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Mizar Analysis of Algorithms: Preliminaries[1]

Grzegorz Bancerek
Department of Theoretical Computer Science
Białystok Technical University
Poland

Summary. Algorithms and its parts – instructions – are formalized as elements of if-while algebras. An if-while algebra is a (1-sorted) universal algebra which has 4 operations: a constant – the empty instruction, a binary catenation of instructions, a ternary conditional instruction, and a binary while instruction. An execution function is defined on pairs (s, I), where s is a state (an element of certain set of states) and I is an instruction, and results in states. The execution function obeys control structures using the set of distinguished true states, i.e. a condition instruction is executed and the continuation of execution depends on if the resulting state is in true states or not. Termination is also defined for pairs (s, I) and depends on the execution function. The existence of execution function determined on elementary instructions and its uniqueness for terminating instructions are shown.

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The articles [42], [26], [47], [36], [6], [45], [49], [22], [50], [25], [23], [19], [29], [28], [11], [34], [33], [20], [1], [5], [41], [21], [43], [12], [39], [4], [7], [8], [3], [31], [16], [30], [40], [24], [2], [15], [27], [48], [35], [18], [32], [37], [10], [14], [17], [9], [13], [44], [38], and [46] provide the terminology and notation for this paper.

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1. Binary Operations, Orbits, and Iterations

(1) Let f, g, h be functions and A be a set. Suppose $A \subseteq \operatorname{dom} f$ and $A \subseteq \operatorname{dom} g$ and $\operatorname{rng} h \subseteq A$ and for every set x such that $x \in A$ holds $f(x) = g(x)$. Then $f \cdot h = g \cdot h$.

Let x, y be non empty sets. Observe that $\langle x, y \rangle$ is non-empty.

Let p, q be non-empty finite sequences. One can check that $p \frown q$ is non-empty.

Let f be a homogeneous function and let x be a set. We say that x is a unity w.r.t. f if and only if:

(Def. 1) For all sets y, z such that $\langle y, z \rangle \in \operatorname{dom} f$ or $\langle z, y \rangle \in \operatorname{dom} f$ holds $\langle x, y \rangle \in \operatorname{dom} f$ and $f(\langle x, y \rangle) = y$ and $\langle y, x \rangle \in \operatorname{dom} f$ and $f(\langle y, x \rangle) = y$.

Let f be a homogeneous function. We say that f is associative if and only if:

(Def. 2) For all sets x, y, z such that $\langle x, y \rangle \in \operatorname{dom} f$ and $\langle y, z \rangle \in \operatorname{dom} f$ and $\langle f(\langle x, y \rangle), z \rangle \in \operatorname{dom} f$ and $\langle x, f(\langle y, z \rangle) \rangle \in \operatorname{dom} f$ holds $f(\langle f(\langle x, y \rangle), z \rangle) = f(\langle x, f(\langle y, z \rangle) \rangle)$.

We say that f is unital if and only if:

(Def. 3) There exists a set which is a unity w.r.t. f.

Let X be a set, let Y be a non empty set, let Z be a set of finite sequences of X, and let y be an element of Y. Then $Z \longmapsto y$ is a partial function from X^* to Y.

Let X be a non empty set, let x be an element of X, and let n be a natural number. Observe that $X^n \longmapsto x$ is non empty, quasi total, and homogeneous.

One can prove the following proposition

(2) For every non empty set X and for every element x of X and for every natural number n holds $\operatorname{arity}(X^n \longmapsto x) = n$.

Let X be a non empty set and let x be an element of X. One can check the following observations:

* $X^0 \longmapsto x$ is nullary,
* $X^1 \longmapsto x$ is unary,
* $X^2 \longmapsto x$ is binary, and
* $X^3 \longmapsto x$ is ternary.

Let X be a non empty set. One can check the following observations:

* there exists a non empty quasi total homogeneous partial function from X^* to X which is binary, associative, and unital,
* there exists a non empty quasi total homogeneous partial function from X^* to X which is nullary, and

* there exists a non empty quasi total homogeneous partial function from X^* to X which is ternary.

Next we state the proposition

(3) Let X be a non empty set, p be a finite sequence of elements of $\mathrm{FinTrees}(X)$, and x, t be sets. If $t \in \operatorname{rng} p$, then $t \neq x\text{-tree}(p)$.

Let f, g be functions and let X be a set. The functor $f{+\cdot}^X g$ yields a function and is defined as follows:

(Def. 4) $f{+\cdot}^X g = g{+\cdot} f{\restriction} X$.

We now state two propositions:

(4) For all functions f, g and for all sets x, X such that $x \in X$ and $X \subseteq \operatorname{dom} f$ holds $(f{+\cdot}^X g)(x) = f(x)$.

(5) For all functions f, g and for all sets x, X such that $x \notin X$ and $x \in \operatorname{dom} g$ holds $(f{+\cdot}^X g)(x) = g(x)$.

Let X, Y be non empty sets, let f, g be elements of Y^X, and let A be a set. Then $f{+\cdot}^A g$ is an element of Y^X.

Let X, Y, Z be non empty sets, let f be an element of Y^X, and let g be an element of Z^Y. Then $g \cdot f$ is an element of Z^X.

Let f be a function and let x be a set. The functor $f\text{-orbit}(x)$ is defined by:

(Def. 5) $f\text{-orbit}(x) = \{f^n(x); n$ ranges over elements of $\mathbb{N}$: $x \in \operatorname{dom}(f^n)\}$.

We now state four propositions:

(6) For every function f and for every set x such that $x \in \operatorname{dom} f$ holds $x \in f\text{-orbit}(x)$.

(7) For every function f and for all sets x, y such that $\operatorname{rng} f \subseteq \operatorname{dom} f$ and $y \in f\text{-orbit}(x)$ holds $f(y) \in f\text{-orbit}(x)$.

(8) For every function f and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) \in f\text{-orbit}(x)$.

(9) For every function f and for every set x such that $x \in \operatorname{dom} f$ and $f(x) \in \operatorname{dom} f$ holds $f\text{-orbit}(f(x)) \subseteq f\text{-orbit}(x)$.

Let f be a function. Let us assume that $\operatorname{rng} f \subseteq \operatorname{dom} f$. Let A be a set and let x be a set. The functor $f^*_{A \to x}$ yielding a function is defined by the conditions (Def. 6).

(Def. 6)(i) $\operatorname{dom}(f^*_{A \to x}) = \operatorname{dom} f$, and

(ii) for every set a such that $a \in \operatorname{dom} f$ holds if $f\text{-orbit}(a) \subseteq A$, then $f^*_{A \to x}(a) = x$ and for every natural number n such that $f^n(a) \notin A$ and for every natural number i such that $i < n$ holds $f^i(a) \in A$ holds $f^*_{A \to x}(a) = f^n(a)$.

Let f be a function. Let us assume that $\operatorname{rng} f \subseteq \operatorname{dom} f$. Let A be a set and let g be a function. The functor $f^*_{A \to g}$ yields a function and is defined by the conditions (Def. 7).

(Def. 7)(i) $\mathrm{dom}(f^*_{A \to g}) = \mathrm{dom}\, f$, and

(ii) for every set a such that $a \in \mathrm{dom}\, f$ holds if f-orbit$(a) \subseteq A$, then $f^*_{A \to g}(a) = g(a)$ and for every natural number n such that $f^n(a) \notin A$ and for every natural number i such that $i < n$ holds $f^i(a) \in A$ holds $f^*_{A \to g}(a) = f^n(a)$.

The following propositions are true:

(10) Let f, g be functions and a, A be sets. Suppose $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$ and $a \in \mathrm{dom}\, f$. Suppose f-orbit$(a) \not\subseteq A$. Then there exists a natural number n such that $f^*_{A \to g}(a) = f^n(a)$ and $f^n(a) \notin A$ and for every natural number i such that $i < n$ holds $f^i(a) \in A$.

(11) Let f, g be functions and a, A be sets. If $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$ and $a \in \mathrm{dom}\, f$ and $g \cdot f = g$, then if $a \in A$, then $f^*_{A \to g}(a) = f^*_{A \to g}(f(a))$.

(12) For all functions f, g and for all sets a, A such that $\mathrm{rng}\, f \subseteq \mathrm{dom}\, f$ and $a \in \mathrm{dom}\, f$ holds if $a \notin A$, then $f^*_{A \to g}(a) = a$.

Let X be a non empty set, let f be an element of X^X, let A be a set, and let g be an element of X^X. Then $f^*_{A \to g}$ is an element of X^X.

2. Free Universal Algebras

We now state three propositions:

(13) Let X be a non empty set and S be a non empty finite sequence of elements of $\mathbb{N}$. Then there exists a universal algebra A such that the carrier of $A = X$ and signature $A = S$.

(14) Let S be a non empty finite sequence of elements of $\mathbb{N}$. Then there exists a universal algebra A such that

(i) the carrier of $A = \mathbb{N}$,

(ii) signature $A = S$, and

(iii) for all natural numbers i, j such that $i \in \mathrm{dom}\, S$ and $j = S(i)$ holds (the characteristic of $A)(i) = \mathbb{N}^j \longmapsto i$.

(15) Let S be a non empty finite sequence of elements of $\mathbb{N}$ and i, j be natural numbers. Suppose $i \in \mathrm{dom}\, S$ and $j = S(i)$. Let X be a non empty set and f be a function from X^j into X. Then there exists a universal algebra A such that the carrier of $A = X$ and signature $A = S$ and (the characteristic of $A)(i) = f$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Observe that every element of FreeUnivAlgNSG(f, D) is relation-like and function-like.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. One can verify that every element of FreeUnivAlgNSG(f, D)

is decorated tree-like and every finite sequence of elements of FreeUnivAlgNSG(f, D) is decorated tree yielding.

We now state two propositions:

(16) Let G be a non empty tree construction structure and t be a set. Suppose $t \in \mathrm{TS}(G)$. Then

(i) there exists a symbol d of G such that $d \in$ the terminals of G and $t =$ the root tree of d, or

(ii) there exists a symbol o of G and there exists a finite sequence p of elements of $\mathrm{TS}(G)$ such that $o \Rightarrow$ the roots of p and $t = o$-tree(p).

(17) Let X be a missing $\mathbb{N}$ non empty set, S be a non empty finite sequence of elements of $\mathbb{N}$, and i be a natural number. Suppose $i \in \mathrm{dom}\, S$. Let p be a finite sequence of elements of FreeUnivAlgNSG(S, X). If $\mathrm{len}\, p = S(i)$, then (Den($i(\in$ dom (the characteristic of FreeUnivAlgNSG(S, X))), FreeUnivAlgNSG(S, X)))(p) $= i$-tree(p).

Let A be a non-empty universal algebra structure, let B be a subset of A, and let n be a natural number. The functor B^n yielding a subset of A is defined by the condition (Def. 8).

(Def. 8) There exists a function F from $\mathbb{N}$ into $2^{\text{the carrier of } A}$ such that

(i) $B^n = F(n)$,

(ii) $F(0) = B$, and

(iii) for every natural number n holds $F(n+1) = F(n) \cup \{(\mathrm{Den}(o, A))(p); o$ ranges over elements of dom (the characteristic of A), p ranges over elements of (the carrier of A)*: $p \in \mathrm{dom}\,\mathrm{Den}(o, A) \ \wedge \ \mathrm{rng}\, p \subseteq F(n)\}$.

Next we state several propositions:

(18) For every universal algebra A and for every subset B of A holds $B^0 = B$.

(19) Let A be a universal algebra, B be a subset of A, and n be a natural number. Then $B^{n+1} = B^n \cup \{(\mathrm{Den}(o, A))(p); o$ ranges over elements of dom (the characteristic of A), p ranges over elements of (the carrier of A)*: $p \in \mathrm{dom}\,\mathrm{Den}(o, A) \ \wedge \ \mathrm{rng}\, p \subseteq B^n\}$.

(20) Let A be a universal algebra, B be a subset of A, n be a natural number, and x be a set. Then $x \in B^{n+1}$ if and only if one of the following conditions is satisfied:

(i) $x \in B^n$, or

(ii) there exists an element o of dom (the characteristic of A) and there exists an element p of (the carrier of A)* such that $x = (\mathrm{Den}(o, A))(p)$ and $p \in \mathrm{dom}\,\mathrm{Den}(o, A)$ and $\mathrm{rng}\, p \subseteq B^n$.

(21) Let A be a universal algebra, B be a subset of A, and n, m be natural numbers. If $n \leq m$, then $B^n \subseteq B^m$.

(22) Let A be a universal algebra and B_1, B_2 be subsets of A. If $B_1 \subseteq B_2$, then for every natural number n holds ${B_1}^n \subseteq {B_2}^n$.

(23) Let A be a universal algebra, B be a subset of A, n be a natural number, and x be a set. Then $x \in B^{n+1}$ if and only if one of the following conditions is satisfied:

(i) $x \in B$, or

(ii) there exists an element o of dom (the characteristic of A) and there exists an element p of (the carrier of A)* such that $x = (\mathrm{Den}(o, A))(p)$ and $p \in \mathrm{dom}\,\mathrm{Den}(o, A)$ and $\mathrm{rng}\, p \subseteq B^n$.

The scheme *MaxVal* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a natural number n such that for every element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ holds $\mathcal{P}[x, n]$

provided the following conditions are satisfied:

- $\mathcal{B}$ is finite,
- For every element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ there exists a natural number n such that $\mathcal{P}[x, n]$, and
- For every element x of $\mathcal{A}$ and for all natural numbers n, m such that $\mathcal{P}[x, n]$ and $n \leq m$ holds $\mathcal{P}[x, m]$.

We now state two propositions:

(24) Let A be a universal algebra and B be a subset of A. Then there exists a subset C of A such that $C = \bigcup\{B^n : n$ ranges over elements of $\mathbb{N}\}$ and C is operations closed.

(25) Let A be a universal algebra and B, C be subsets of A. Suppose C is operations closed and $B \subseteq C$. Then $\bigcup\{B^n : n$ ranges over elements of $\mathbb{N}\} \subseteq C$.

Let A be a universal algebra. The functor Generators A yielding a subset of A is defined by:

(Def. 9) Generators $A =$ (the carrier of A) $\setminus \bigcup\{\mathrm{rng}\, o : o$ ranges over elements of Operations$(A)\}$.

Next we state several propositions:

(26) Let A be a universal algebra and a be an element of A. Then $a \in$ Generators A if and only if it is not true that there exists an element o of Operations(A) such that $a \in \mathrm{rng}\, o$.

(27) For every universal algebra A and for every subset B of A such that B is operations closed holds Constants$(A) \subseteq B$.

(28) For every universal algebra A such that Constants$(A) = \emptyset$ holds $\emptyset_A$ is operations closed.

(29) For every universal algebra A such that Constants$(A) = \emptyset$ and for every generator set G of A holds $G \neq \emptyset$.

(30) Let A be a universal algebra and G be a subset of A. Then G is a generator set of A if and only if for every element I of A there exists a

natural number n such that $I \in G^n$.

(31) Let A be a universal algebra, B be a subset of A, and G be a generator set of A. If $G \subseteq B$, then B is a generator set of A.

(32) Let A be a universal algebra, G be a generator set of A, and a be an element of A. If it is not true that there exists an element o of $\mathrm{Operations}(A)$ such that $a \in \mathrm{rng}\, o$, then $a \in G$.

(33) For every universal algebra A and for every generator set G of A holds $\mathrm{Generators}\, A \subseteq G$.

(34) For every free universal algebra A and for every free generator set G of A holds $G = \mathrm{Generators}\, A$.

Let A be a free universal algebra. Note that $\mathrm{Generators}\, A$ is free.

Let A be a free universal algebra. Then $\mathrm{Generators}\, A$ is a generator set of A.

Let A, B be sets. Note that $[\!: A, B :\!]$ is missing $\mathbb{N}$.

One can prove the following propositions:

(35) Let A be a free universal algebra, G be a generator set of A, B be a universal algebra, and h_1, h_2 be functions from A into B. Suppose h_1 is a homomorphism of A into B and h_2 is a homomorphism of A into B and $h_1 {\restriction} G = h_2 {\restriction} G$. Then $h_1 = h_2$.

(36) Let A be a free universal algebra, o_1, o_2 be operation symbols of A, and p_1, p_2 be finite sequences. If $p_1 \in \mathrm{dom}\, \mathrm{Den}(o_1, A)$ and $p_2 \in \mathrm{dom}\, \mathrm{Den}(o_2, A)$, then if $(\mathrm{Den}(o_1, A))(p_1) = (\mathrm{Den}(o_2, A))(p_2)$, then $o_1 = o_2$ and $p_1 = p_2$.

(37) Let A be a free universal algebra, o_1, o_2 be elements of $\mathrm{Operations}(A)$, and p_1, p_2 be finite sequences. If $p_1 \in \mathrm{dom}\, o_1$ and $p_2 \in \mathrm{dom}\, o_2$, then if $o_1(p_1) = o_2(p_2)$, then $o_1 = o_2$ and $p_1 = p_2$.

(38) Let A be a free universal algebra, o be an operation symbol of A, and p be a finite sequence. If $p \in \mathrm{dom}\, \mathrm{Den}(o, A)$, then for every set a such that $a \in \mathrm{rng}\, p$ holds $a \neq (\mathrm{Den}(o, A))(p)$.

(39) Let A be a free universal algebra, G be a generator set of A, and o be an operation symbol of A. Suppose that for every operation symbol o' of A and for every finite sequence p such that $p \in \mathrm{dom}\, \mathrm{Den}(o', A)$ and $(\mathrm{Den}(o', A))(p) \in G$ holds $o' \neq o$. Let p be a finite sequence. Suppose $p \in \mathrm{dom}\, \mathrm{Den}(o, A)$. Let n be a natural number. If $(\mathrm{Den}(o, A))(p) \in G^{n+1}$, then $\mathrm{rng}\, p \subseteq G^n$.

(40) Let A be a free universal algebra, o be an operation symbol of A, and p be a finite sequence. Suppose $p \in \mathrm{dom}\, \mathrm{Den}(o, A)$. Let n be a natural number. If $(\mathrm{Den}(o, A))(p) \in (\mathrm{Generators}\, A)^{n+1}$, then $\mathrm{rng}\, p \subseteq (\mathrm{Generators}\, A)^n$.

3. If-while Algebra

Let S be a non empty universal algebra structure. We say that S has empty-instruction if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) $1 \in \mathrm{dom}$ (the characteristic of S), and

(ii) (the characteristic of S)(1) is a nullary non empty homogeneous quasi total partial function from (the carrier of S)* to the carrier of S.

We say that S has catenation if and only if the conditions (Def. 11) are satisfied.

(Def. 11)(i) $2 \in \mathrm{dom}$ (the characteristic of S), and

(ii) (the characteristic of S)(2) is a binary non empty homogeneous quasi total partial function from (the carrier of S)* to the carrier of S.

We say that S has if-instruction if and only if the conditions (Def. 12) are satisfied.

(Def. 12)(i) $3 \in \mathrm{dom}$ (the characteristic of S), and

(ii) (the characteristic of S)(3) is a ternary non empty homogeneous quasi total partial function from (the carrier of S)* to the carrier of S.

We say that S has while-instruction if and only if the conditions (Def. 13) are satisfied.

(Def. 13)(i) $4 \in \mathrm{dom}$ (the characteristic of S), and

(ii) (the characteristic of S)(4) is a binary non empty homogeneous quasi total partial function from (the carrier of S)* to the carrier of S.

We say that S is associative if and only if the condition (Def. 14) is satisfied.

(Def. 14) (The characteristic of S)(2) is a binary associative non empty homogeneous quasi total partial function from (the carrier of S)* to the carrier of S.

Let S be a non-empty universal algebra structure. We say that S is unital if and only if the condition (Def. 15) is satisfied.

(Def. 15) There exists a binary non empty homogeneous quasi total partial function f from (the carrier of S)* to the carrier of S such that $f =$ (the characteristic of S)(2) and (Den(1($\in$ dom (the characteristic of S)), S))($\emptyset$) is a unity w.r.t. f.

One can prove the following proposition

(41) Let X be a non empty set, x be an element of X, and c be a binary associative unital non empty quasi total homogeneous partial function from X^* to X. Suppose x is a unity w.r.t. c. Let i be a ternary non empty quasi total homogeneous partial function from X^* to X and w be a binary non empty quasi total homogeneous partial function from X^* to X. Then there exists a non-empty strict universal algebra structure S such that

(i) the carrier of $S = X$,

(ii) the characteristic of $S = \langle X^0 \longmapsto x, c\rangle \frown \langle i, w\rangle$, and

(iii) S is unital, associative, quasi total, and partial and has empty-instruction, catenation, if-instruction, and while-instruction.

Let us note that there exists a quasi total partial non-empty strict universal algebra structure which is unital and associative and has empty-instruction, catenation, if-instruction, and while-instruction.

A pre-if-while algebra is a universal algebra with empty-instruction, catenation, if-instruction, and while-instruction.

For simplicity, we use the following convention: A is a pre-if-while algebra, C, I, J are elements of A, S is a non empty set, T is a subset of S, and s is an element of S.

Let A be a non empty universal algebra structure. An algorithm of A is an element of A.

The following proposition is true

(42) Let A be a non-empty universal algebra structure with empty-instruction. Then $\mathrm{dom}\,\mathrm{Den}(1(\in \mathrm{dom}\,(\text{the characteristic of } A)), A) = \{\emptyset\}$.

Let A be a non-empty universal algebra structure with empty-instruction. The functor $\mathrm{EmptyIns}_A$ yielding an algorithm of A is defined as follows:

(Def. 16) $\mathrm{EmptyIns}_A = (\mathrm{Den}(1(\in \mathrm{dom}\,(\text{the characteristic of } A)), A))(\emptyset)$.

The following two propositions are true:

(43) Let A be a universal algebra with empty-instruction and o be an element of $\mathrm{Operations}(A)$. If $o = \mathrm{Den}(1(\in \mathrm{dom}\,(\text{the characteristic of } A)), A)$, then $\mathrm{arity}\, o = 0$ and $\mathrm{EmptyIns}_A \in \mathrm{rng}\, o$.

(44) Let A be a non-empty universal algebra structure with catenation. Then $\mathrm{dom}\,\mathrm{Den}(2(\in \mathrm{dom}\,(\text{the characteristic of } A)), A) = (\text{the carrier of } A)^2$.

Let A be a non-empty universal algebra structure with catenation and let I_1, I_2 be algorithms of A. The functor $I_1; I_2$ yielding an algorithm of A is defined as follows:

(Def. 17) $I_1; I_2 = (\mathrm{Den}(2(\in \mathrm{dom}\,(\text{the characteristic of } A)), A))(\langle I_1, I_2\rangle)$.

The following propositions are true:

(45) Let A be a unital non-empty universal algebra structure with empty-instruction and catenation and I be an element of A. Then $\mathrm{EmptyIns}_A; I = I$ and $I; \mathrm{EmptyIns}_A = I$.

(46) Let A be an associative non-empty universal algebra structure with catenation and I_1, I_2, I_3 be elements of A. Then $(I_1; I_2); I_3 = I_1; (I_2; I_3)$.

(47) Let A be a non-empty universal algebra structure with if-instruction. Then $\mathrm{dom}\,\mathrm{Den}(3(\in \mathrm{dom}\,(\text{the characteristic of } A)), A) = (\text{the carrier of } A)^3$.

Let A be a non-empty universal algebra structure with if-instruction and let C, I_1, I_2 be algorithms of A. The functor $\mathrm{if}\, C\, \mathrm{then}\, I_1\, \mathrm{else}\, I_2$ yields an algorithm of A and is defined as follows:

(Def. 18) $\mathrm{if}\, C \,\mathrm{then}\, I_1 \,\mathrm{else}\, I_2 = (\mathrm{Den}(3(\in \mathrm{dom}\,(\text{the characteristic of } A)), A))(\langle C, I_1, I_2\rangle)$.

Let A be a non-empty universal algebra structure with empty-instruction and if-instruction and let C, I be algorithms of A. The functor $\mathrm{if}\, C \,\mathrm{then}\, I$ yields an algorithm of A and is defined as follows:

(Def. 19) $\mathrm{if}\, C \,\mathrm{then}\, I = \mathrm{if}\, C \,\mathrm{then}\, I \,\mathrm{else}\, (\mathrm{EmptyIns}_A)$.

We now state the proposition

(48) Let A be a non-empty universal algebra structure with while-instruction. Then $\mathrm{dom}\,\mathrm{Den}(4(\in \mathrm{dom}\,(\text{the characteristic of } A)), A) = (\text{the carrier of } A)^2$.

Let A be a non-empty universal algebra structure with while-instruction and let C, I be algorithms of A. The functor $\mathrm{while}\, C \,\mathrm{do}\, I$ yields an algorithm of A and is defined as follows:

(Def. 20) $\mathrm{while}\, C \,\mathrm{do}\, I = (\mathrm{Den}(4(\in \mathrm{dom}\,(\text{the characteristic of } A)), A))(\langle C, I\rangle)$.

Let A be a pre-if-while algebra and let I_0, C, I, J be elements of A. The functor $\mathrm{for}\, I_0 \,\mathrm{until}\, C \,\mathrm{step}\, J \,\mathrm{do}\, I$ yields an element of A and is defined by:

(Def. 21) $\mathrm{for}\, I_0 \,\mathrm{until}\, C \,\mathrm{step}\, J \,\mathrm{do}\, I = I_0; \mathrm{while}\, C \,\mathrm{do}\, (I; J)$.

Let A be a pre-if-while algebra. The functor $\mathrm{ElementaryInstructions}_A$ yields a subset of A and is defined by the condition (Def. 22).

(Def. 22) $\mathrm{ElementaryInstructions}_A = (\text{the carrier of } A) \setminus \{\mathrm{EmptyIns}_A\} \setminus \mathrm{rng}\,\mathrm{Den}(3(\in \mathrm{dom}\,(\text{the characteristic of } A)), A) \setminus \mathrm{rng}\,\mathrm{Den}(4(\in \mathrm{dom}\,(\text{the characteristic of } A)), A) \setminus \{I_1; I_2; I_1$ ranges over algorithms of A, I_2 ranges over algorithms of A: $I_1 \neq I_1; I_2 \ \wedge\ I_2 \neq I_1; I_2\}$.

Next we state several propositions:

(49) For every pre-if-while algebra A holds $\mathrm{EmptyIns}_A \notin \mathrm{ElementaryInstructions}_A$.

(50) For every pre-if-while algebra A and for all elements I_1, I_2 of A such that $I_1 \neq I_1; I_2$ and $I_2 \neq I_1; I_2$ holds $I_1; I_2 \notin \mathrm{ElementaryInstructions}_A$.

(51) For every pre-if-while algebra A and for all elements C, I_1, I_2 of A holds $\mathrm{if}\, C \,\mathrm{then}\, I_1 \,\mathrm{else}\, I_2 \notin \mathrm{ElementaryInstructions}_A$.

(52) For every pre-if-while algebra A and for all elements C, I of A holds $\mathrm{while}\, C \,\mathrm{do}\, I \notin \mathrm{ElementaryInstructions}_A$.

(53) Let A be a pre-if-while algebra and I be an element of A. Suppose $I \notin \mathrm{ElementaryInstructions}_A$. Then

(i) $I = \mathrm{EmptyIns}_A$, or

(ii) there exist elements I_1, I_2 of A such that $I = I_1; I_2$ and $I_1 \neq I_1; I_2$ and $I_2 \neq I_1; I_2$, or

(iii) there exist elements C, I_1, I_2 of A such that $I = \mathrm{if}\, C \,\mathrm{then}\, I_1 \,\mathrm{else}\, I_2$, or

(iv) there exist elements C, J of A such that $I = \text{while}\, C \,\text{do}\, J$.

Let A be a pre-if-while algebra. We say that A is infinite if and only if:

(Def. 23) $\text{ElementaryInstructions}_A$ is infinite.

We say that A is degenerated if and only if the conditions (Def. 24) are satisfied.

(Def. 24)(i) There exist elements I_1, I_2 of A such that $I_1 \neq \text{EmptyIns}_A$ and $I_1; I_2 = I_2$ or $I_2 \neq \text{EmptyIns}_A$ and $I_1; I_2 = I_1$ or $I_1 \neq \text{EmptyIns}_A$ or $I_2 \neq \text{EmptyIns}_A$ but $I_1; I_2 = \text{EmptyIns}_A$, or

(ii) there exist elements C, I_1, I_2 of A such that $\text{if}\, C \,\text{then}\, I_1 \,\text{else}\, I_2 = \text{EmptyIns}_A$, or

(iii) there exist elements C, I of A such that $\text{while}\, C \,\text{do}\, I = \text{EmptyIns}_A$, or

(iv) there exist elements I_1, I_2, C, J_1, J_2 of A such that $I_1 \neq \text{EmptyIns}_A$ and $I_2 \neq \text{EmptyIns}_A$ and $I_1; I_2 = \text{if}\, C \,\text{then}\, J_1 \,\text{else}\, J_2$, or

(v) there exist elements I_1, I_2, C, J of A such that $I_1 \neq \text{EmptyIns}_A$ and $I_2 \neq \text{EmptyIns}_A$ and $I_1; I_2 = \text{while}\, C \,\text{do}\, J$, or

(vi) there exist elements C_1, I_1, I_2, C_2, J of A such that $\text{if}\, C_1 \,\text{then}\, I_1 \,\text{else}\, I_2 = \text{while}\, C_2 \,\text{do}\, J$.

We say that A is well founded if and only if:

(Def. 25) $\text{ElementaryInstructions}_A$ is a generator set of A.

The non empty finite sequence ECIW-signature of elements of $\mathbb{N}$ is defined by:

(Def. 26) $\text{ECIW-signature} = \langle 0, 2\rangle \frown \langle 3, 2\rangle$.

We now state the proposition

(54) $\text{len}\,\text{ECIW-signature} = 4$ and $\text{dom}\,\text{ECIW-signature} = \text{Seg}\,4$ and $(\text{ECIW-signature})(1) = 0$ and $(\text{ECIW-signature})(2) = 2$ and $(\text{ECIW-signature})(3) = 3$ and $(\text{ECIW-signature})(4) = 2$.

Let A be a partial non-empty non empty universal algebra structure. We say that A is E.C.I.W.-strict if and only if:

(Def. 27) $\text{signature}\, A = \text{ECIW-signature}$.

Next we state the proposition

(55) Let A be a partial non-empty non empty universal algebra structure. Suppose A is E.C.I.W.-strict. Let o be an operation symbol of A. Then $o = 1$ or $o = 2$ or $o = 3$ or $o = 4$.

Let X be a missing $\mathbb{N}$ non empty set. One can verify that $\text{FreeUnivAlgNSG}(\text{ECIW-signature}, X)$ has empty-instruction, catenation, if-instruction, and while-instruction.

We now state a number of propositions:

(56) Let X be a missing $\mathbb{N}$ non empty set and I be an element of $\text{FreeUnivAlgNSG}(\text{ECIW-signature}, X)$. Then

(i) there exists an element x of X such that $I =$ the root tree of x, or

(ii) there exists a natural number n and there exists a finite sequence p of elements of FreeUnivAlgNSG(ECIW-signature, X) such that $n \in \operatorname{Seg} 4$ and $I = n$-tree(p) and $\operatorname{len} p = (\text{ECIW-signature})(n)$.

(57) For every missing $\mathbb{N}$ non empty set X holds $\text{EmptyIns}_{\text{FreeUnivAlgNSG(ECIW-signature},X)} = 1\text{-tree}(\emptyset)$.

(58) Let X be a missing $\mathbb{N}$ non empty set and p be a finite sequence of elements of FreeUnivAlgNSG(ECIW-signature, X). If 1-tree(p) is an element of FreeUnivAlgNSG(ECIW-signature, X), then $p = \emptyset$.

(59) For every missing $\mathbb{N}$ non empty set X and for all elements I_1, I_2 of FreeUnivAlgNSG(ECIW-signature, X) holds $I_1; I_2 = 2\text{-tree}(I_1, I_2)$.

(60) Let X be a missing $\mathbb{N}$ non empty set and p be a finite sequence of elements of FreeUnivAlgNSG(ECIW-signature, X). Suppose 2-tree(p) is an element of FreeUnivAlgNSG(ECIW-signature, X). Then there exist elements I_1, I_2 of FreeUnivAlgNSG(ECIW-signature, X) such that $p = \langle I_1, I_2 \rangle$.

(61) For every missing $\mathbb{N}$ non empty set X and for all elements I_1, I_2 of FreeUnivAlgNSG(ECIW-signature, X) holds $I_1; I_2 \neq I_1$ and $I_1; I_2 \neq I_2$.

(62) Let X be a missing $\mathbb{N}$ non empty set and I_1, I_2, J_1, J_2 be elements of FreeUnivAlgNSG(ECIW-signature, X). If $I_1; I_2 = J_1; J_2$, then $I_1 = J_1$ and $I_2 = J_2$.

(63) For every missing $\mathbb{N}$ non empty set X and for all elements C, I_1, I_2 of FreeUnivAlgNSG(ECIW-signature, X) holds $\text{if}\, C \text{ then } I_1 \text{ else } I_2 = 3\text{-tree}(\langle C, I_1, I_2 \rangle)$.

(64) Let X be a missing $\mathbb{N}$ non empty set and p be a finite sequence of elements of FreeUnivAlgNSG(ECIW-signature, X). Suppose 3-tree(p) is an element of FreeUnivAlgNSG(ECIW-signature, X). Then there exist elements C, I_1, I_2 of FreeUnivAlgNSG(ECIW-signature, X) such that $p = \langle C, I_1, I_2 \rangle$.

(65) Let X be a missing $\mathbb{N}$ non empty set and C_1, C_2, I_1, I_2, J_1, J_2 be elements of FreeUnivAlgNSG(ECIW-signature, X). If $\text{if}\, C_1 \text{then} I_1 \text{else} I_2 = \text{if}\, C_2 \text{ then } J_1 \text{ else } J_2$, then $C_1 = C_2$ and $I_1 = J_1$ and $I_2 = J_2$.

(66) For every missing $\mathbb{N}$ non empty set X and for all elements C, I of FreeUnivAlgNSG(ECIW-signature, X) holds $\text{while}\, C \text{ do } I = 4\text{-tree}(C, I)$.

(67) Let X be a missing $\mathbb{N}$ non empty set and p be a finite sequence of elements of FreeUnivAlgNSG(ECIW-signature, X). Suppose 4-tree(p) is an element of FreeUnivAlgNSG(ECIW-signature, X).
Then there exist elements C, I of FreeUnivAlgNSG(ECIW-signature, X) such that $p = \langle C, I \rangle$.

(68) Let X be a missing $\mathbb{N}$ non empty set and I be an element of FreeUnivAlgNSG(ECIW-signature, X).
If $I \in \text{ElementaryInstructions}_{\text{FreeUnivAlgNSG(ECIW-signature},X)}$, then there exists an element x of X such that $I = x\text{-tree}(\emptyset)$.

(69) Let X be a missing $\mathbb{N}$ non empty set, p be a finite sequence of elements of FreeUnivAlgNSG(ECIW-signature, X), and x be an element of X. If x--tree(p) is an element of FreeUnivAlgNSG(ECIW-signature, X), then $p{=}\emptyset$.

(70) For every missing $\mathbb{N}$ non empty set X holds ElementaryInstructions$_{\text{FreeUnivAlgNSG(ECIW-signature},X)}$ = FreeGenSetNSG(ECIW-signature, X) and $\overline{\overline{X}} = \overline{\overline{\text{FreeGenSetNSG(ECIW-signature}, X)}}$.

Let us observe that there exists a set which is infinite and missing $\mathbb{N}$.

Let X be an infinite missing $\mathbb{N}$ set. One can check that FreeUnivAlgNSG(ECIW-signature, X) is infinite.

Let X be a missing $\mathbb{N}$ non empty set. Note that FreeUnivAlgNSG (ECIW-signature, X) is E.C.I.W.-strict.

The following propositions are true:

(71) For every pre-if-while algebra A holds Generators $A \subseteq$ ElementaryInstructions$_A$.

(72) Let A be a pre-if-while algebra. Suppose A is free. Let C, I_1, I_2 be elements of A. Then EmptyIns$_A \neq I_1; I_2$ and EmptyIns$_A \neq$ if C then I_1 else I_2 and EmptyIns$_A \neq$ while C do I_1.

(73) Let A be a pre-if-while algebra. Suppose A is free. Let I_1, I_2, C, J_1, J_2 be elements of A. Then $I_1; I_2 \neq I_1$ and $I_1; I_2 \neq I_2$ and if $I_1; I_2 = J_1; J_2$, then $I_1 = J_1$ and $I_2 = J_2$ and $I_1; I_2 \neq$ if C then J_1 else J_2 and $I_1; I_2 \neq$ while C do J_1.

(74) Let A be a pre-if-while algebra. Suppose A is free. Let C, I_1, I_2, D, J_1, J_2 be elements of A. Then if CthenI_1else$I_2 \neq C$ and if CthenI_1else$I_2 \neq I_1$ and if C then I_1 else $I_2 \neq I_2$ and if C then I_1 else $I_2 \neq$ while D do J_1 and if if C then I_1 else $I_2 =$ if D then J_1 else J_2, then $C = D$ and $I_1 = J_1$ and $I_2 = J_2$.

(75) Let A be a pre-if-while algebra. Suppose A is free. Let C, I, D, J be elements of A. Then while C do $I \neq C$ and while C do $I \neq I$ and if while C do $I =$ while D do J, then $C = D$ and $I = J$.

Let us note that every pre-if-while algebra which is free is also well founded and non degenerated.

Let us mention that there exists a pre-if-while algebra which is infinite, non degenerated, well founded, E.C.I.W.-strict, free, and strict.

An if-while algebra is a non degenerated well founded E.C.I.W.-strict pre-if-while algebra.

Let A be an infinite pre-if-while algebra.

Observe that ElementaryInstructions$_A$ is infinite.

One can prove the following four propositions:

(76) Let A be a pre-if-while algebra, B be a subset of A, and n be a natural number. Then

(i) $\mathrm{EmptyIns}_A \in B^{n+1}$, and

(ii) for all elements C, I_1, I_2 of A such that $C \in B^n$ and $I_1 \in B^n$ and $I_2 \in B^n$ holds $I_1; I_2 \in B^{n+1}$ and $\mathrm{if}\, C\, \mathrm{then}\, I_1\, \mathrm{else}\, I_2 \in B^{n+1}$ and $\mathrm{while}\, C\, \mathrm{do}\, I_1 \in B^{n+1}$.

(77) Let A be an E.C.I.W.-strict pre-if-while algebra, x be a set, and n be a natural number. Suppose $x \in \mathrm{ElementaryInstructions}_A{}^{n+1}$. Then

(i) $x \in \mathrm{ElementaryInstructions}_A{}^{n}$, or

(ii) $x = \mathrm{EmptyIns}_A$, or

(iii) there exist elements I_1, I_2 of A such that $x = I_1; I_2$ and $I_1 \in \mathrm{ElementaryInstructions}_A{}^{n}$ and $I_2 \in \mathrm{ElementaryInstructions}_A{}^{n}$, or

(iv) there exist elements C, I_1, I_2 of A such that $x = \mathrm{if}\, C\, \mathrm{then}\, I_1\, \mathrm{else}\, I_2$ and $C \in \mathrm{ElementaryInstructions}_A{}^{n}$ and $I_1 \in \mathrm{ElementaryInstructions}_A{}^{n}$ and $I_2 \in \mathrm{ElementaryInstructions}_A{}^{n}$, or

(v) there exist elements C, I of A such that $x = \mathrm{while}\, C\, \mathrm{do}\, I$ and $C \in \mathrm{ElementaryInstructions}_A{}^{n}$ and $I \in \mathrm{ElementaryInstructions}_A{}^{n}$.

(78) For every universal algebra A and for every subset B of A holds $\mathrm{Constants}(A) \subseteq B^1$.

(79) Let A be a pre-if-while algebra. Then A is well founded if and only if for every element I of A there exists a natural number n such that $I \in \mathrm{ElementaryInstructions}_A{}^{n}$.

The scheme *StructInd* deals with a well founded E.C.I.W.-strict pre-if-while algebra $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following conditions are satisfied:

- For every element I of $\mathcal{A}$ such that $I \in \mathrm{ElementaryInstructions}_{\mathcal{A}}$ holds $\mathcal{P}[I]$,
- $\mathcal{P}[\mathrm{EmptyIns}_{\mathcal{A}}]$,
- For all elements I_1, I_2 of $\mathcal{A}$ such that $\mathcal{P}[I_1]$ and $\mathcal{P}[I_2]$ holds $\mathcal{P}[I_1; I_2]$,
- For all elements C, I_1, I_2 of $\mathcal{A}$ such that $\mathcal{P}[C]$ and $\mathcal{P}[I_1]$ and $\mathcal{P}[I_2]$ holds $\mathcal{P}[\mathrm{if}\, C\, \mathrm{then}\, I_1\, \mathrm{else}\, I_2]$, and
- For all elements C, I of $\mathcal{A}$ such that $\mathcal{P}[C]$ and $\mathcal{P}[I]$ holds $\mathcal{P}[\mathrm{while}\, C\, \mathrm{do}\, I]$.

4. Execution Function

Let A be a pre-if-while algebra, let S be a non empty set, and let f be a function from $[: S, \text{the carrier of } A :]$ into S. We say that f is complying-with-empty-instruction if and only if:

(Def. 28) For every element s of S holds $f(s, \mathrm{EmptyIns}_A) = s$.

We say that f is complying-with-catenation if and only if:

(Def. 29) For every element s of S and for all elements I_1, I_2 of A holds $f(s, I_1; I_2) = f(f(s, I_1), I_2)$.

Let A be a pre-if-while algebra, let S be a non empty set, let T be a subset of S, and let f be a function from $[\!: S, \text{the carrier of } A :\!]$ into S. We say that f complies with **if** w.r.t. T if and only if the condition (Def. 30) is satisfied.

(Def. 30) Let s be an element of S and C, I_1, I_2 be elements of A. Then

(i) if $f(s, C) \in T$, then $f(s, \text{if } C \text{ then } I_1 \text{ else } I_2) = f(f(s, C), I_1)$, and

(ii) if $f(s, C) \notin T$, then $f(s, \text{if } C \text{ then } I_1 \text{ else } I_2) = f(f(s, C), I_2)$.

We say that f complies with **while** w.r.t. T if and only if the condition (Def. 31) is satisfied.

(Def. 31) Let s be an element of S and C, I be elements of A. Then

(i) if $f(s, C) \in T$, then $f(s, \text{while } C \text{ do } I) = f(f(f(s, C), I), \text{while } C \text{ do } I)$, and

(ii) if $f(s, C) \notin T$, then $f(s, \text{while } C \text{ do } I) = f(s, C)$.

One can prove the following two propositions:

(80) Let f be a function from $[\!: S, \text{the carrier of } A :\!]$ into S. Suppose f is complying-with-empty-instruction and f complies with **if** w.r.t. T. Let s be an element of S. If $f(s, C) \notin T$, then $f(s, \text{if } C \text{ then } I) = f(s, C)$.

(81)(i) $\pi_1(S \times \text{the carrier of } A)$ is complying-with-empty-instruction,

(ii) $\pi_1(S \times \text{the carrier of } A)$ is complying-with-catenation,

(iii) $\pi_1(S \times \text{the carrier of } A)$ complies with **if** w.r.t. T, and

(iv) $\pi_1(S \times \text{the carrier of } A)$ complies with **while** w.r.t. T.

Let A be a pre-if-while algebra, let S be a non empty set, and let T be a subset of S. A function from $[\!: S, \text{the carrier of } A :\!]$ into S is said to be an execution function of A over S and T if it satisfies the conditions (Def. 32).

(Def. 32)(i) It is complying-with-empty-instruction,

(ii) it is complying-with-catenation,

(iii) it complies with **if** w.r.t. T, and

(iv) it complies with **while** w.r.t. T.

Let A be a pre-if-while algebra, let S be a non empty set, and let T be a subset of S. One can verify that every execution function of A over S and T is complying-with-empty-instruction and complying-with-catenation.

Let A be a pre-if-while algebra, let I be an element of A, let S be a non empty set, let s be an element of S, let T be a subset of S, and let f be an execution function of A over S and T. We say that iteration of f started in I terminates w.r.t. s if and only if the condition (Def. 33) is satisfied.

(Def. 33) There exists a non empty finite sequence r of elements of S such that $r(1) = s$ and $r(\operatorname{len} r) \notin T$ and for every natural number i such that $1 \leq i$

and $i < \operatorname{len} r$ holds $r(i) \in T$ and $r(i+1) = f(r(i), I)$.

Let A be a pre-if-while algebra, let I be an element of A, let S be a non empty set, let s be an element of S, let T be a subset of S, and let f be an execution function of A over S and T. The functor termination-degree(I, s, f) yields an extended real number and is defined by:

(Def. 34)(i) There exists a non empty finite sequence r of elements of S such that termination-degree$(I, s, f) = \operatorname{len} r - 1$ and $r(1) = s$ and $r(\operatorname{len} r) \notin T$ and for every natural number i such that $1 \leq i$ and $i < \operatorname{len} r$ holds $r(i) \in T$ and $r(i+1) = f(r(i), I)$ if iteration of f started in I terminates w.r.t. s,

(ii) termination-degree$(I, s, f) = +\infty$, otherwise.

In the sequel f denotes an execution function of A over S and T.

We now state four propositions:

(82) Iteration of f started in I terminates w.r.t. s iff termination-degree$(I, s, f) < +\infty$.

(83) If $s \notin T$, then iteration of f started in I terminates w.r.t. s and termination-degree$(I, s, f) = 0$.

(84) Suppose $s \in T$. Then

(i) iteration of f started in I terminates w.r.t. s iff iteration of f started in I terminates w.r.t. $f(s, I)$, and

(ii) termination-degree$(I, s, f) = \overline{1} +$ termination-degree$(I, f(s, I), f)$.

(85) termination-degree$(I, s, f) \geq 0$.

Now we present two schemes. The scheme *Termination* deals with a pre-if-while algebra $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a non empty set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{C}$, a subset $\mathcal{E}$ of $\mathcal{C}$, an execution function $\mathcal{F}$ of $\mathcal{A}$ over $\mathcal{C}$ and $\mathcal{E}$, a unary functor $\mathcal{F}$ yielding a natural number, and a unary predicate $\mathcal{P}$, and states that:

Iteration of $\mathcal{F}$ started in $\mathcal{B}$ terminates w.r.t. $\mathcal{D}$

provided the parameters meet the following requirements:

- $\mathcal{D} \in \mathcal{E}$ iff $\mathcal{P}[\mathcal{D}]$, and
- For every element s of $\mathcal{C}$ such that $\mathcal{P}[s]$ holds $\mathcal{P}[\mathcal{F}(s, \mathcal{B})]$ iff $\mathcal{F}(s, \mathcal{B}) \in \mathcal{E}$ and $\mathcal{F}(\mathcal{F}(s, \mathcal{B})) < \mathcal{F}(s)$.

The scheme *Termination2* deals with a pre-if-while algebra $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a non empty set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{C}$, a subset $\mathcal{E}$ of $\mathcal{C}$, an execution function $\mathcal{F}$ of $\mathcal{A}$ over $\mathcal{C}$ and $\mathcal{E}$, a unary functor $\mathcal{F}$ yielding a natural number, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

Iteration of $\mathcal{F}$ started in $\mathcal{B}$ terminates w.r.t. $\mathcal{D}$

provided the following requirements are met:

- $\mathcal{P}[\mathcal{D}]$,
- $\mathcal{D} \in \mathcal{E}$ iff $\mathcal{Q}[\mathcal{D}]$, and
- Let s be an element of $\mathcal{C}$. Suppose $\mathcal{P}[s]$ and $s \in \mathcal{E}$ and $\mathcal{Q}[s]$. Then $\mathcal{P}[\mathcal{F}(s, \mathcal{B})]$ and $\mathcal{Q}[\mathcal{F}(s, \mathcal{B})]$ iff $\mathcal{F}(s, \mathcal{B}) \in \mathcal{E}$ and $\mathcal{F}(\mathcal{F}(s, \mathcal{B})) < \mathcal{F}(s)$.

Next we state two propositions:

(86) Let r be a non empty finite sequence of elements of S. Suppose $r(1) = f(s, C)$ and $r(\mathrm{len}\, r) \notin T$ and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, r$ holds $r(i) \in T$ and $r(i + 1) = f(r(i), I; C)$. Then $f(s, \mathrm{while}\, C\, \mathrm{do}\, I) = r(\mathrm{len}\, r)$.

(87) Let I be an element of A and s be an element of S. Then iteration of f started in I does not terminate w.r.t. s if and only if $(\mathrm{curry}'\, f)(I)\text{-orbit}(s) \subseteq T$.

Now we present two schemes. The scheme *InvariantSch* deals with a pre-if-while algebra $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, a non empty set $\mathcal{D}$, an element $\mathcal{E}$ of $\mathcal{D}$, a subset $\mathcal{F}$ of $\mathcal{D}$, an execution function $\mathcal{G}$ of $\mathcal{A}$ over $\mathcal{D}$ and $\mathcal{F}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$$\mathcal{P}[\mathcal{G}(\mathcal{E}, \mathrm{while}\, \mathcal{B}\, \mathrm{do}\, \mathcal{C})] \text{ and not } \mathcal{Q}[\mathcal{G}(\mathcal{E}, \mathrm{while}\, \mathcal{B}\, \mathrm{do}\, \mathcal{C})]$$

provided the following conditions are met:

- $\mathcal{P}[\mathcal{E}]$,
- Iteration of $\mathcal{G}$ started in $\mathcal{C}; \mathcal{B}$ terminates w.r.t. $\mathcal{G}(\mathcal{E}, \mathcal{B})$,
- For every element s of $\mathcal{D}$ such that $\mathcal{P}[s]$ and $s \in \mathcal{F}$ and $\mathcal{Q}[s]$ holds $\mathcal{P}[\mathcal{G}(s, \mathcal{C})]$, and
- For every element s of $\mathcal{D}$ such that $\mathcal{P}[s]$ holds $\mathcal{P}[\mathcal{G}(s, \mathcal{B})]$ and $\mathcal{G}(s, \mathcal{B}) \in \mathcal{F}$ iff $\mathcal{Q}[\mathcal{G}(s, \mathcal{B})]$.

The scheme *coInvariantSch* deals with a pre-if-while algebra $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, a non empty set $\mathcal{D}$, an element $\mathcal{E}$ of $\mathcal{D}$, a subset $\mathcal{F}$ of $\mathcal{D}$, an execution function $\mathcal{G}$ of $\mathcal{A}$ over $\mathcal{D}$ and $\mathcal{F}$, and a unary predicate $\mathcal{P}$, and states that:

$$\mathcal{P}[\mathcal{E}]$$

provided the following conditions are met:

- $\mathcal{P}[\mathcal{G}(\mathcal{E}, \mathrm{while}\, \mathcal{B}\, \mathrm{do}\, \mathcal{C})]$,
- Iteration of $\mathcal{G}$ started in $\mathcal{C}; \mathcal{B}$ terminates w.r.t. $\mathcal{G}(\mathcal{E}, \mathcal{B})$,
- For every element s of $\mathcal{D}$ such that $\mathcal{P}[\mathcal{G}(\mathcal{G}(s, \mathcal{B}), \mathcal{C})]$ and $\mathcal{G}(s, \mathcal{B}) \in \mathcal{F}$ holds $\mathcal{P}[\mathcal{G}(s, \mathcal{B})]$, and
- For every element s of $\mathcal{D}$ such that $\mathcal{P}[\mathcal{G}(s, \mathcal{B})]$ holds $\mathcal{P}[s]$.

Next we state three propositions:

(88) Let A be a free pre-if-while algebra, I_1, I_2 be elements of A, and n be a natural number. Suppose $I_1; I_2 \in \mathrm{ElementaryInstructions}_A{}^n$. Then there exists a natural number i such that $n = i + 1$ and $I_1 \in \mathrm{ElementaryInstructions}_A{}^i$ and $I_2 \in \mathrm{ElementaryInstructions}_A{}^i$.

(89) Let A be a free pre-if-while algebra, C, I_1, I_2 be elements of A, and n be a natural number. Suppose $\mathrm{if}\, C\, \mathrm{then}\, I_1\, \mathrm{else}\, I_2 \in \mathrm{ElementaryInstructions}_A{}^n$. Then there exists a natural number i such that $n = i + 1$ and $C \in \mathrm{ElementaryInstructions}_A{}^i$ and $I_1 \in \mathrm{ElementaryInstructions}_A{}^i$ and $I_2 \in \mathrm{ElementaryInstructions}_A{}^i$.

(90) Let A be a free pre-if-while algebra, C, I be elements of A, and n be a natural number. Suppose while C do $I \in \mathrm{ElementaryInstructions}_A{}^n$. Then there exists a natural number i such that $n = i + 1$ and $C \in \mathrm{ElementaryInstructions}_A{}^i$ and $I \in \mathrm{ElementaryInstructions}_A{}^i$.

5. Existence and Uniqueness of Execution Function and Termination

The scheme *IndDef* deals with a free E.C.I.W.-strict pre-if-while algebra $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, two binary functors $\mathcal{G}$ and $\mathcal{H}$ yielding elements of $\mathcal{B}$, and a ternary functor $\mathcal{I}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function f from the carrier of $\mathcal{A}$ into $\mathcal{B}$ such that

(i) for every element I of $\mathcal{A}$ such that $I \in \mathrm{ElementaryInstructions}_{\mathcal{A}}$ holds $f(I) = \mathcal{F}(I)$,

(ii) $f(\mathrm{EmptyIns}_{\mathcal{A}}) = \mathcal{C}$,

(iii) for all elements I_1, I_2 of $\mathcal{A}$ holds $f(I_1; I_2) = \mathcal{G}(f(I_1), f(I_2))$,

(iv) for all elements C, I_1, I_2 of $\mathcal{A}$ holds $f(\mathrm{if}\, C\, \mathrm{then}\, I_1\, \mathrm{else}\, I_2) = \mathcal{I}(f(C), f(I_1), f(I_2))$, and

(v) for all elements C, I of $\mathcal{A}$ holds $f(\mathrm{while}\, C\, \mathrm{do}\, I) = \mathcal{H}(f(C), f(I))$

provided the following requirement is met:

- For every element I of $\mathcal{A}$ such that $I \in \mathrm{ElementaryInstructions}_{\mathcal{A}}$ holds $\mathcal{F}(I) \in \mathcal{B}$.

We now state three propositions:

(91) Let A be a free E.C.I.W.-strict pre-if-while algebra, g be a function from $[\!: S, \mathrm{ElementaryInstructions}_A :\!]$ into S, and s_0 be an element of S. Then there exists an execution function f of A over S and T such that

(i) $f \restriction [\!: S, \mathrm{ElementaryInstructions}_A :\!] = g$, and

(ii) for every element s of S and for all elements C, I of A such that iteration of f started in $I; C$ does not terminate w.r.t. $f(s, C)$ holds $f(s, \mathrm{while}\, C\, \mathrm{do}\, I) = s_0$.

(92) Let A be a free E.C.I.W.-strict pre-if-while algebra, g be a function from $[\!: S, \mathrm{ElementaryInstructions}_A :\!]$ into S, and F be a function from S^S into S^S. Suppose that for every element h of S^S holds $F(h) \cdot h = F(h)$. Then there exists an execution function f of A over S and T such that

(i) $f \restriction [\!: S, \mathrm{ElementaryInstructions}_A :\!] = g$, and

(ii) for all elements C, I of A and for every element s of S such that iteration of f started in $I; C$ does not terminate w.r.t. $f(s, C)$ holds $f(s, \mathrm{while}\, C\, \mathrm{do}\, I) = F((\mathrm{curry}'\, f)(I; C))(f(s, C))$.

(93) Let A be a free E.C.I.W.-strict pre-if-while algebra and f_1, f_2 be execution functions of A over S and T. Suppose that

(i) $f_1 \restriction [\!: S, \mathrm{ElementaryInstructions}_A :\!] = f_2 \restriction [\!: S, \mathrm{ElementaryInstructions}_A :\!]$, and

(ii) for every element s of S and for all elements C, I of A such that iteration of f_1 started in $I;C$ does not terminate w.r.t. $f_1(s, C)$ holds $f_1(s, \mathrm{while}\, C\, \mathrm{do}\, I) = f_2(s, \mathrm{while}\, C\, \mathrm{do}\, I)$.

Then $f_1 = f_2$.

Let A be a pre-if-while algebra, let S be a non empty set, let T be a subset of S, and let f be an execution function of A over S and T. The functor $\mathrm{TerminatingPrograms}(A, S, T, f)$ yielding a subset of $[\!: S, \text{the carrier of } A :\!]$ is defined by the conditions (Def. 35).

(Def. 35)(i) $[\!: S, \mathrm{ElementaryInstructions}_A :\!] \subseteq \mathrm{TerminatingPrograms}(A, S, T, f)$,

(ii) $[\!: S, \{\mathrm{EmptyIns}_A\} :\!] \subseteq \mathrm{TerminatingPrograms}(A, S, T, f)$,

(iii) for every element s of S and for all elements C, I, J of A holds if $\langle s, I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and $\langle f(s, I), J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$, then $\langle s, I;J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and if $\langle s, C \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and $\langle f(s, C), I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and $f(s, C) \in T$, then $\langle s, \mathrm{if}\, C\, \mathrm{then}\, I\, \mathrm{else}\, J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and if $\langle s, C \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and $\langle f(s, C), J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and $f(s, C) \notin T$, then $\langle s, \mathrm{if}\, C\, \mathrm{then}\, I\, \mathrm{else}\, J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and if $\langle s, C \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and there exists a non empty finite sequence r of elements of S such that $r(1) = f(s, C)$ and $r(\mathrm{len}\, r) \notin T$ and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, r$ holds $r(i) \in T$ and $\langle r(i), I;C \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and $r(i+1) = f(r(i), I;C)$, then $\langle s, \mathrm{while}\, C\, \mathrm{do}\, I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$, and

(iv) for every subset P of $[\!: S, \text{the carrier of } A :\!]$ such that $[\!: S, \mathrm{ElementaryInstructions}_A :\!] \subseteq P$ and $[\!: S, \{\mathrm{EmptyIns}_A\} :\!] \subseteq P$ and for every element s of S and for all elements C, I, J of A holds if $\langle s, I \rangle \in P$ and $\langle f(s, I), J \rangle \in P$, then $\langle s, I;J \rangle \in P$ and if $\langle s, C \rangle \in P$ and $\langle f(s, C), I \rangle \in P$ and $f(s, C) \in T$, then $\langle s, \mathrm{if}\, C\, \mathrm{then}\, I\, \mathrm{else}\, J \rangle \in P$ and if $\langle s, C \rangle \in P$ and $\langle f(s, C), J \rangle \in P$ and $f(s, C) \notin T$, then $\langle s, \mathrm{if}\, C\, \mathrm{then}\, I\, \mathrm{else}\, J \rangle \in P$ and if $\langle s, C \rangle \in P$ and there exists a non empty finite sequence r of elements of S such that $r(1) = f(s, C)$ and $r(\mathrm{len}\, r) \notin T$ and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, r$ holds $r(i) \in T$ and $\langle r(i), I;C \rangle \in P$ and $r(i+1) = f(r(i), I;C)$, then $\langle s, \mathrm{while}\, C\, \mathrm{do}\, I \rangle \in P$ holds $\mathrm{TerminatingPrograms}(A, S, T, f) \subseteq P$.

Let A be a pre-if-while algebra and let I be an element of A. We say that I is absolutely-terminating if and only if the condition (Def. 36) is satisfied.

(Def. 36) Let S be a non empty set, s be an element of S, T be a subset of S, and f be an execution function of A over S and T. Then $\langle s, I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$.

Let A be a pre-if-while algebra, let S be a non empty set, let T be a subset of S, let I be an element of A, and let f be an execution function of A over S and T. We say that I is terminating w.r.t. f if and only if:

(Def. 37) For every element s of S holds $\langle s, I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$.

Let A be a pre-if-while algebra, let S be a non empty set, let T be a subset of S, let I be an element of A, let f be an execution function of A over S and T, and let Z be a set. We say that I is terminating w.r.t. f and Z if and only if:

(Def. 38) For every element s of S such that $s \in Z$ holds $\langle s, I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$.

We say that Z is invariant w.r.t. I and f if and only if:

(Def. 39) For every element s of S such that $s \in Z$ holds $f(s, I) \in Z$.

One can prove the following propositions:

(94) If $I \in \mathrm{ElementaryInstructions}_A$, then $\langle s, I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$.

(95) If $I \in \mathrm{ElementaryInstructions}_A$, then I is absolutely-terminating.

(96) $\langle s, \mathrm{EmptyIns}_A \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$.

Let us consider A. Observe that $\mathrm{EmptyIns}_A$ is absolutely-terminating.

Let us consider A. Observe that there exists an element of A which is absolutely-terminating.

Next we state the proposition

(97) If A is free and $\langle s, I; J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$, then $\langle s, I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and $\langle f(s, I), J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$.

Let us consider A and let I, J be absolutely-terminating elements of A. One can verify that $I; J$ is absolutely-terminating.

We now state the proposition

(98) Suppose A is free and $\langle s, \text{if } C \text{ then } I \text{ else } J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$. Then $\langle s, C \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and if $f(s, C) \in T$, then $\langle f(s, C), I \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$ and if $f(s, C) \notin T$, then $\langle f(s, C), J \rangle \in \mathrm{TerminatingPrograms}(A, S, T, f)$.

Let us consider A and let C, I, J be absolutely-terminating elements of A. Note that if C then I else J is absolutely-terminating.

Let us consider A and let C, I be absolutely-terminating elements of A. Note that if C then I is absolutely-terminating.

The following propositions are true:

(99) Suppose A is free and $\langle s, \text{while}\, C \text{do} I\rangle \in \text{TerminatingPrograms}(A, S, T, f)$. Then

(i) $\langle s, C\rangle \in \text{TerminatingPrograms}(A, S, T, f)$, and

(ii) there exists a non empty finite sequence r of elements of S such that $r(1) = f(s, C)$ and $r(\text{len}\, r) \notin T$ and for every natural number i such that $1 \leq i$ and $i < \text{len}\, r$ holds $r(i) \in T$ and $\langle r(i), I;C\rangle \in \text{TerminatingPrograms}(A, S, T, f)$ and $r(i+1) = f(r(i), I;C)$.

(100) If A is free and $\langle s, \text{while}\, C \,\text{do}\, I\rangle \in \text{TerminatingPrograms}(A, S, T, f)$ and $f(s, C) \in T$, then $\langle f(s, C), I\rangle \in \text{TerminatingPrograms}(A, S, T, f)$.

(101) Let C, I be absolutely-terminating elements of A. Suppose iteration of f started in $I;C$ terminates w.r.t. $f(s, C)$. Then $\langle s, \text{while}\, C \,\text{do}\, I\rangle \in \text{TerminatingPrograms}(A, S, T, f)$.

(102) Let A be a free E.C.I.W.-strict pre-if-while algebra and f_1, f_2 be execution functions of A over S and T. If $f_1 \restriction [\!: S, \text{ElementaryInstructions}_A :\!] = f_2 \restriction [\!: S, \text{ElementaryInstructions}_A :\!]$, then $\text{TerminatingPrograms}(A, S, T, f_1) = \text{TerminatingPrograms}(A, S, T, f_2)$.

(103) Let A be a free E.C.I.W.-strict pre-if-while algebra and f_1, f_2 be execution functions of A over S and T. Suppose $f_1 \restriction [\!: S, \text{ElementaryInstructions}_A :\!] = f_2 \restriction [\!: S, \text{ElementaryInstructions}_A :\!]$. Let s be an element of S and I be an element of A. If $\langle s, I\rangle \in \text{TerminatingPrograms}(A, S, T, f_1)$, then $f_1(s, I) = f_2(s, I)$.

(104) Every absolutely-terminating element of A is terminating w.r.t. f.

(105) For every element I of A holds I is terminating w.r.t. f iff I is terminating w.r.t. f and S.

(106) Let I be an element of A. Suppose I is terminating w.r.t. f. Let P be a set. Then I is terminating w.r.t. f and P.

(107) For every absolutely-terminating element I of A and for every set P holds I is terminating w.r.t. f and P.

(108) For every element I of A holds S is invariant w.r.t. I and f.

(109) Let P be a set and I, J be elements of A. Suppose P is invariant w.r.t. I and f and invariant w.r.t. J and f. Then P is invariant w.r.t. $I;J$ and f.

(110) Let I, J be elements of A. Suppose I is terminating w.r.t. f and J is terminating w.r.t. f. Then $I;J$ is terminating w.r.t. f.

(111) Let P be a set and I, J be elements of A. Suppose I is terminating w.r.t. f and P and J is terminating w.r.t. f and P and P is invariant w.r.t. I and f. Then $I;J$ is terminating w.r.t. f and P.

(112) Let C, I, J be elements of A. Suppose C is terminating w.r.t. f and I is terminating w.r.t. f and J is terminating w.r.t. f. Then if C then I else J is terminating w.r.t. f.

(113) Let P be a set and C, I, J be elements of A. Suppose that
(i) C is terminating w.r.t. f and P,
(ii) I is terminating w.r.t. f and P,
(iii) J is terminating w.r.t. f and P, and
(iv) P is invariant w.r.t. C and f.
Then if C then I else J is terminating w.r.t. f and P.

(114) Let C, I be elements of A. Suppose that
(i) C is terminating w.r.t. f,
(ii) I is terminating w.r.t. f, and
(iii) iteration of f started in $I; C$ terminates w.r.t. $f(s, C)$.
Then $\langle s, \text{while}\, C \text{ do } I\rangle \in \text{TerminatingPrograms}(A, S, T, f)$.

(115) Let P be a set and C, I be elements of A. Suppose that
(i) C is terminating w.r.t. f and P,
(ii) I is terminating w.r.t. f and P,
(iii) P is invariant w.r.t. C and f and invariant w.r.t. I and f,
(iv) iteration of f started in $I; C$ terminates w.r.t. $f(s, C)$, and
(v) $s \in P$.
Then $\langle s, \text{while}\, C \text{ do } I\rangle \in \text{TerminatingPrograms}(A, S, T, f)$.

(116) Let P be a set and C, I be elements of A. Suppose that
(i) C is terminating w.r.t. f,
(ii) I is terminating w.r.t. f and P,
(iii) P is invariant w.r.t. C and f,
(iv) for every s such that $s \in P$ and $f(f(s, I), C) \in T$ holds $f(s, I) \in P$,
(v) iteration of f started in $I; C$ terminates w.r.t. $f(s, C)$, and
(vi) $s \in P$.
Then $\langle s, \text{while}\, C \text{ do } I\rangle \in \text{TerminatingPrograms}(A, S, T, f)$.

(117) Let C, I be elements of A. Suppose that
(i) C is terminating w.r.t. f,
(ii) I is terminating w.r.t. f, and
(iii) for every s holds iteration of f started in $I; C$ terminates w.r.t. s.
Then while C do I is terminating w.r.t. f.

(118) Let P be a set and C, I be elements of A. Suppose that
(i) C is terminating w.r.t. f,
(ii) I is terminating w.r.t. f and P,
(iii) P is invariant w.r.t. C and f,
(iv) for every s such that $s \in P$ and $f(f(s, I), C) \in T$ holds $f(s, I) \in P$, and
(v) for every s such that $f(s, C) \in P$ holds iteration of f started in $I; C$ terminates w.r.t. $f(s, C)$.
Then while C do I is terminating w.r.t. f and P.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.
[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[4] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.
[5] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[6] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[7] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.
[8] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Formalized Mathematics*, 3(**2**):195–204, 1992.
[9] Grzegorz Bancerek. Joining of decorated trees. *Formalized Mathematics*, 4(**1**):77–82, 1993.
[10] Grzegorz Bancerek. Minimal signature for partial algebra. *Formalized Mathematics*, 5(**3**):405–414, 1996.
[11] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[12] Grzegorz Bancerek and Yatsuka Nakamura. Full adder circuit. Part I. *Formalized Mathematics*, 5(**3**):367–380, 1996.
[13] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(**1**):91–101, 1993.
[14] Grzegorz Bancerek and Piotr Rudnicki. The set of primitive recursive functions. *Formalized Mathematics*, 9(**4**):705–720, 2001.
[15] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.
[16] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.
[17] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[18] Ewa Burakowska. Subalgebras of the universal algebra. Lattices of subalgebras. *Formalized Mathematics*, 4(**1**):23–27, 1993.
[19] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[20] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[21] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[22] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[23] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[24] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[25] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[26] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[27] Czesław Byliński. Subcategories and products of categories. *Formalized Mathematics*, 1(**4**):725–732, 1990.
[28] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar – part 1. *Formalized Mathematics*, 2(**5**):683–687, 1991.
[29] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[30] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(**3**):495–500, 2001.
[31] Andrzej Kondracki. The Chinese Remainder Theorem. *Formalized Mathematics*, 6(**4**):573–577, 1997.
[32] Małgorzata Korolkiewicz. Homomorphisms of algebras. Quotient universal algebra. *Formalized Mathematics*, 4(**1**):109–113, 1993.
[33] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[34] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[35] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(**2**):251–253, 1992.
[36] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[37] Beata Perkowska. Free universal algebra construction. *Formalized Mathematics*, 4(**1**):115–120, 1993.
[38] Beata Perkowska. Free many sorted universal algebra. *Formalized Mathematics*, 5(**1**):67–74, 1996.
[39] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[40] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[41] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[42] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[43] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[44] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[45] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.
[46] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[47] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[48] Edmund Woronowicz. Many–argument relations. *Formalized Mathematics*, 1(**4**):733–737, 1990.
[49] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[50] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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