Chordal Graphs¹

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Summary. We are formalizing [9, pp. 81–84] where chordal graphs are defined and their basic characterization is given. This formalization is a part of the M.Sc. work of the first author under supervision of the second author.

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The terminology and notation used here are introduced in the following articles: [18], [21], [3], [16], [22], [5], [6], [4], [1], [8], [19], [2], [12], [11], [10], [7], [14], [17], [20], [15], and [13].

1. Preliminaries

One can prove the following propositions:

- (1) For every non zero natural number n holds n-1 is a natural number and $1 \le n$.
- (2) For every odd natural number n holds n-1 is a natural number and $1 \le n$.
- (3) For all odd integers n, m such that n < m holds $n \le m 2$.
- (4) For all odd integers n, m such that m < n holds $m + 2 \le n$.
- (5) For every odd natural number n such that $1 \neq n$ there exists an odd natural number m such that m + 2 = n.
- (6) For every odd natural number n such that $n \leq 2$ holds n = 1.
- (7) For every odd natural number n such that $n \leq 4$ holds n = 1 or n = 3.
- (8) For every odd natural number n such that $n \leq 6$ holds n = 1 or n = 3 or n = 5.

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BRODERICK ARNESON AND PIOTR RUDNICKI

- (9) For every odd natural number n such that $n \le 8$ holds n = 1 or n = 3 or n = 5 or n = 7.
- (10) For every even natural number n such that $n \leq 1$ holds n = 0.
- (11) For every even natural number n such that $n \leq 3$ holds n = 0 or n = 2.
- (12) For every even natural number n such that $n \leq 5$ holds n = 0 or n = 2 or n = 4.
- (13) For every even natural number n such that $n \leq 7$ holds n = 0 or n = 2 or n = 4 or n = 6.
- (14) For every finite sequence p and for every non zero natural number n such that p is one-to-one and $n \leq \ln p$ holds $p(n) \leftrightarrow p = n$.
- (15) Let p be a non empty finite sequence and T be a non empty subset of rng p. Then there exists a set x such that $x \in T$ and for every set y such that $y \in T$ holds $x \leftrightarrow p \leq y \leftrightarrow p$.

Let p be a finite sequence and let n be a natural number. The functor p.followSet(n) yields a finite set and is defined as follows:

(Def. 1) $p.\text{followSet}(n) = \operatorname{rng}\langle p(n), \dots, p(\operatorname{len} p) \rangle.$

The following three propositions are true:

- (16) Let p be a finite sequence, x be a set, and n be a natural number. Suppose $x \in \operatorname{rng} p$ and $n \in \operatorname{dom} p$ and p is one-to-one. Then $x \in p.\operatorname{followSet}(n)$ if and only if $x \leftrightarrow p \ge n$.
- (17) Let p, q be finite sequences and x be a set. If $p = \langle x \rangle \cap q$, then for every non zero natural number n holds p.followSet(n + 1) = q.followSet(n).
- (18) Let X be a set, f be a finite sequence of elements of X, and g be a FinSubsequence of f. If len Seq g = len f, then Seq g = f.

2. MISCELLANY ON GRAPHS

Next we state a number of propositions:

- (19) Let G be a graph, S be a subset of the vertices of G, H be a subgraph of G induced by S, and u, v be sets. Suppose $u \in S$ and $v \in S$. Let e be a set. If e joins u and v in G, then e joins u and v in H.
- (20) For every graph G and for every walk W of G holds W is trail-like iff $\operatorname{len} W = 2 \cdot \operatorname{card}(W.\operatorname{edges}()) + 1.$
- (21) Let G be a graph, S be a subset of the vertices of G, H be a subgraph of G with vertices S removed, and W be a walk of G. Suppose that for every odd natural number n such that $n \leq \text{len } W$ holds $W(n) \notin S$. Then W is a walk of H.

- (22) Let G be a graph and a, b be sets. Suppose $a \neq b$. Let W be a walk of G. If W.vertices() = $\{a, b\}$, then there exists a set e such that e joins a and b in G.
- (23) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and W be a walk of G. If W.vertices() \subseteq S, then W is a walk of H.
- (24) Let G_1 , G_2 be graphs. Suppose $G_1 =_G G_2$. Let W_1 be a walk of G_1 and W_2 be a walk of G_2 . If $W_1 = W_2$, then if W_1 is cycle-like, then W_2 is cycle-like.
- (25) Let G be a graph, P be a path of G, and m, n be odd natural numbers. Suppose $m \leq \text{len } P$ and $n \leq \text{len } P$ and P(m) = P(n). Then m = n or m = 1 and n = len P or m = len P and n = 1.
- (26) Let G be a graph and P be a path of G. Suppose P is open. Let a, e, b be sets. Suppose $a \notin P$.vertices() and b = P.first() and e joins a and b in G. Then (G.walkOf(a, e, b)).append(P) is path-like.
- (27) Let G be a graph and P, H be paths of G. Suppose P.edges() misses H.edges() and P is non trivial and open and H is non trivial and open and $P.vertices() \cap H.vertices() = \{P.first(), P.last()\}$ and H.first() = P.last() and H.last() = P.first(). Then P.append(H) is cycle-like.
- (28) For every graph G and for all walks W_1 , W_2 of G such that W_1 .last() = W_2 .first() holds $(W_1.append(W_2)).length() = W_1.length() + W_2.length()$.
- (29) Let G be a graph and A, B be non empty subsets of the vertices of G. Suppose $B \subseteq A$. Let H_1 be a subgraph of G induced by A. Then every subgraph of H_1 induced by B is a subgraph of G induced by B.
- (30) Let G be a graph and A, B be non empty subsets of the vertices of G. Suppose $B \subseteq A$. Let H_1 be a subgraph of G induced by A. Then every subgraph of G induced by B is a subgraph of H_1 induced by B.
- (31) Let G be a graph and S, T be non empty subsets of the vertices of G. If $T \subseteq S$, then for every subgraph G_2 of G induced by S holds G_2 .edgesBetween(T) = G.edgesBetween(T).

The scheme FinGraphOrderCompInd concerns a unary predicate \mathcal{P} , and states that:

For every finite graph G holds $\mathcal{P}[G]$

provided the parameters meet the following condition:

• Let k be a non zero natural number. Suppose that for every finite graph G_3 such that G_3 .order() < k holds $\mathcal{P}[G_3]$. Let G_4 be a finite graph. If G_4 .order() = k, then $\mathcal{P}[G_4]$.

We now state two propositions:

(32) For every graph G and for every walk W of G such that W is open and path-like holds W is vertex-distinct.

(33) Let G be a graph and P be a path of G. Suppose P is open and len P > 3. Let e be a set. If e joins P.last() and P.first() in G, then P.addEdge(e) is cycle-like.

3. Shortest Topological Path

Let G be a graph and let W be a walk of G. We say that W is minimum length if and only if:

(Def. 2) For every walk W_2 of G such that W_2 is walk from W.first() to W.last() holds len $W_2 \ge \text{len } W$.

The following propositions are true:

- (34) For every graph G and for every walk W of G and for every subwalk S of W such that S.first() = W.first() and S.edgeSeq() = W.edgeSeq() holds S = W.
- (35) For every graph G and for every walk W of G and for every subwalk S of W such that len S = len W holds S = W.
- (36) For every graph G and for every walk W of G such that W is minimum length holds W is path-like.
- (37) For every graph G and for every walk W of G such that W is minimum length holds W is path-like.
- (38) Let G be a graph and W be a walk of G. Suppose that for every path P of G such that P is walk from W.first() to W.last() holds len $P \ge \text{len } W$. Then W is minimum length.
- (39) For every graph G and for every walk W of G holds there exists a path of G which is walk from W.first() to W.last() and minimum length.
- (40) Let G be a graph and W be a walk of G. Suppose W is minimum length. Let m, n be odd natural numbers. Suppose m + 2 < n and $n \leq \text{len } W$. Then it is not true that there exists a set e such that e joins W(m) and W(n) in G.
- (41) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and W be a walk of H. Suppose W is minimum length. Let m, n be odd natural numbers. Suppose m + 2 < n and $n \leq \text{len } W$. Then it is not true that there exists a set e such that e joins W(m) and W(n) in G.
- (42) Let G be a graph and W be a walk of G. Suppose W is minimum length. Let m, n be odd natural numbers. If $m \leq n$ and $n \leq \text{len } W$, then $W.\operatorname{cut}(m,n)$ is minimum length.
- (43) Let G be a graph. Suppose G is connected. Let A, B be non empty subsets of the vertices of G. Suppose A misses B. Then there exists a path P of G such that

- (i) P is minimum length and non trivial,
- (ii) $P.\text{first}() \in A$,
- (iii) $P.\text{last}() \in B$, and
- (iv) for every odd natural number n such that 1 < n and n < len P holds $P(n) \notin A$ and $P(n) \notin B$.

4. Adjacency and Complete Graphs

Let G be a graph and let a, b be vertices of G. We say that a and b are adjacent if and only if:

- (Def. 3) There exists a set e such that e joins a and b in G.
 - Let us note that the predicate a and b are adjacent is symmetric. Next we state several propositions:
 - (44) Let G_1 , G_2 be graphs. Suppose $G_1 =_G G_2$. Let u_1 , v_1 be vertices of G_1 . Suppose u_1 and v_1 are adjacent. Let u_2 , v_2 be vertices of G_2 . If $u_1 = u_2$ and $v_1 = v_2$, then u_2 and v_2 are adjacent.
 - (45) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, u, v be vertices of G, and t, w be vertices of H. Suppose u = t and v = w. Then u and v are adjacent if and only if t and w are adjacent.
 - (46) For every graph G and for every walk W of G such that $W.first() \neq W.last()$ and W.first() and W.last() are not adjacent holds $W.length() \geq 2$.
 - (47) Let G be a graph and v_1 , v_2 , v_3 be vertices of G. Suppose $v_1 \neq v_2$ and $v_1 \neq v_3$ and $v_2 \neq v_3$ and v_1 and v_2 are adjacent and v_2 and v_3 are adjacent. Then there exists a path P of G and there exist sets e_1 , e_2 such that P is open and len P = 5 and P.length() = 2 and e_1 joins v_1 and v_2 in G and e_2 joins v_2 and v_3 in G and P.edges() = $\{e_1, e_2\}$ and P.vertices() = $\{v_1, v_2, v_3\}$ and $P(1) = v_1$ and $P(3) = v_2$ and $P(5) = v_3$.
 - (48) Let G be a graph and v_1 , v_2 , v_3 , v_4 be vertices of G. Suppose that $v_1 \neq v_2$ and $v_1 \neq v_3$ and $v_2 \neq v_3$ and $v_2 \neq v_4$ and $v_3 \neq v_4$ and v_1 and v_2 are adjacent and v_2 and v_3 are adjacent and v_3 and v_4 are adjacent. Then there exists a path P of G such that len P = 7 and P.length() = 3 and P.vertices() = $\{v_1, v_2, v_3, v_4\}$ and $P(1) = v_1$ and $P(3) = v_2$ and $P(5) = v_3$ and $P(7) = v_4$.

Let G be a graph and let S be a set. The functor G.adjacentSet(S) yields a subset of the vertices of G and is defined as follows:

(Def. 4) $G.adjacentSet(S) = \{u; u \text{ ranges over vertices of } G: u \notin S \land \bigvee_{v: vertex of G} (v \in S \land u \text{ and } v \text{ are adjacent})\}.$ One can prove the following propositions:

BRODERICK ARNESON AND PIOTR RUDNICKI

- (49) For every graph G and for all sets S, x such that $x \in G.adjacentSet(S)$ holds $x \notin S$.
- (50) Let G be a graph, S be a set, and u be a vertex of G. Then $u \in G.$ adjacentSet(S) if and only if the following conditions are satisfied:
 - (i) $u \notin S$, and
- (ii) there exists a vertex v of G such that $v \in S$ and u and v are adjacent.
- (51) For all graphs G_1 , G_2 such that $G_1 =_G G_2$ and for every set S holds G_1 .adjacentSet $(S) = G_2$.adjacentSet(S).
- (52) For every graph G and for all vertices u, v of G holds $u \in G.adjacentSet(\{v\})$ iff $u \neq v$ and v and u are adjacent.
- (53) For every graph G and for all sets x, y holds $x \in G.adjacentSet(\{y\})$ iff $y \in G.adjacentSet(\{x\})$.
- (54) Let G be a graph and C be a path of G. Suppose C is cycle-like and C.length() > 3. Let x be a vertex of G. Suppose $x \in C.$ vertices(). Then there exist odd natural numbers m, n such that m + 2 < n and $n \leq \text{len } C$ and m = 1 and n = len C and m = 1 and n = len C 2 and m = 3 and n = len C and $C(m) \neq C(n)$ and $C(m) \in G.$ adjacentSet($\{x\}$).
- (55) Let G be a graph and C be a path of G. Suppose C is cycle-like and C.length() > 3. Let x be a vertex of G. Suppose $x \in C.\text{vertices}()$. Then there exist odd natural numbers m, n such that
 - (i) m + 2 < n,
 - (ii) $n \leq \operatorname{len} C$,
- (iii) $C(m) \neq C(n)$,
- (iv) $C(m) \in G.adjacentSet(\{x\}),$
- (v) $C(n) \in G.adjacentSet(\{x\}), and$
- (vi) for every set e such that $e \in C.edges()$ holds e does not join C(m) and C(n) in G.
- (56) For every loopless graph G and for every vertex u of G holds $G.adjacentSet(\{u\}) = \emptyset$ iff u is isolated.
- (57) Let G be a graph, G_0 be a subgraph of G, S be a non empty subset of the vertices of G, x be a vertex of G, G_1 be a subgraph of G induced by S, and G_2 be a subgraph of G induced by $S \cup \{x\}$. If G_1 is connected and $x \in G$.adjacentSet(the vertices of G_1), then G_2 is connected.
- (58) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and u be a vertex of G. Suppose $u \in S$ and G.adjacentSet($\{u\}$) \subseteq S. Let v be a vertex of H. If u = v, then G.adjacentSet($\{u\}$) = H.adjacentSet($\{v\}$).

Let G be a graph and let S be a set. A subgraph of G is called an adjacency graph of S in G if:

(Def. 5) It is a subgraph of G induced by G.adjacentSet(S) if S is a subset of the vertices of G.

Next we state two propositions:

- (59) Let G_1 , G_2 be graphs. Suppose $G_1 =_G G_2$. Let u_1 be a vertex of G_1 and u_2 be a vertex of G_2 . Suppose $u_1 = u_2$. Let H_1 be an adjacency graph of $\{u_1\}$ in G_1 and H_2 be an adjacency graph of $\{u_2\}$ in G_2 . Then $H_1 =_G H_2$.
- (60) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and u be a vertex of G. Suppose $u \in S$ and G.adjacentSet($\{u\}$) $\subseteq S$ and G.adjacentSet($\{u\}$) $\neq \emptyset$. Let v be a vertex of H. Suppose u = v. Let G_5 be an adjacency graph of $\{u\}$ in G and H_3 be an adjacency graph of $\{v\}$ in H. Then $G_5 =_G H_3$.

Let G be a graph. We say that G is complete if and only if:

- (Def. 6) For all vertices u, v of G such that $u \neq v$ holds u and v are adjacent. We now state the proposition
 - (61) For every graph G such that G is trivial holds G is complete.

One can check that every graph which is trivial is also complete.

Let us note that there exists a graph which is trivial, simple, and complete and there exists a graph which is non trivial, finite, simple, and complete.

The following propositions are true:

- (62) For all graphs G_1 , G_2 such that $G_1 =_G G_2$ holds if G_1 is complete, then G_2 is complete.
- (63) For every complete graph G and for every subset S of the vertices of G holds every subgraph of G induced by S is complete.

5. SIMPLICIAL VERTEX

Let G be a graph and let v be a vertex of G. We say that v is simplicial if and only if:

(Def. 7) If G.adjacentSet($\{v\}$) $\neq \emptyset$, then every adjacency graph of $\{v\}$ in G is complete.

The following propositions are true:

- (64) For every complete graph G holds every vertex of G is simplicial.
- (65) For every trivial graph G holds every vertex of G is simplicial.
- (66) Let G_1 , G_2 be graphs. Suppose $G_1 =_G G_2$. Let u_1 be a vertex of G_1 and u_2 be a vertex of G_2 . If $u_1 = u_2$ and u_1 is simplicial, then u_2 is simplicial.
- (67) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, and u be a vertex of G. Suppose $u \in S$ and G.adjacentSet($\{u\}$) $\subseteq S$. Let v be a vertex of H. If u = v, then u is simplicial iff v is simplicial.

BRODERICK ARNESON AND PIOTR RUDNICKI

- (68) Let G be a graph and v be a vertex of G. Suppose v is simplicial. Let a, b be sets. Suppose $a \neq b$ and $a \in G$.adjacentSet($\{v\}$) and $b \in G$.adjacentSet($\{v\}$). Then there exists a set e such that e joins a and b in G.
- (69) Let G be a graph and v be a vertex of G. Suppose v is not simplicial. Then there exist vertices a, b of G such that $a \neq b$ and $v \neq a$ and $v \neq b$ and v and a are adjacent and v and b are adjacent and a and b are not adjacent.

6. Vertex Separator

Let G be a graph and let a, b be vertices of G. Let us assume that $a \neq b$ and a and b are not adjacent. A subset of the vertices of G is said to be a vertex separator of a and b if:

(Def. 8) $a \notin it$ and $b \notin it$ and for every subgraph G_2 of G with vertices it removed holds there exists no walk of G_2 which is walk from a to b.

Next we state several propositions:

- (70) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Then every vertex separator of a and b is a vertex separator of b and a.
- (71) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a subset of the vertices of G. Then S is a vertex separator of a and b if and only if $a \notin S$ and $b \notin S$ and for every walk W of G such that W is walk from a to b there exists a vertex x of G such that $x \in S$ and $x \in W$.vertices().
- (72) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b and W be a walk of G. Suppose W is walk from a to b. Then there exists an odd natural number k such that 1 < k and $k < \operatorname{len} W$ and $W(k) \in S$.
- (73) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. If $S = \emptyset$, then there exists no walk of G which is walk from a to b.
- (74) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent and there exists no walk of G which is walk from a to b. Then \emptyset is a vertex separator of a and b.
- (75) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b, G_2 be a subgraph of G with vertices S removed, and a_2 be a vertex of G_2 . If $a_2 = a$, then $(G_2.\text{reachableFrom}(a_2)) \cap S = \emptyset$.

CHORDAL GRAPHS

- (76) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b, G_2 be a subgraph of G with vertices S removed, and a_2 , b_2 be vertices of G_2 . If $a_2 = a$ and $b_2 = b$, then $(G_2.reachableFrom(a_2)) \cap (G_2.reachableFrom(b_2)) = \emptyset$.
- (77) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b and G_2 be a subgraph of G with vertices S removed. Then a is a vertex of G_2 and b is a vertex of G_2 .

Let G be a graph, let a, b be vertices of G, and let S be a vertex separator of a and b. We say that S is minimal if and only if:

(Def. 9) For every subset T of S such that $T \neq S$ holds T is not a vertex separator of a and b.

Next we state several propositions:

- (78) Let G be a graph, a, b be vertices of G, and S be a vertex separator of a and b. If $S = \emptyset$, then S is minimal.
- (79) For every finite graph G and for all vertices a, b of G holds there exists a vertex separator of a and b which is minimal.
- (80) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let T be a vertex separator of b and a. If S = T, then T is minimal.
- (81) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let x be a vertex of G. If $x \in S$, then there exists a walk W of G such that W is walk from a to b and $x \in W$.vertices().
- (82) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let H be a subgraph of G with vertices S removed and a_1 be a vertex of H. Suppose $a_1 = a$. Let x be a vertex of G. Suppose $x \in S$. Then there exists a vertex y of G such that $y \in H$.reachableFrom (a_1) and x and y are adjacent.
- (83) Let G be a graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let H be a subgraph of G with vertices S removed and a_1 be a vertex of H. Suppose $a_1 = b$. Let x be a vertex of G. Suppose $x \in S$. Then there exists a vertex y of G such that $y \in H$.reachableFrom (a_1) and x and y are adjacent.

7. CHORDAL GRAPHS

Let G be a graph and let W be a walk of G. We say that W is chordal if and only if the condition (Def. 10) is satisfied.

(Def. 10) There exist odd natural numbers m, n such that

- (i) m + 2 < n,
- (ii) $n \leq \operatorname{len} W$,
- (iii) $W(m) \neq W(n)$,
- (iv) there exists a set e such that e joins W(m) and W(n) in G, and
- (v) for every set f such that $f \in W$.edges() holds f does not join W(m) and W(n) in G.

Let G be a graph and let W be a walk of G. We introduce W is chordless as an antonym of W is chordal.

Next we state a number of propositions:

- (84) Let G be a graph and W be a walk of G. Suppose W is chordal. Then there exist odd natural numbers m, n such that
 - (i) m + 2 < n,
 - (ii) $n \leq \operatorname{len} W$,
- (iii) $W(m) \neq W(n)$,
- (iv) there exists a set e such that e joins W(m) and W(n) in G, and
- (v) if W is cycle-like, then m = 1 and $n = \operatorname{len} W$ and m = 1 and $n = \operatorname{len} W 2$ and m = 3 and $n = \operatorname{len} W$.
- (85) Let G be a graph and P be a path of G. Given odd natural numbers m, n such that
 - (i) m + 2 < n,
 - (ii) $n \leq \operatorname{len} P$,
- (iii) there exists a set e such that e joins P(m) and P(n) in G, and
- (iv) if P is cycle-like, then m = 1 and n = len P and m = 1 and n = len P-2and m = 3 and n = len P. Then P is chordal.
- (86) Let G_1 , G_2 be graphs. Suppose $G_1 =_G G_2$. Let W_1 be a walk of G_1 and W_2 be a walk of G_2 . If $W_1 = W_2$, then if W_1 is chordal, then W_2 is chordal.
- (87) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, W_1 be a walk of G, and W_2 be a walk of H. If $W_1 = W_2$, then W_2 is chordal iff W_1 is chordal.
- (88) Let G be a graph and W be a walk of G. Suppose W is cycle-like and chordal and W.length() = 4. Then there exists a set e such that e joins W(1) and W(5) in G or e joins W(3) and W(7) in G.
- (89) For every graph G and for every walk W of G such that W is minimum length holds W is chordless.

- (90) Let G be a graph and W be a walk of G. Suppose W is open and len W = 5 and W.first() and W.last() are not adjacent. Then W is chordless.
- (91) For every graph G and for every walk W of G holds W is chordal iff W.reverse() is chordal.
- (92) Let G be a graph and P be a path of G. Suppose P is open and chordless. Let m, n be odd natural numbers. Suppose m < n and $n \leq \text{len } P$. Then there exists a set e such that e joins P(m) and P(n) in G if and only if m + 2 = n.
- (93) Let G be a graph and P be a path of G. Suppose P is open and chordless. Let m, n be odd natural numbers. If m < n and $n \leq \text{len } P$, then P.cut(m, n) is chordless and P.cut(m, n) is open.
- (94) Let G be a graph, S be a non empty subset of the vertices of G, H be a subgraph of G induced by S, W be a walk of G, and V be a walk of H. If W = V, then W is chordless iff V is chordless.

Let G be a graph. We say that G is chordal if and only if:

(Def. 11) For every walk P of G such that P.length() > 3 and P is cycle-like holds P is chordal.

Next we state two propositions:

- (95) For all graphs G_1 , G_2 such that $G_1 =_G G_2$ holds if G_1 is chordal, then G_2 is chordal.
- (96) For every finite graph G such that card (the vertices of G) ≤ 3 holds G is chordal.

One can verify the following observations:

- * there exists a graph which is trivial, finite, and chordal,
- * there exists a graph which is non trivial, finite, simple, and chordal, and
- * every graph which is complete is also chordal.

Let G be a chordal graph and let V be a set. One can check that every subgraph of G induced by V is chordal.

Next we state several propositions:

- (97) Let G be a chordal graph and P be a path of G. Suppose P is open and chordless. Let x, e be sets. Suppose $x \notin P$.vertices() and e joins P.last() and x in G and it is not true that there exists a set f such that f joins P(len P - 2) and x in G. Then P.addEdge(e) is path-like and P.addEdge(e) is open and P.addEdge(e) is chordless.
- (98) Let G be a chordal graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. If S is minimal and non empty, then every subgraph of G induced by S is complete.
- (99) Let G be a finite graph. Suppose that for all vertices a, b of G such that

 $a \neq b$ and a and b are not adjacent and for every vertex separator S of a and b such that S is minimal and non empty holds every subgraph of G induced by S is complete. Then G is chordal.

- (100) Let G be a finite chordal graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let H be a subgraph of G with vertices S removed and a_3 be a vertex of H. Suppose $a = a_3$. Then there exists a vertex c of G such that $c \in H$.reachableFrom (a_3) and for every vertex x of G such that $x \in S$ holds c and x are adjacent.
- (101) Let G be a finite chordal graph and a, b be vertices of G. Suppose $a \neq b$ and a and b are not adjacent. Let S be a vertex separator of a and b. Suppose S is minimal. Let H be a subgraph of G with vertices S removed and a_3 be a vertex of H. Suppose $a = a_3$. Let x, y be vertices of G. Suppose $x \in S$ and $y \in S$. Then there exists a vertex c of G such that $c \in H$.reachableFrom (a_3) and c and x are adjacent and c and y are adjacent.
- (102) Let G be a non trivial finite chordal graph. Suppose G is not complete. Then there exist vertices a, b of G such that $a \neq b$ and a and b are not adjacent and a is simplicial and b is simplicial.
- (103) For every finite chordal graph G holds there exists a vertex of G which is simplicial.

8. VERTEX ELIMINATION SCHEME

Let G be a finite graph. A finite sequence of elements of the vertices of G is said to be a vertex scheme of G if:

(Def. 12) It is one-to-one and rng it = the vertices of G.

Let G be a finite graph. Note that every vertex scheme of G is non empty. The following three propositions are true:

- (104) For every finite graph G and for every vertex scheme S of G holds len S = card (the vertices of G).
- (105) For every finite graph G and for every vertex scheme S of G holds $1 \leq \text{len } S$.
- (106) For all finite graphs G, H and for every vertex scheme g of G such that $G =_G H$ holds g is a vertex scheme of H.

Let G be a finite graph, let S be a vertex scheme of G, and let x be a vertex of G. Then $x \leftrightarrow S$ is a non zero element of \mathbb{N} .

Let G be a finite graph, let S be a vertex scheme of G, and let n be a natural number. Then S.followSet(n) is a subset of the vertices of G.

Next we state the proposition

(107) Let G be a finite graph, S be a vertex scheme of G, and n be a non zero natural number. If $n \leq \text{len } S$, then S.followSet(n) is non empty.

Let G be a finite graph and let S be a vertex scheme of G. We say that S is perfect if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let n be a non zero natural number. Suppose $n \leq \text{len } S$. Let G_6 be a subgraph of G induced by S.followSet(n) and v be a vertex of G_6 . If v = S(n), then v is simplicial.

One can prove the following propositions:

- (108) Let G be a finite trivial graph and v be a vertex of G. Then there exists a vertex scheme S of G such that $S = \langle v \rangle$ and S is perfect.
- (109) Let G be a finite graph and V be a vertex scheme of G. Then V is perfect if and only if for all vertices a, b, c of G such that $b \neq c$ and a and b are adjacent and a and c are adjacent and for all natural numbers v_5 , v_6 , v_7 such that $v_5 \in \text{dom } V$ and $v_6 \in \text{dom } V$ and $v_7 \in \text{dom } V$ and $V(v_5) = a$ and $V(v_6) = b$ and $V(v_7) = c$ and $v_5 < v_6$ and $v_5 < v_7$ holds b and c are adjacent.

Let G be a finite chordal graph. One can check that there exists a vertex scheme of G which is perfect.

The following propositions are true:

- (110) Let G, H be finite chordal graphs and g be a perfect vertex scheme of G. If $G =_G H$, then g is a perfect vertex scheme of H.
- (111) For every finite graph G such that there exists a vertex scheme of G which is perfect holds G is chordal.

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