1$.

2. Cantor's Theorem on Complete Spaces

The following propositions are true:

(9) Let M be a non empty metric space and C be a sequence of M. Then there exists a non-empty closed sequence S of subsets of M such that
 (i) S is descending,
 (ii) if C is Cauchy, then S is bounded and $\lim \varnothing S = 0$, and
 (iii) for every i there exists a subset U of M_{top} such that $U = \{C(j); j$ ranges over elements of $\mathbb{N}$: $j \geq i\}$ and $S(i) = \overline{U}$.

(10) Let M be a non empty metric space. Then M is complete if and only if for every non-empty bounded closed sequence S of subsets of M such that S is descending and $\lim \varnothing S = 0$ holds $\bigcap S$ is non empty.

(11) Let T be a non empty topological space and S be a non-empty sequence of subsets of T. Suppose S is descending. Let F be a family of subsets of T. If $F = \operatorname{rng} S$, then F is centered.

(12) Let M be a non empty metric structure, S be a sequence of subsets of M, and F be a family of subsets of M_{top} such that $F = \operatorname{rng} S$. Then
 (i) if S is open, then F is open, and
 (ii) if S is closed, then F is closed.

(13) Let T be a non empty topological space, F be a family of subsets of T, and S be a sequence of subsets of T. Suppose $\operatorname{rng} S \subseteq F$. Then there exists a sequence R of subsets of T such that

(i) R is descending,
(ii) if F is centered, then R is non-empty,
(iii) if F is open, then R is open,
(iv) if F is closed, then R is closed, and
(v) for every i holds $R(i) = \bigcap\{S(j); j$ ranges over elements of $\mathbb{N}$: $j \leq i\}$.

(14) Let M be a non empty metric space. Then M is complete if and only if for every family F of subsets of $M_{\rm top}$ such that F is closed and centered and for every real number r such that $r > 0$ there exists a subset A of M such that $A \in F$ and A is bounded and $\varnothing A < r$ holds $\bigcap F$ is non empty.

(15) Let M be a non empty metric space, A be a non empty subset of M, B be a subset of M, and B' be a subset of $M \restriction A$. If $B = B'$, then B' is bounded iff B is bounded.

(16) Let M be a non empty metric space, A be a non empty subset of M, B be a subset of M, and B' be a subset of $M \restriction A$. If $B = B'$ and B is bounded, then $\varnothing B' \leq \varnothing B$.

(17) For every non empty metric space M and for every non empty subset A of M holds every sequence of $M \restriction A$ is a sequence of M.

(18) Let M be a non empty metric space, A be a non empty subset of M, S be a sequence of $M \restriction A$, and S' be a sequence of M. If $S = S'$, then S' is Cauchy iff S is Cauchy.

(19) Let M be a non empty metric space. Suppose M is complete. Let A be a non empty subset of M and A' be a subset of $M_{\rm top}$. If $A = A'$, then $M \restriction A$ is complete iff A' is closed.

3. Countable Compact Spaces

Let T be a topological structure. We say that T is countably-compact if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let F be a family of subsets of T. Suppose F is a cover of T, open, and countable. Then there exists a family G of subsets of T such that $G \subseteq F$ and G is a cover of T and finite.

We now state a number of propositions:

(20) For every topological structure T such that T is compact holds T is countably-compact.

(21) Let T be a non empty topological space. Then T is countably-compact if and only if for every family F of subsets of T such that F is centered, closed, and countable holds $\bigcap F \neq \emptyset$.

(22) Let T be a non empty topological space. Then T is countably-compact if and only if for every non-empty closed sequence S of subsets of T such that S is descending holds $\bigcap S \neq \emptyset$.

(23) Let T be a non empty topological space, F be a family of subsets of T, and S be a sequence of subsets of T. Suppose $\operatorname{rng} S \subseteq F$ and S is non-empty. Then there exists a non-empty closed sequence R of subsets of T such that

(i) R is descending,

(ii) if F is locally finite and S is one-to-one, then $\bigcap R = \emptyset$, and

(iii) for every i there exists a family S_1 of subsets of T such that $R(i) = \overline{\bigcup S_1}$ and $S_1 = \{S(j); j$ ranges over elements of $\mathbb{N}$: $j \geq i\}$.

(24) For every function F such that $\operatorname{dom} F$ is infinite and $\operatorname{rng} F$ is finite there exists x such that $x \in \operatorname{rng} F$ and $F^{-1}(\{x\})$ is infinite.

(25) Let X be a non empty set and F be a sequence of subsets of X. Suppose F is descending. Let S be a function from $\mathbb{N}$ into X. If for every n holds $S(n) \in F(n)$, then if $\operatorname{rng} S$ is finite, then $\bigcap F$ is non empty.

(26) Let T be a non empty topological space. Then T is countably-compact if and only if for every family F of subsets of T such that F is locally finite and has non empty elements holds F is finite.

(27) Let T be a non empty topological space. Then T is countably-compact if and only if for every family F of subsets of T such that F is locally finite and for every subset A of T such that $A \in F$ holds $\overline{\overline{A}} = 1$ holds F is finite.

(28) Let T be a T_1 non empty topological space. Then T is countably-compact if and only if for every subset A of T such that A is infinite holds $\operatorname{Der} A$ is non empty.

(29) Let T be a T_1 non empty topological space. Then T is countably-compact if and only if for every subset A of T such that A is infinite and countable holds $\operatorname{Der} A$ is non empty.

The scheme *Th39* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a subset A of $\mathcal{A}$ such that

(i) for all elements x, y of $\mathcal{A}$ such that $x, y \in A$ and $x \neq y$ holds $\mathcal{P}[x, y]$, and

(ii) for every element x of $\mathcal{A}$ there exists an element y of $\mathcal{A}$ such that $y \in A$ and not $\mathcal{P}[x, y]$

provided the following conditions are satisfied:

- For all elements x, y of $\mathcal{A}$ holds $\mathcal{P}[x, y]$ iff $\mathcal{P}[y, x]$, and
- For every element x of $\mathcal{A}$ holds not $\mathcal{P}[x, x]$.

We now state several propositions:

(30) Let M be a reflexive symmetric non empty metric structure and r be a real number. Suppose $r > 0$. Then there exists a subset A of M such that

(i) for all points p, q of M such that $p \neq q$ and $p, q \in A$ holds $\rho(p, q) \geq r$, and

(ii) for every point p of M there exists a point q of M such that $q \in A$ and $p \in \mathrm{Ball}(q, r)$.

(31) Let M be a reflexive symmetric triangle non empty metric structure. Then M is totally bounded if and only if for every real number r and for every subset A of M such that $r > 0$ and for all points p, q of M such that $p \neq q$ and p, $q \in A$ holds $\rho(p, q) \geq r$ holds A is finite.

(32) Let M be a reflexive symmetric triangle non empty metric structure. If M_{top} is countably-compact, then M is totally bounded.

(33) For every non empty metric space M such that M is totally bounded holds M_{top} is second-countable.

(34) Let T be a non empty topological space. Suppose T is second-countable. Let F be a family of subsets of T. Suppose F is a cover of T and open. Then there exists a family G of subsets of T such that $G \subseteq F$ and G is a cover of T and countable.

4. The Main Theorem

The following three propositions are true:

(35) For every non empty metric space M holds M_{top} is compact iff M_{top} is countably-compact.

(36) Let X be a set and F be a family of subsets of X. Suppose F is finite. Let A be a subset of X. Suppose A is infinite and $A \subseteq \bigcup F$. Then there exists a subset Y of X such that $Y \in F$ and $Y \cap A$ is infinite.

(37) For every non empty metric space M holds M_{top} is compact iff M is totally bounded and complete.

5. Well Spaces

Let T be a set, let S be a function from $\mathbb{N}$ into T, and let i be a natural number. Then $S(i)$ is an element of T.

The following proposition is true

(38) Let M be a metric structure, a be a point of M, and given x. Then $x \in X \times ((\text{the carrier of } M) \setminus \{a\}) \cup \{\langle X, a\rangle\}$ if and only if there exists a set y and there exists a point b of M such that $x = \langle y, b\rangle$ but $y \in X$ and $b \neq a$ or $y = X$ and $b = a$.

Let M be a metric structure, let a be a point of M, and let X be a set. The functor well-dist(a, X) yields a function from $(X \times ((\text{the carrier of } M) \setminus \{a\}) \cup \{\langle X, a\rangle\}) \times (X \times ((\text{the carrier of } M) \setminus \{a\}) \cup \{\langle X, a\rangle\})$ into $\mathbb{R}$ and is defined by the condition (Def. 10).

(Def. 10) Let x, y be elements of $X \times ((\text{the carrier of } M) \setminus \{a\}) \cup \{\langle X, a\rangle\}$, x_1, y_1 be sets, and x_2, y_2 be points of M such that $x = \langle x_1, x_2\rangle$ and $y = \langle y_1, y_2\rangle$. Then

(i) if $x_1 = y_1$, then $(\text{well-dist}(a, X))(x, y) = \rho(x_2, y_2)$, and

(ii) if $x_1 \neq y_1$, then $(\text{well-dist}(a, X))(x, y) = \rho(x_2, a) + \rho(a, y_2)$.

We now state the proposition

(39) Let M be a metric structure, a be a point of M, and X be a non empty set. Then

(i) if well-dist(a, X) is reflexive, then M is reflexive,

(ii) if well-dist(a, X) is symmetric, then M is symmetric,

(iii) if well-dist(a, X) is triangle and reflexive, then M is triangle, and

(iv) if well-dist(a, X) is discernible and reflexive, then M is discernible.

Let M be a metric structure, let a be a point of M, and let X be a set. The functor WellSpace(a, X) yields a strict metric structure and is defined as follows:

(Def. 11) $\text{WellSpace}(a, X) = \langle X \times ((\text{the carrier of } M) \setminus \{a\}) \cup \{\langle X, a\rangle\}, \text{well-dist}(a, X)\rangle$.

Let M be a metric structure, let a be a point of M, and let X be a set. One can check that WellSpace(a, X) is non empty.

Let M be a reflexive metric structure, let a be a point of M, and let X be a set. Note that WellSpace(a, X) is reflexive.

Let M be a symmetric metric structure, let a be a point of M, and let X be a set. Observe that WellSpace(a, X) is symmetric.

Let M be a symmetric triangle reflexive metric structure, let a be a point of M, and let X be a set. One can verify that WellSpace(a, X) is triangle.

Let M be a metric space, let a be a point of M, and let X be a set. Observe that WellSpace(a, X) is discernible.

We now state several propositions:

(40) Let M be a triangle reflexive non empty metric structure, a be a point of M, and X be a non empty set. If WellSpace(a, X) is complete, then M is complete.

(41) Let M be a symmetric triangle reflexive non empty metric structure, a be a point of M, and S be a sequence of WellSpace(a, X). Suppose S is Cauchy. Then

(i) for every point X_1 of WellSpace(a, X) such that $X_1 = \langle X, a\rangle$ and for every r such that $r > 0$ there exists n such that for every m such that $m \geq n$ holds $\rho(S(m), X_1) < r$, or

(ii) there exist n, Y such that for every m such that $m \geq n$ there exists a point p of M such that $S(m) = \langle Y, p\rangle$.

(42) Let M be a symmetric triangle reflexive non empty metric structure and a be a point of M. If M is complete, then WellSpace(a, X) is complete.

(43) Let M be a symmetric triangle reflexive non empty metric structure. Suppose M is complete. Let a be a point of M. Given a point b of M such that $\rho(a,b) \neq 0$. Let X be an infinite set. Then

(i) WellSpace(a, X) is complete, and

(ii) there exists a non-empty bounded sequence S of subsets of WellSpace(a, X) such that S is closed and descending and $\bigcap S$ is empty.

(44) There exists a non empty metric space M such that

(i) M is complete, and

(ii) there exists a non-empty bounded sequence S of subsets of M such that S is closed and descending and $\bigcap S$ is empty.

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