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Combinatorial Grassmannians

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Summary. In the paper I construct the configuration G which is a partial linear space. It consists of k-element subsets of some base set as points and $(k+1)$-element subsets as lines. The incidence is given by inclusion. I also introduce automorphisms of partial linear spaces and show that automorphisms of G are generated by permutations of the base set.

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The articles [15], [17], [3], [14], [7], [11], [13], [8], [18], [19], [4], [12], [16], [9], [5], [6], [10], [2], and [1] provide the notation and terminology for this paper.

1. Preliminaries

We follow the rules: k, n denote elements of $\mathbb{N}$ and X, Y, Z denote sets.

One can prove the following propositions:

(1) For all sets a, b such that $a \neq b$ and $\overline{\overline{a}} = n$ and $\overline{\overline{b}} = n$ holds $\overline{\overline{a \cap b}} < n$ and $n + 1 \leq \overline{\overline{a \cup b}}$.

(2) For all sets a, b such that $\overline{\overline{a}} = n + k$ and $\overline{\overline{b}} = n + k$ holds $\overline{\overline{a \cap b}} = n$ iff $\overline{\overline{a \cup b}} = n + 2 \cdot k$.

(3) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ iff there exists a function f such that f is one-to-one and $X \subseteq \operatorname{dom} f$ and $f^\circ X \subseteq Y$.

(4) For every function f such that f is one-to-one and $X \subseteq \operatorname{dom} f$ holds $\overline{\overline{f^\circ X}} = \overline{\overline{X}}$.

(5) If $X \setminus Y = X \setminus Z$ and $Y \subseteq X$ and $Z \subseteq X$, then $Y = Z$.

(6) Let Y be a non empty set and p be a function from X into Y. Suppose p is one-to-one. Let x_1, x_2 be subsets of X. If $x_1 \neq x_2$, then $p^\circ x_1 \neq p^\circ x_2$.

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(7) Let a, b, c be sets such that $\overline{\overline{a}} = n-1$ and $\overline{\overline{b}} = n-1$ and $\overline{\overline{c}} = n-1$ and $\overline{\overline{a \cap b}} = n-2$ and $\overline{\overline{a \cap c}} = n-2$ and $\overline{\overline{b \cap c}} = n-2$ and $2 \leq n$. Then

(i) if $3 \leq n$, then $\overline{\overline{a \cap b \cap c}} = n-2$ and $\overline{\overline{a \cup b \cup c}} = n+1$ or $\overline{\overline{a \cap b \cap c}} = n-3$ and $\overline{\overline{a \cup b \cup c}} = n$, and

(ii) if $n = 2$, then $\overline{\overline{a \cap b \cap c}} = n-2$ and $\overline{\overline{a \cup b \cup c}} = n+1$.

(8) Let P_1, P_2 be projective incidence structures. Suppose the projective incidence structure of P_1 = the projective incidence structure of P_2. Let A_1 be a point of P_1 and A_2 be a point of P_2. Suppose $A_1 = A_2$. Let L_1 be a line of P_1 and L_2 be a line of P_2. If $L_1 = L_2$, then if A_1 lies on L_1, then A_2 lies on L_2.

(9) Let P_1, P_2 be projective incidence structures. Suppose the projective incidence structure of P_1 = the projective incidence structure of P_2. Let A_1 be a subset of the points of P_1 and A_2 be a subset of the points of P_2. Suppose $A_1 = A_2$. Let L_1 be a line of P_1 and L_2 be a line of P_2. If $L_1 = L_2$, then if A_1 lies on L_1, then A_2 lies on L_2.

Let us note that there exists a projective incidence structure which is linear, up-2-rank, and strict and has non-trivial-lines.

2. Configuration G

A partial linear space is an up-2-rank projective incidence structure with non-trivial-lines.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. The functor $\mathrm{G}_k(X)$ yields a strict partial linear space and is defined by the conditions (Def. 1).

(Def. 1)(i) The points of $\mathrm{G}_k(X) = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k\}$,

(ii) the lines of $\mathrm{G}_k(X) = \{L; L$ ranges over subsets of X: $\overline{\overline{L}} = k+1\}$, and

(iii) the incidence of $\mathrm{G}_k(X) = \subseteq_{2^X} \cap [\!:$ the points of $\mathrm{G}_k(X)$, the lines of $\mathrm{G}_k(X) :\!]$.

One can prove the following four propositions:

(10) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let A be a point of $\mathrm{G}_k(X)$ and L be a line of $\mathrm{G}_k(X)$. Then A lies on L if and only if $A \subseteq L$.

(11) For every element k of $\mathbb{N}$ and for every non empty set X such that $0 < k$ and $k + 1 \leq \overline{\overline{X}}$ holds $\mathrm{G}_k(X)$ is Vebleian.

(12) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let A_1, A_2, A_3, A_4, A_5, A_6 be points of $\mathrm{G}_k(X)$ and L_1, L_2, L_3, L_4 be lines of $\mathrm{G}_k(X)$. Suppose that A_1 lies on L_1 and A_2 lies on L_1 and A_3 lies on L_2 and A_4 lies on L_2 and A_5 lies on L_1 and A_5 lies on

L_2 and A_1 lies on L_3 and A_3 lies on L_3 and A_2 lies on L_4 and A_4 lies on L_4 and A_5 does not lie on L_3 and A_5 does not lie on L_4 and $L_1 \neq L_2$ and $L_3 \neq L_4$. Then there exists a point A_6 of $\mathrm{G}_k(X)$ such that A_6 lies on L_3 and A_6 lies on L_4 and $A_6 = A_1 \cap A_2 \cup A_3 \cap A_4$.

(13) For every element k of $\mathbb{N}$ and for every non empty set X such that $0 < k$ and $k + 1 \leq \overline{\overline{X}}$ holds $\mathrm{G}_k(X)$ is Desarguesian.

Let S be a projective incidence structure and let K be a subset of the points of S. We say that K is a clique if and only if:

(Def. 2) For all points A, B of S such that $A \in K$ and $B \in K$ there exists a line L of S such that $\{A, B\}$ lies on L.

Let S be a projective incidence structure and let K be a subset of the points of S. We say that K is a maximal-clique if and only if:

(Def. 3) K is a clique and for every subset U of the points of S such that U is a clique and $K \subseteq U$ holds $U = K$.

Let k be an element of $\mathbb{N}$, let X be a non empty set, and let T be a subset of the points of $\mathrm{G}_k(X)$. We say that T is a star if and only if:

(Def. 4) There exists a subset S of X such that $\overline{\overline{S}} = k - 1$ and $T = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k \ \wedge \ S \subseteq A\}$.

We say that T is a top if and only if:

(Def. 5) There exists a subset S of X such that $\overline{\overline{S}} = k + 1$ and $T = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k \ \wedge \ A \subseteq S\}$.

Next we state two propositions:

(14) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k + 2 \leq \overline{\overline{X}}$. Let K be a subset of the points of $\mathrm{G}_k(X)$. If K is a star or a top, then K is a maximal-clique.

(15) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k + 2 \leq \overline{\overline{X}}$. Let K be a subset of the points of $\mathrm{G}_k(X)$. If K is a maximal-clique, then K is a star or a top.

3. Automorphisms

Let S_1, S_2 be projective incidence structures. We consider maps between projective spaces S_1 and S_2 as systems

$\langle$ a point-map, a line-map $\rangle$,

where the point-map is a function from the points of S_1 into the points of S_2 and the line-map is a function from the lines of S_1 into the lines of S_2.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2, and let a be a point of S_1. The functor $F(a)$ yields a point of S_2 and is defined as follows:

(Def. 6) $F(a) = ($the point-map of $F)(a)$.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2, and let L be a line of S_1. The functor $F(L)$ yields a line of S_2 and is defined by:

(Def. 7) $F(L) = ($the line-map of $F)(L)$.

Next we state the proposition

(16) Let S_1, S_2 be projective incidence structures and F_1, F_2 be maps between projective spaces S_1 and S_2. Suppose for every point A of S_1 holds $F_1(A) = F_2(A)$ and for every line L of S_1 holds $F_1(L) = F_2(L)$. Then the map of $F_1 =$ the map of F_2.

Let S_1, S_2 be projective incidence structures and let F be a map between projective spaces S_1 and S_2. We say that F preserves incidence strongly if and only if:

(Def. 8) For every point A_1 of S_1 and for every line L_1 of S_1 holds A_1 lies on L_1 iff $F(A_1)$ lies on $F(L_1)$.

The following proposition is true

(17) Let S_1, S_2 be projective incidence structures and F_1, F_2 be maps between projective spaces S_1 and S_2. Suppose the map of $F_1 =$ the map of F_2. If F_1 preserves incidence strongly, then F_2 preserves incidence strongly.

Let S be a projective incidence structure and let F be a map between projective spaces S and S. We say that F is automorphism if and only if:

(Def. 9) The line-map of F is bijective and the point-map of F is bijective and F preserves incidence strongly.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2, and let K be a subset of the points of S_1. The functor $F^\circ K$ yielding a subset of the points of S_2 is defined by:

(Def. 10) $F^\circ K = ($the point-map of $F)^\circ K$.

Let S_1, S_2 be projective incidence structures, let F be a map between projective spaces S_1 and S_2, and let K be a subset of the points of S_2. The functor $F^{-1}(K)$ yielding a subset of the points of S_1 is defined as follows:

(Def. 11) $F^{-1}(K) = ($the point-map of $F)^{-1}(K)$.

Let X be a set and let A be a finite set. The functor $\uparrow(A, X)$ yielding a subset of 2^X is defined as follows:

(Def. 12) $\uparrow(A, X) = \{B; B$ ranges over subsets of X: $\overline{\overline{B}} = \operatorname{card} A + 1 \ \wedge \ A \subseteq B\}$.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k + 1 \leq \overline{\overline{X}}$. Let A be a finite set. Let us assume that $\overline{\overline{A}} = k - 1$ and $A \subseteq X$. The functor $\uparrow(A, X, k)$ yields a subset of the points of $\mathrm{G}_k(X)$ and is defined as follows:

(Def. 13) $\uparrow(A, X, k) = \uparrow(A, X)$.

The following propositions are true:

(18) Let S_1, S_2 be projective incidence structures, F be a map between projective spaces S_1 and S_2, and K be a subset of the points of S_1. Then $F^\circ K = \{B; B$ ranges over points of S_2: $\bigvee_{A\,:\,\text{point of } S_1} (A \in K \wedge F(A) = B)\}$.

(19) Let S_1, S_2 be projective incidence structures, F be a map between projective spaces S_1 and S_2, and K be a subset of the points of S_2. Then $F^{-1}(K) = \{A; A$ ranges over points of S_1: $\bigvee_{B\,:\,\text{point of } S_2} (B \in K \wedge F(A) = B)\}$.

(20) Let S be a projective incidence structure, F be a map between projective spaces S and S, and K be a subset of the points of S. If F preserves incidence strongly and K is a clique, then $F^\circ K$ is a clique.

(21) Let S be a projective incidence structure, F be a map between projective spaces S and S, and K be a subset of the points of S. Suppose F preserves incidence strongly and the line-map of F is onto and K is a clique. Then $F^{-1}(K)$ is a clique.

(22) Let S be a projective incidence structure, F be a map between projective spaces S and S, and K be a subset of the points of S. Suppose F is automorphism and K is a maximal-clique. Then $F^\circ K$ is a maximal-clique and $F^{-1}(K)$ is a maximal-clique.

(23) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k+2 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose F is automorphism. Let K be a subset of the points of $\mathrm{G}_k(X)$. If K is a star, then $F^\circ K$ is a star and $F^{-1}(K)$ is a star.

Let k be an element of $\mathbb{N}$ and let X be a non empty set. Let us assume that $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let s be a permutation of X. The functor incprojmap(k, s) yielding a strict map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$ is defined as follows:

(Def. 14) For every point A of $\mathrm{G}_k(X)$ holds $(\text{incprojmap}(k, s))(A) = s^\circ A$ and for every line L of $\mathrm{G}_k(X)$ holds $(\text{incprojmap}(k, s))(L) = s^\circ L$.

One can prove the following propositions:

(24) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $k = 1$ and $k + 1 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \text{incprojmap}(k, s)$.

(25) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $1 < k$ and $\overline{\overline{X}} = k + 1$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \text{incprojmap}(k, s)$.

(26) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$

and $k+1 \leq \overline{\overline{X}}$. Let T be a subset of the points of $\mathrm{G}_k(X)$ and S be a subset of X. If $\overline{\overline{S}} = k-1$ and $T = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k \ \wedge \ S \subseteq A\}$, then $S = \bigcap T$.

(27) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let T be a subset of the points of $\mathrm{G}_k(X)$. Suppose T is a star. Let S be a subset of X. If $S = \bigcap T$, then $\overline{\overline{S}} = k-1$ and $T = \{A; A$ ranges over subsets of X: $\overline{\overline{A}} = k \ \wedge \ S \subseteq A\}$.

(28) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let T_1, T_2 be subsets of the points of $\mathrm{G}_k(X)$. If T_1 is a star and T_2 is a star and $\bigcap T_1 = \bigcap T_2$, then $T_1 = T_2$.

(29) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let A be a finite subset of X. If $\overline{\overline{A}} = k-1$, then $\uparrow(A, X, k)$ is a star.

(30) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let A be a finite subset of X. If $\overline{\overline{A}} = k-1$, then $\bigcap \uparrow(A, X, k) = A$.

(31) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+3 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_{(k+1)}(X)$ and $\mathrm{G}_{(k+1)}(X)$. Suppose F is automorphism. Then there exists a map H between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$ such that
 (i) H is automorphism,
 (ii) the line-map of $H =$ the point-map of F, and
 (iii) for every point A of $\mathrm{G}_k(X)$ and for every finite set B such that $B = A$ holds $H(A) = \bigcap (F^\circ \uparrow(B, X, k+1))$.

(32) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$ and $k+3 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_{(k+1)}(X)$ and $\mathrm{G}_{(k+1)}(X)$. Suppose F is automorphism. Let H be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose that
 (i) H is automorphism,
 (ii) the line-map of $H =$ the point-map of F, and
 (iii) for every point A of $\mathrm{G}_k(X)$ and for every finite set B such that $B = A$ holds $H(A) = \bigcap (F^\circ \uparrow(B, X, k+1))$.
 Let f be a permutation of X. If the map of $H = \mathrm{incprojmap}(k, f)$, then the map of $F = \mathrm{incprojmap}(k+1, f)$.

(33) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $2 \leq k$ and $k+2 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Suppose F is automorphism. Then there exists a permutation s of X such that the map of $F = \mathrm{incprojmap}(k, s)$.

(34) Let k be an element of $\mathbb{N}$ and X be a non empty set. Suppose $0 < k$

and $k+1 \leq \overline{\overline{X}}$. Let s be a permutation of X. Then incprojmap(k,s) is automorphism.

(35) Let X be a non empty set. Suppose $0 < k$ and $k+1 \leq \overline{\overline{X}}$. Let F be a map between projective spaces $\mathrm{G}_k(X)$ and $\mathrm{G}_k(X)$. Then F is automorphism if and only if there exists a permutation s of X such that the map of $F = \mathrm{incprojmap}(k,s)$.

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