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On the Permanent of a Matrix

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Summary. We introduce the notion of a permanent [13] of a square matrix. It is a notion somewhat related to a determinant, so we follow closely the approach and theorems already introduced in the Mizar Mathematical Library for the determinant. Unfortunately, the formalization of the latter notion is at its early stage, so we had to prove many very elementary auxiliary facts.

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The articles [18], [25], [14], [1], [16], [9], [26], [4], [6], [5], [2], [3], [15], [20], [21], [12], [23], [17], [24], [7], [19], [10], [22], [8], [11], and [27] provide the terminology and notation for this paper.

1. Preliminaries

In this paper i, n are natural numbers and K is a field.

We now state the proposition

(1) For all sets a, A such that $a \in A$ holds $\{a\} \in \operatorname{Fin} A$.

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Let n be a natural number. Observe that there exists an element of Fin (the permutations of n-element set) which is non empty.

The scheme *NonEmptyFiniteX* deals with a natural number $\mathcal{A}$, a non empty element $\mathcal{B}$ of Fin the permutations of $\mathcal{A}$-element set, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following conditions are met:

- For every element x of the permutations of $\mathcal{A}$-element set such that $x \in \mathcal{B}$ holds $\mathcal{P}[\{x\}]$, and
- Let x be an element of the permutations of $\mathcal{A}$-element set and B be a non empty element of Fin (the permutations of $\mathcal{A}$-element set). If $x \in \mathcal{B}$ and $B \subseteq \mathcal{B}$ and $x \notin B$ and $\mathcal{P}[B]$, then $\mathcal{P}[B \cup \{x\}]$.

Let us consider n. Observe that there exists a function from $\operatorname{Seg} n$ into $\operatorname{Seg} n$ which is one-to-one and finite sequence-like.

Let us consider n. Observe that $\mathrm{id}_{\operatorname{Seg} n}$ is finite sequence-like.

One can prove the following two propositions:

(2) $(\operatorname{Rev}(\operatorname{idseq}(2)))(1) = 2$ and $(\operatorname{Rev}(\operatorname{idseq}(2)))(2) = 1$.

(3) For every one-to-one function f such that $\operatorname{dom} f = \operatorname{Seg} 2$ and $\operatorname{rng} f = \operatorname{Seg} 2$ holds $f = \mathrm{id}_{\operatorname{Seg} 2}$ or $f = \operatorname{Rev}(\mathrm{id}_{\operatorname{Seg} 2})$.

2. Permutations

One can prove the following propositions:

(4) $\operatorname{Rev}(\operatorname{idseq}(n)) \in$ the permutations of n-element set.

(5) Let f be a finite sequence. Suppose $n \neq 0$ and $f \in$ the permutations of n-element set. Then $\operatorname{Rev}(f) \in$ the permutations of n-element set.

(6) The permutations of 2-element set $= \{\operatorname{idseq}(2), \operatorname{Rev}(\operatorname{idseq}(2))\}$.

3. The Permanent of a Matrix

Let us consider n, K and let M be a matrix over K of dimension n. The functor $\operatorname{PPath} M$ yielding a function from the permutations of n-element set into the carrier of K is defined by:

(Def. 1) For every element p of the permutations of n-element set holds $(\operatorname{PPath} M)(p) = (\text{the multiplication of } K) \circledast (p\text{-Path}\, M)$.

Let us consider n, K and let M be a matrix over K of dimension n. The functor $\operatorname{Per} M$ yielding an element of K is defined as follows:

(Def. 2) $\operatorname{Per} M = (\text{the addition of } K)\text{-}\sum_{\Omega^{\mathrm{f}}_{\text{the permutations of } n\text{-element set}}} \operatorname{PPath} M$.

In the sequel a, b, c, d denote elements of K.

The following propositions are true:

(7) $\mathrm{Per}\langle\langle a\rangle\rangle = a$.

(8) For every field K and for every natural number n such that $n \geq 1$ holds $\mathrm{Per}(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n}) = 0_K$.

(9) For every element p of the permutations of 2-element set such that $p = \mathrm{idseq}(2)$ holds p-Path $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \langle a, d\rangle$.

(10) For every element p of the permutations of 2-element set such that $p = \mathrm{Rev}(\mathrm{idseq}(2))$ holds p-Path $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \langle b, c\rangle$.

(11) (The multiplication of K) $\circledast \langle a, b\rangle = a \cdot b$.

4. Matrices with the Dimension 2 and 3

One can check that there exists a permutation of $\mathrm{Seg}\,2$ which is odd.

Let n be a natural number. Observe that there exists a permutation of $\mathrm{Seg}\,n$ which is even.

One can prove the following four propositions:

(12) $\langle 2, 1\rangle$ is an odd permutation of $\mathrm{Seg}\,2$.

(13) $\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - b \cdot c$.

(14) $\mathrm{Per}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d + b \cdot c$.

(15) $\mathrm{Rev}(\mathrm{idseq}(3)) = \langle 3, 2, 1\rangle$.

In the sequel D is a non empty set.

One can prove the following propositions:

(16) For all elements x, y, z of D and for every finite sequence f of elements of D such that $f = \langle x, y, z\rangle$ holds $\mathrm{Rev}(f) = \langle z, y, x\rangle$.

(17) Let f, g be finite sequences. Suppose $f \frown g \in$ the permutations of n-element set. Then $f \frown \mathrm{Rev}(g) \in$ the permutations of n-element set.

(18) Let f, g be finite sequences. Suppose $f \frown g \in$ the permutations of n-element set. Then $g \frown f \in$ the permutations of n-element set.

(19) The permutations of 3-element set $= \{\langle 1, 2, 3\rangle, \langle 3, 2, 1\rangle, \langle 1, 3, 2\rangle, \langle 2, 3, 1\rangle, \langle 2, 1, 3\rangle, \langle 3, 1, 2\rangle\}$.

(20) Let a, b, c, d, e, f, g, h, i be elements of K and M be a matrix over K of dimension 3. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Let p be an element of the permutations of 3-element set. If $p = \langle 1, 2, 3\rangle$, then p-Path $M = \langle a, e, i\rangle$.

(21) Let a, b, c, d, e, f, g, h, i be elements of K and M be a matrix over K of dimension 3. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Let p be an element of the permutations of 3-element set. If $p = \langle 3, 2, 1\rangle$, then p-Path $M = \langle c, e, g\rangle$.

(22) Let a, b, c, d, e, f, g, h, i be elements of K and M be a matrix over K of dimension 3. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Let p be an element of the permutations of 3-element set. If $p = \langle 1, 3, 2\rangle$, then p-Path $M = \langle a, f, h\rangle$.

(23) Let a, b, c, d, e, f, g, h, i be elements of K and M be a matrix over K of dimension 3. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Let p be an element of the permutations of 3-element set. If $p = \langle 2, 3, 1\rangle$, then p-Path $M = \langle b, f, g\rangle$.

(24) Let a, b, c, d, e, f, g, h, i be elements of K and M be a matrix over K of dimension 3. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Let p be an element of the permutations of 3-element set. If $p = \langle 2, 1, 3\rangle$, then p-Path $M = \langle b, d, i\rangle$.

(25) Let a, b, c, d, e, f, g, h, i be elements of K and M be a matrix over K of dimension 3. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Let p be an element of the permutations of 3-element set. If $p = \langle 3, 1, 2\rangle$, then p-Path $M = \langle c, d, h\rangle$.

(26) (The multiplication of K) $\circledast \langle a, b, c\rangle = a \cdot b \cdot c$.

(27)(i) $\langle 1, 3, 2\rangle \in$ the permutations of 3-element set,
(ii) $\langle 2, 3, 1\rangle \in$ the permutations of 3-element set,
(iii) $\langle 2, 1, 3\rangle \in$ the permutations of 3-element set,
(iv) $\langle 3, 1, 2\rangle \in$ the permutations of 3-element set,
(v) $\langle 1, 2, 3\rangle \in$ the permutations of 3-element set, and
(vi) $\langle 3, 2, 1\rangle \in$ the permutations of 3-element set.

(28) $\langle 2, 3, 1\rangle^{-1} = \langle 3, 1, 2\rangle$.

(29) For every element a of A_3 such that $a = \langle 2, 3, 1\rangle$ holds $a^{-1} = \langle 3, 1, 2\rangle$.

5. Transpositions

The following propositions are true:

(30) For every permutation p of Seg 3 such that $p = \langle 1, 3, 2\rangle$ holds p is a transposition.

(31) For every permutation p of Seg 3 such that $p = \langle 2, 1, 3\rangle$ holds p is a transposition.

(32) For every permutation p of Seg 3 such that $p = \langle 3, 2, 1\rangle$ holds p is a transposition.

(33) For every permutation p of $\operatorname{Seg} n$ such that $p = \mathrm{id}_{\operatorname{Seg} n}$ holds p is not a transposition.

(34) For every permutation p of $\operatorname{Seg} 3$ such that $p = \langle 3, 1, 2\rangle$ holds p is not a transposition.

(35) For every permutation p of $\operatorname{Seg} 3$ such that $p = \langle 2, 3, 1\rangle$ holds p is not a transposition.

6. Even and Odd Permutations

One can prove the following propositions:

(36) Every permutation of $\operatorname{Seg} n$ is a finite sequence of elements of $\operatorname{Seg} n$.

(37) $\langle 2, 1, 3\rangle \cdot \langle 1, 3, 2\rangle = \langle 2, 3, 1\rangle$ and $\langle 1, 3, 2\rangle \cdot \langle 2, 1, 3\rangle = \langle 3, 1, 2\rangle$ and $\langle 2, 1, 3\rangle \cdot \langle 3, 2, 1\rangle = \langle 3, 1, 2\rangle$ and $\langle 3, 2, 1\rangle \cdot \langle 2, 1, 3\rangle = \langle 2, 3, 1\rangle$ and $\langle 3, 2, 1\rangle \cdot \langle 3, 2, 1\rangle = \langle 1, 2, 3\rangle$ and $\langle 2, 1, 3\rangle \cdot \langle 2, 1, 3\rangle = \langle 1, 2, 3\rangle$ and $\langle 1, 3, 2\rangle \cdot \langle 1, 3, 2\rangle = \langle 1, 2, 3\rangle$ and $\langle 1, 3, 2\rangle \cdot \langle 2, 3, 1\rangle = \langle 3, 2, 1\rangle$ and $\langle 2, 3, 1\rangle \cdot \langle 2, 3, 1\rangle = \langle 3, 1, 2\rangle$ and $\langle 2, 3, 1\rangle \cdot \langle 3, 1, 2\rangle = \langle 1, 2, 3\rangle$ and $\langle 3, 1, 2\rangle \cdot \langle 2, 3, 1\rangle = \langle 1, 2, 3\rangle$ and $\langle 3, 1, 2\rangle \cdot \langle 3, 1, 2\rangle = \langle 2, 3, 1\rangle$ and $\langle 1, 3, 2\rangle \cdot \langle 3, 2, 1\rangle = \langle 2, 3, 1\rangle$ and $\langle 3, 2, 1\rangle \cdot \langle 1, 3, 2\rangle = \langle 3, 1, 2\rangle$.

(38) For every permutation p of $\operatorname{Seg} 3$ such that p is a transposition holds $p = \langle 2, 1, 3\rangle$ or $p = \langle 1, 3, 2\rangle$ or $p = \langle 3, 2, 1\rangle$.

(39) For all elements f, g of the permutations of n-element set holds $f \cdot g \in$ the permutations of n-element set.

(40) Let l be a finite sequence of elements of A_n. Suppose that

(i) $\operatorname{len} l \bmod 2 = 0$, and

(ii) for every natural number i such that $i \in \operatorname{dom} l$ there exists an element q of the permutations of n-element set such that $l(i) = q$ and q is a transposition.

Then $\prod l$ is an even permutation of $\operatorname{Seg} n$.

(41) Let l be a finite sequence of elements of A_3. Suppose that

(i) $\operatorname{len} l \bmod 2 = 0$, and

(ii) for every natural number i such that $i \in \operatorname{dom} l$ there exists an element q of the permutations of 3-element set such that $l(i) = q$ and q is a transposition.

Then $\prod l = \langle 1, 2, 3\rangle$ or $\prod l = \langle 2, 3, 1\rangle$ or $\prod l = \langle 3, 1, 2\rangle$.

Let us mention that there exists a permutation of $\operatorname{Seg} 3$ which is odd.

We now state four propositions:

(42) $\langle 3, 2, 1\rangle$ is an odd permutation of $\operatorname{Seg} 3$.

(43) $\langle 2, 1, 3\rangle$ is an odd permutation of $\operatorname{Seg} 3$.

(44) $\langle 1, 3, 2\rangle$ is an odd permutation of $\operatorname{Seg} 3$.

(45) For every odd permutation p of $\operatorname{Seg} 3$ holds $p = \langle 3, 2, 1\rangle$ or $p = \langle 1, 3, 2\rangle$ or $p = \langle 2, 1, 3\rangle$.

7. Determinant and Permanent

One can prove the following propositions:

(46) Let a, b, c, d, e, f, g, h, i be elements of K and M be a matrix over K of dimension 3. If $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$, then $\operatorname{Det} M = (((a \cdot e \cdot i - c \cdot e \cdot g - a \cdot f \cdot h) + b \cdot f \cdot g) - b \cdot d \cdot i) + c \cdot d \cdot h$.

(47) Let a, b, c, d, e, f, g, h, i be elements of K and M be a matrix over K of dimension 3. If $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$, then $\operatorname{Per} M = a \cdot e \cdot i + c \cdot e \cdot g + a \cdot f \cdot h + b \cdot f \cdot g + b \cdot d \cdot i + c \cdot d \cdot h$.

(48) Let i, n be natural numbers and p be an element of the permutations of n-element set. If $i \in \operatorname{Seg} n$, then there exists a natural number k such that $k \in \operatorname{Seg} n$ and $i = p(k)$.

(49) Let M be a matrix over K of dimension n. Given a natural number i such that $i \in \operatorname{Seg} n$ and for every natural number k such that $k \in \operatorname{Seg} n$ holds $M_{\square,i}(k) = 0_K$. Let p be an element of the permutations of n-element set. Then there exists a natural number l such that $l \in \operatorname{Seg} n$ and $(p\text{-Path}\, M)(l) = 0_K$.

(50) Let p be an element of the permutations of n-element set and M be a matrix over K of dimension n. Given a natural number i such that $i \in \operatorname{Seg} n$ and for every natural number k such that $k \in \operatorname{Seg} n$ holds $M_{\square,i}(k) = 0_K$. Then (the product on paths of $M)(p) = 0_K$.

(51) Let M be a matrix over K of dimension n. Given a natural number i such that $i \in \operatorname{Seg} n$ and for every natural number k such that $k \in \operatorname{Seg} n$ holds $M_{\square,i}(k) = 0_K$. Then (the addition of K)-$\sum_{\Omega^{\mathrm{f}}_{\text{the permutations of } n\text{-element set}}}$(the product on paths of $M) = 0_K$.

(52) Let p be an element of the permutations of n-element set and M be a matrix over K of dimension n. Given a natural number i such that $i \in \operatorname{Seg} n$ and for every natural number k such that $k \in \operatorname{Seg} n$ holds $M_{\square,i}(k) = 0_K$. Then $(\operatorname{PPath} M)(p) = 0_K$.

(53) Let M be a matrix over K of dimension n. Given a natural number i such that $i \in \operatorname{Seg} n$ and for every natural number k such that $k \in \operatorname{Seg} n$ holds $M_{\square,i}(k) = 0_K$. Then $\operatorname{Det} M = 0_K$.

(54) Let M be a matrix over K of dimension n. Given a natural number i such that $i \in \operatorname{Seg} n$ and for every natural number k such that $k \in \operatorname{Seg} n$ holds $M_{\square,i}(k) = 0_K$. Then $\operatorname{Per} M = 0_K$.

8. On the Paths of Matrices

One can prove the following two propositions:

(55) Let M, N be matrices over K of dimension n. Suppose $i \in \mathrm{Seg}\,n$. Let p be an element of the permutations of n-element set. Then there exists a natural number k such that $k \in \mathrm{Seg}\,n$ and $i = p(k)$ and $(N_{\square,i})_k = (p\text{-Path}\,N)_k$.

(56) Let a be an element of K and M, N be matrices over K of dimension n. Given a natural number i such that

(i) $i \in \mathrm{Seg}\,n$,

(ii) for every natural number k such that $k \in \mathrm{Seg}\,n$ holds $M_{\square,i}(k) = a \cdot (N_{\square,i})_k$, and

(iii) for every natural number l such that $l \neq i$ and $l \in \mathrm{Seg}\,n$ holds $M_{\square,l} = N_{\square,l}$.

Let p be an element of the permutations of n-element set. Then there exists a natural number l such that $l \in \mathrm{Seg}\,n$ and $(p\text{-Path}\,M)_l = a \cdot (p\text{-Path}\,N)_l$.

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