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Several Classes of BCI-algebras and their Properties

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Summary. I have formalized the BCI-algebras closely following the book [6], sections 1.1 to 1.3, 1.6, 2.1 to 2.3, and 2.7. In this article the general theory of BCI-algebras and several classes of BCI-algebras are given.

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The articles [10], [4], [13], [9], [3], [12], [2], [11], [5], [7], [8], [1], and [14] provide the notation and terminology for this paper.

1. The Basics of General Theory of BCI-algebras

We introduce BCI structures which are extensions of 1-sorted structure and are systems

⟨ a carrier, an internal complement ⟩,

where the carrier is a set and the internal complement is a binary operation on the carrier.

Let us note that there exists a BCI structure which is non empty and strict.

Let A be a BCI structure and let x, y be elements of A. The functor $x \setminus y$ yielding an element of A is defined by:

(Def. 1) $x \setminus y =$ (the internal complement of A)(x, y).

We introduce BCI structures with 0 which are extensions of BCI structure and zero structure and are systems

⟨ a carrier, an internal complement, a zero ⟩,

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where the carrier is a set, the internal complement is a binary operation on the carrier, and the zero is an element of the carrier.

Let us note that there exists a BCI structure with 0 which is non empty and strict.

Let I_1 be a non empty BCI structure with 0 and let x be an element of I_1. The functor x^{c} yields an element of I_1 and is defined by:

(Def. 2) $x^{\mathrm{c}} = 0_{(I_1)} \setminus x$.

Let I_1 be a non empty BCI structure with 0. We say that I_1 is B if and only if:

(Def. 3) For all elements x, y, z of I_1 holds $x \setminus y \setminus (z \setminus y) \setminus (x \setminus z) = 0_{(I_1)}$.

We say that I_1 is C if and only if:

(Def. 4) For all elements x, y, z of I_1 holds $x \setminus y \setminus z \setminus (x \setminus z \setminus y) = 0_{(I_1)}$.

We say that I_1 is I if and only if:

(Def. 5) For every element x of I_1 holds $x \setminus x = 0_{(I_1)}$.

We say that I_1 is K if and only if:

(Def. 6) For all elements x, y of I_1 holds $x \setminus y \setminus x = 0_{(I_1)}$.

We say that I_1 is BCI-4 if and only if:

(Def. 7) For all elements x, y of I_1 such that $x \setminus y = 0_{(I_1)}$ and $y \setminus x = 0_{(I_1)}$ holds $x = y$.

We say that I_1 is BCK-5 if and only if:

(Def. 8) For every element x of I_1 holds $x^{\mathrm{c}} = 0_{(I_1)}$.

The BCI structure BCI-EXAMPLE with 0 is defined as follows:

(Def. 9) BCI-EXAMPLE $= \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_0 \rangle$.

Let us note that BCI-EXAMPLE is strict and non empty.

One can verify that there exists a non empty BCI structure with 0 which is strict, B, C, I, and BCI-4.

A BCI-algebra is B C I BCI-4 non empty BCI structure with 0.

Let X be a BCI-algebra. A BCI-algebra is called a subalgebra of X if it satisfies the conditions (Def. 10).

(Def. 10)(i) $0_{\mathrm{it}} = 0_X$,

(ii) the carrier of it $\subseteq$ the carrier of X, and

(iii) the internal complement of it $=$ (the internal complement of X) $\restriction$ (the carrier of it).

The following proposition is true

(1) Let X be a non empty BCI structure with 0. Then X is a BCI-algebra if and only if the following conditions are satisfied:

(i) X is I and BCI-4, and

(ii) for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus (x \setminus y) \setminus y = 0_X$.

One can check that there exists a BCI-algebra which is strict and BCK-5.

A BCK-algebra is BCK-5 BCI-algebra.

Let I_1 be a non empty BCI structure with 0 and let x, y be elements of I_1. The predicate $x \leq y$ is defined as follows:

(Def. 11) $x \setminus y = 0_{(I_1)}$.

We use the following convention: X denotes a BCI-algebra, x, y, z, u, a, b denote elements of X, and I_1 denotes a non empty subset of X.

We now state a number of propositions:

(2) $x \setminus 0_X = x$.

(3) If $x \setminus y = 0_X$ and $y \setminus z = 0_X$, then $x \setminus z = 0_X$.

(4) If $x \setminus y = 0_X$, then $x \setminus z \setminus (y \setminus z) = 0_X$ and $z \setminus y \setminus (z \setminus x) = 0_X$.

(5) If $x \leq y$, then $x \setminus z \leq y \setminus z$ and $z \setminus y \leq z \setminus x$.

(6) If $x \setminus y = 0_X$, then $(y \setminus x)^{\rm c} = 0_X$.

(7) $x \setminus y \setminus z = x \setminus z \setminus y$.

(8) $x \setminus (x \setminus (x \setminus y)) = x \setminus y$.

(9) $(x \setminus y)^{\rm c} = x^{\rm c} \setminus y^{\rm c}$.

(10) $x \setminus (x \setminus y) \setminus (y \setminus x) \setminus (x \setminus (x \setminus (y \setminus (y \setminus x)))) = 0_X$.

(11) Let X be a non empty BCI structure with 0. Then X is a BCI-algebra if and only if the following conditions are satisfied:

(i) X is BCI-4, and

(ii) for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$.

(12) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus (x \setminus y) = y \setminus (y \setminus x)$, then X is a BCK-algebra.

(13) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus y \setminus y = x \setminus y$, then X is a BCK-algebra.

(14) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus (y \setminus x) = x$, then X is a BCK-algebra.

(15) If for every BCI-algebra X and for all elements x, y, z of X holds $(x \setminus y) \setminus y = x \setminus z \setminus (y \setminus z)$, then X is a BCK-algebra.

(16) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus y \setminus (y \setminus x) = x \setminus y$, then X is a BCK-algebra.

(17) If for every BCI-algebra X and for all elements x, y of X holds $x \setminus y \setminus (x \setminus y \setminus (y \setminus x)) = 0_X$, then X is a BCK-algebra.

(18) For every BCI-algebra X holds X is K iff X is a BCK-algebra.

Let X be a BCI-algebra. The functor BCK-part X yielding a non empty subset of X is defined by:

(Def. 12) BCK-part $X = \{x; x$ ranges over elements of X: $0_X \leq x\}$.

Next we state the proposition

(19) $0_X \in$ BCK-part X.

Let us consider X. Note that 0_X

Next we state three propositions:

(20) For all elements x, y of BCK-part X holds $x \setminus y \in$ BCK-part X.

(21) For every element x of X and for every element y of BCK-part X holds $x \setminus y \leq x$.

(22) X is a subalgebra of X.

Let X be a BCI-algebra and let I_1 be a subalgebra of X. We say that I_1 is proper if and only if:

(Def. 13) $I_1 \neq X$.

Let us consider X. Note that there exists a subalgebra of X which is non proper.

Let X be a BCI-algebra and let I_1 be an element of X. We say that I_1 is atom if and only if:

(Def. 14) For every element z of X such that $z \setminus I_1 = 0_X$ holds $z = I_1$.

Let X be a BCI-algebra. The functor AtomSet X yields a non empty subset of X and is defined by:

(Def. 15) AtomSet $X = \{x; x$ ranges over elements of X: x is atom$\}$.

One can prove the following propositions:

(23) $0_X \in$ AtomSet X.

(24) For every element x of X holds $x \in$ AtomSet X iff for every element z of X holds $z \setminus (z \setminus x) = x$.

(25) For every element x of X holds $x \in$ AtomSet X iff for all elements u, z of X holds $z \setminus u \setminus (z \setminus x) = x \setminus u$.

(26) For every element x of X holds $x \in$ AtomSet X iff for all elements y, z of X holds $x \setminus (z \setminus y) \leq y \setminus (z \setminus x)$.

(27) For every element x of X holds $x \in$ AtomSet X iff for all elements y, z, u of X holds $(x \setminus u) \setminus (z \setminus y) \leq y \setminus u \setminus (z \setminus x)$.

(28) For every element x of X holds $x \in$ AtomSet X iff for every element z of X holds $z^{\mathrm{c}} \setminus x^{\mathrm{c}} = x \setminus z$.

(29) For every element x of X holds $x \in$ AtomSet X iff $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

(30) For every element x of X holds $x \in$ AtomSet X iff for every element z of X holds $(z \setminus x)^{\mathrm{c}} = x \setminus z$.

(31) For every element x of X holds $x \in$ AtomSet X iff for every element z of X holds $((x \setminus z)^{\mathrm{c}})^{\mathrm{c}} = x \setminus z$.

(32) For every element x of X holds $x \in$ AtomSet X iff for all elements z, u of X holds $z \setminus (z \setminus (x \setminus u)) = x \setminus u$.

(33) For every element a of AtomSet X and for every element x of X holds $a \setminus x \in$ AtomSet X.

Let X be a BCI-algebra and let a, b be elements of AtomSet X. Then $a \setminus b$ is an element of AtomSet X.

One can prove the following propositions:

(34) For every element x of X holds $x^{\mathrm{c}} \in$ AtomSet X.

(35) For every element x of X there exists an element a of AtomSet X such that $a \leq x$.

Let X be a BCI-algebra. We say that X is generated by atom if and only if:

(Def. 16) For every element x of X there exists an element a of AtomSet X such that $a \leq x$.

Let X be a BCI-algebra and let a be an element of AtomSet X. The functor BranchV a yields a non empty subset of X and is defined as follows:

(Def. 17) BranchV $a = \{x; x$ ranges over elements of X: $a \leq x\}$.

We now state several propositions:

(36) Every BCI-algebra is generated by atom.

(37) For all elements a, b of AtomSet X and for every element x of BranchV b holds $a \setminus x = a \setminus b$.

(38) For every element a of AtomSet X and for every element x of BCK-part X holds $a \setminus x = a$.

(39) For all elements a, b of AtomSet X and for every element x of BranchV a and for every element y of BranchV b holds $x \setminus y \in$ BranchV$(a \setminus b)$.

(40) For every element a of AtomSet X and for all elements x, y of BranchV a holds $x \setminus y \in$ BCK-part X.

(41) For all elements a, b of AtomSet X and for every element x of BranchV a and for every element y of BranchV b such that $a \neq b$ holds $x \setminus y \notin$ BCK-part X.

(42) For all elements a, b of AtomSet X such that $a \neq b$ holds BranchV $a \cap$ BranchV $b = \emptyset$.

Let X be a BCI-algebra. A non empty subset of X is said to be an ideal of X if:

(Def. 18) $0_X \in$ it and for all elements x, y of X such that $x \setminus y \in$ it and $y \in$ it holds $x \in$ it.

Let X be a BCI-algebra and let I_1 be an ideal of X. We say that I_1 is closed if and only if:

(Def. 19) For every element x of I_1 holds $x^{\mathrm{c}} \in I_1$.

Let us consider X. Note that there exists an ideal of X which is closed.

Next we state four propositions:

(43) $\{0_X\}$ is a closed ideal of X.

(44) The carrier of X is a closed ideal of X.

(45) BCK-part X is a closed ideal of X.

(46) If I_1 is an ideal of X, then for all elements x, y of X such that $x \in I_1$ and $y \leq x$ holds $y \in I_1$.

2. Associative BCI-algebras

Let I_1 be a BCI-algebra. We say that I_1 is associative if and only if:

(Def. 20) For all elements x, y, z of I_1 holds $(x \setminus y) \setminus z = x \setminus (y \setminus z)$.

We say that I_1 is quasi-associative if and only if:

(Def. 21) For every element x of I_1 holds $(x^{\mathrm{c}})^{\mathrm{c}} = x^{\mathrm{c}}$.

We say that I_1 is positive-implicative if and only if:

(Def. 22) For all elements x, y of I_1 holds $(x \setminus (x \setminus y)) \setminus (y \setminus x) = x \setminus (x \setminus (y \setminus (y \setminus x)))$.

We say that I_1 is weakly-positive-implicative if and only if:

(Def. 23) For all elements x, y, z of I_1 holds $(x \setminus y) \setminus z = x \setminus z \setminus z \setminus (y \setminus z)$.

We say that I_1 is implicative if and only if:

(Def. 24) For all elements x, y of I_1 holds $(x \setminus (x \setminus y)) \setminus (y \setminus x) = y \setminus (y \setminus x)$.

We say that I_1 is weakly-implicative if and only if:

(Def. 25) For all elements x, y of I_1 holds $x \setminus (y \setminus x) \setminus (y \setminus x)^{\mathrm{c}} = x$.

We say that I_1 is p-semisimple if and only if:

(Def. 26) For all elements x, y of I_1 holds $x \setminus (x \setminus y) = y$.

We say that I_1 is alternative if and only if:

(Def. 27) For all elements x, y of I_1 holds $x \setminus (x \setminus y) = (x \setminus x) \setminus y$ and $(x \setminus y) \setminus y = x \setminus (y \setminus y)$.

One can check that there exists a BCI-algebra which is implicative, positive-implicative, p-semisimple, associative, weakly-implicative, and weakly-positive-implicative.

Next we state several propositions:

(47) X is associative iff for every element x of X holds $x^{\mathrm{c}} = x$.

(48) For all elements x, y of X holds $y \setminus x = x \setminus y$ iff X is associative.

(49) Let X be a non empty BCI structure with 0. Then X is an associative BCI-algebra if and only if for all elements x, y, z of X holds $y \setminus x \setminus (z \setminus x) = z \setminus y$ and $x \setminus 0_X = x$.

(50) Let X be a non empty BCI structure with 0. Then X is an associative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) = z \setminus y$ and $x^{\mathrm{c}} = x$.

(51) Let X be a non empty BCI structure with 0. Then X is an associative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) = y \setminus z$ and $x \setminus 0_X = x$.

3. p-semisimple BCI-algebras

One can prove the following propositions:

(52) X is p-semisimple iff every element of X is atom.

(53) If X is p-semisimple, then BCK-part $X = \{0_X\}$.

(54) X is p-semisimple iff for every element x of X holds $(x^{\mathrm{c}})^{\mathrm{c}} = x$.

(55) X is p-semisimple iff for all x, y holds $y \setminus (y \setminus x) = x$.

(56) X is p-semisimple iff for all x, y, z holds $z \setminus y \setminus (z \setminus x) = x \setminus y$.

(57) X is p-semisimple iff for all x, y, z holds $x \setminus (z \setminus y) = y \setminus (z \setminus x)$.

(58) X is p-semisimple iff for all x, y, z, u holds $(x \setminus u) \setminus (z \setminus y) = y \setminus u \setminus (z \setminus x)$.

(59) X is p-semisimple iff for all x, z holds $z^{\mathrm{c}} \setminus x^{\mathrm{c}} = x \setminus z$.

(60) X is p-semisimple iff for all x, z holds $((x \setminus z)^{\mathrm{c}})^{\mathrm{c}} = x \setminus z$.

(61) X is p-semisimple iff for all x, u, z holds $z \setminus (z \setminus (x \setminus u)) = x \setminus u$.

(62) X is p-semisimple iff for every x such that $x^{\mathrm{c}} = 0_X$ holds $x = 0_X$.

(63) X is p-semisimple iff for all x, y holds $x \setminus y^{\mathrm{c}} = y \setminus x^{\mathrm{c}}$.

(64) X is p-semisimple iff for all x, y, z, u holds $(x \setminus y) \setminus (z \setminus u) = x \setminus z \setminus (y \setminus u)$.

(65) X is p-semisimple iff for all x, y, z holds $x \setminus y \setminus (z \setminus y) = x \setminus z$.

(66) X is p-semisimple iff for all x, y, z holds $x \setminus (y \setminus z) = (z \setminus y) \setminus x^{\mathrm{c}}$.

(67) X is p-semisimple iff for all x, y, z such that $y \setminus x = z \setminus x$ holds $y = z$.

(68) X is p-semisimple iff for all x, y, z such that $x \setminus y = x \setminus z$ holds $y = z$.

(69) Let X be a non empty BCI structure with 0. Then X is a p-semisimple BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) = z \setminus y$ and $x \setminus 0_X = x$.

(70) Let X be a non empty BCI structure with 0. Then X is a p-semisimple BCI-algebra if and only if X is I and for all elements x, y, z of X holds $x \setminus (y \setminus z) = z \setminus (y \setminus x)$ and $x \setminus 0_X = x$.

4. Quasi-associative BCI-algebras

Next we state several propositions:

(71) X is quasi-associative iff for every element x of X holds $x^{\mathrm{c}} \leq x$.

(72) X is quasi-associative iff for all elements x, y of X holds $(x \setminus y)^{\mathrm{c}} = (y \setminus x)^{\mathrm{c}}$.

(73) X is quasi-associative iff for all elements x, y of X holds $x^{\mathrm{c}} \setminus y = (x \setminus y)^{\mathrm{c}}$.

(74) X is quasi-associative iff for all elements x, y of X holds $x \setminus y \setminus (y \setminus x) \in$ BCK-part X.

(75) X is quasi-associative iff for all elements x, y, z of X holds $(x \setminus y) \setminus z \leq x \setminus (y \setminus z)$.

5. Alternative BCI-algebras

We now state several propositions:

(76) If X is alternative, then $x^{\mathrm{c}} = x$ and $x \setminus (x \setminus y) = y$ and $x \setminus y \setminus y = x$.

(77) If X is alternative and $x \setminus a = x \setminus b$, then $a = b$.

(78) If X is alternative and $a \setminus x = b \setminus x$, then $a = b$.

(79) If X is alternative and $x \setminus y = 0_X$, then $x = y$.

(80) If X is alternative and $x \setminus a \setminus b = 0_X$, then $a = x \setminus b$ and $b = x \setminus a$.

One can check the following observations:

* every BCI-algebra which is alternative is also associative,
* every BCI-algebra which is associative is also alternative, and
* every BCI-algebra which is alternative is also implicative.

The following two propositions are true:

(81) If X is alternative, then $x \setminus (x \setminus y) \setminus (y \setminus x) = x$.

(82) If X is alternative, then $y \setminus (y \setminus (x \setminus (x \setminus y))) = y$.

6. Implicative, Positive-Implicative, and Weakly-Positive-Implicative BCI-algebras

Let us observe that every BCI-algebra which is associative is also weakly-positive-implicative and every BCI-algebra which is p-semisimple is also weakly-positive-implicative.

We now state two propositions:

(83) Let X be a non empty BCI structure with 0. Then X is an implicative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$ and $(x \setminus (x \setminus y)) \setminus (y \setminus x) = y \setminus (y \setminus x)$.

(84) X is weakly-positive-implicative iff for all elements x, y of X holds $x \setminus y = x \setminus y \setminus y \setminus y^{\mathrm{c}}$.

One can verify that every BCI-algebra which is positive-implicative is also weakly-positive-implicative and every BCI-algebra which is alternative is also weakly-positive-implicative.

One can prove the following two propositions:

(85) Suppose X is a weakly-positive-implicative BCI-algebra. Let x, y be elements of X. Then $(x \setminus (x \setminus y)) \setminus (y \setminus x) = y \setminus (y \setminus x) \setminus (y \setminus x) \setminus (x \setminus y)$.

(86) Let X be a non empty BCI structure with 0. Then X is a positive-implicative BCI-algebra if and only if for all elements x, y, z of X holds $x \setminus y \setminus (x \setminus z) \setminus (z \setminus y) = 0_X$ and $x \setminus 0_X = x$ and $x \setminus y = x \setminus y \setminus y \setminus y^{\mathrm{c}}$ and $(x \setminus (x \setminus y)) \setminus (y \setminus x) = y \setminus (y \setminus x) \setminus (y \setminus x) \setminus (x \setminus y)$.

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