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HYBRID CONCEPTS OF LONG-TERM ESTIMATES FOR VALUE AT RISK

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Abstract

There is a growing demand for models which enable to measure and assess the risk in long-term horizons (sometimes more than 2 years). The practical demand for such models is required by the institutions which manage the investments and retirement funds. In the paper the theoretical aspects of risk assessment methodology with the use of *Value at Risk (VaR)* were presented. In this method in order to estimate the long-term *VaR* limits the hybrid model which is the optimum mixture of random walk and mean reversion was used. The application of the presented methodology was exemplified by the estimation of long-term predictions for *VaR* limits for stock prices.

Key words: Value at Risk, capital market, Warsaw Stock Exchange.

JEL classification: C01, C51, C53.

Introduction

Risk assessment methodology was developed and introduced for the first time by *RiskMetrics®* in 1994 and was based upon the *Value at Risk (VaR)* concept[1]. From the very beginning it was regarded as a revolutionary and efficient method of risk assessment for the investors who invest on the financial markets. From the practical point of view this methodology is successfully used in practice both by institutional investors (e.g. banks) as well as the individual ones.

In the classical *RiskMetrics*™ methodology two basic assumptions are accepted[2]:

- expected daily logarithmical returns of financial assets equal zero $E_t[r_{t+1|t}] = 0$ (which can be accepted only for short time horizons),
- assets variation for k time horizon of the days $\sigma_{t+k|t}$ is calculated on the basis of one-day variation $\sigma_{t+1|t}$ and then calibrated, where calibration coefficient equals the estimate period root $\sigma_{t+k|t} = \sqrt{k} \cdot \sigma_{t+1|t}$.

There is a big demand for models which enable to measure and assess the risk in longer time horizons (often exceeding 2 years). Enterprises and firms which deal mainly with financial assets are not as vulnerable to daily changes of market prices of these assets. For such enterprises the more important thing is the risk assessment of the potential losses in the value of investment portfolios in longer time horizon. The demand for long-term estimates model is also reported by institutions which manage the investment funds. The main source of their risk is market changes in prices of financial assets in which these funds invest in long-term time horizon. The long-term estimates are also very important for open retirement funds and insurance companies, which have long-term commitments (pensions, annuities, indemnities) and which have to be paid periodically. For these reasons in 2000 it was published the technical document which described the new methodology of risk assessment for financial investments in long-term time horizon (exceeding the period of two years). This methodology was called by *RiskMetrics*™ the *ClearHorizon*™[3].

In the publication the theoretical aspects of the *ClearHorizon*™ methodology will be presented as well as the practical possibilities of its application taking into account long-term risk assessment in stocks investments.

1. Random Walk Model

Random walk model is one of two basic models of time series, on which the already discussed long-term method of estimation for *VaR* is based. May V_t means price (in time *t*) of the researched financial instrument, then the discrete version of random walk model can be presented in the following way[4]:

$$\Delta p_t = p_t - p_{t-1} = \ln(V_t) - \ln(V_{t-1}) = \mu + \sigma \cdot \varepsilon_t, \tag{1}$$

where:

μ – trend parameter,

σ – variation parameter and is described by means of standard deviation,

ε_t – random errors with the same normal standardized distribution $N(0,1)$.

Usually Δp_t is defined as random walk process whereas time series $p_t = \ln(V_t)$ as a difference stationary series after $p_t - p_{t-1}$.

By transforming the equation (1) we can obtain the following recurrence dependence:

$$p_{t+k} = p_t + k\mu + \sigma\left(\varepsilon_{t+1} + \varepsilon_{t+2} + ... + \varepsilon_{t+k}\right). \tag{2}$$

On the basis of the dependence (2) of the estimate for *k* periods in future for the mean and random variance $p_{t+k} = \ln(V_{t+k})$ we can determine (with the use of characteristics of expected value operators $E_t[\ ]$ and $VAR_t[\ ]$ variance calculated in *t*) in the following way[5]:

$$\begin{aligned} \mu_k^{(RW)} &= E_t\left[p_{t+k}\right] = p_t + k\mu + \sigma \cdot E_t\left[\varepsilon_{t+1} + ... + \varepsilon_{t+k}\right] = p_t + k \cdot \mu \ , \\ \sigma_k^{2(RW)} &= VAR_t\left[p_{t+k}\right] = \sigma^2 \cdot VAR_t\left[\varepsilon_{t+1} + ... + \varepsilon_{t+k}\right] = k \cdot \sigma^2. \end{aligned} \tag{3}$$

It results from the fact that ε_{t+m} are independent random variances with the identical distribution $\varepsilon_{t+m} : N(0,1)$ for each m=1,2,…,k.

2. Mean Reversion Model

Mean reversion model which describes the prices fluctuation V_t of the researched financial assets may be described in the discrete version by the following equation[6]:

$$\Delta p_t = p_t - p_{t-1} = \mu + \theta\left[p_0 + \mu\left(t-1\right) - p_{t-1}\right] + \sigma \cdot \varepsilon_t, \tag{4}$$

where:

$\theta \in [0,1]$ – parameter which describes the speed of the mean reversion,

μ – trend parameter,

σ – variation parameter (standard deviation),

$p_0 = \ln(V_0)$, $p_t = \ln(V_t)$, $p_{t-1} = \ln(V_{t-1})$,

ε_t – independent random variables with the identical normal standardized distribution $N(0,1)$.

After some transformations the discrete version for the model of mean reversion might have the following form:

$$p_t = \alpha_0 + \beta \cdot t + \gamma \cdot p_{t-1} + \sigma \cdot \varepsilon_t, \tag{5}$$

where:

$\alpha_0 = \theta \cdot p_0 + \mu(1-\theta)$,

$\beta = \mu \cdot \theta$, $\gamma = 1-\theta$.

Process $p_t = \ln(V_t)$ is described as the trend stationary series. On the basis of the equation (5) we have the following recurrent dependence:

$$\begin{aligned} p_{t+k} = \frac{\alpha_0\left(1-\gamma^k\right)}{1-\gamma} + \frac{\beta}{1-\gamma}\left[(t+k) - \gamma^k(t+1) - \frac{\gamma\left(1-\gamma^{k-1}\right)}{1-\gamma}\right] + \gamma^k \cdot p_t + \\ \sigma\left(\gamma^{k-1}\cdot\varepsilon_{t+1} + \gamma^{k-2}\cdot\varepsilon_{t+2} + \ldots + \gamma\cdot\varepsilon_{t+k-1} + \varepsilon_{t+k}\right). \end{aligned} \tag{6}$$

With the use of (6) estimate of *k* periods in future for the mean value and the variance of the random variable $p_{t+k} = \ln(V_{t+k})$ can be calculated (similarly as it was in the model of random walk) from the formula[7]:

$$\begin{aligned} \mu_k^{(M\,\mathrm{Rev})} &= E_t\left[p_{t+k}\right] = \frac{\alpha_0\left(1-\gamma^k\right)}{1-\gamma} + \frac{\beta}{1-\gamma}\left[(t+k) - \gamma^k(t+1) - \frac{\gamma\left(1-\gamma^{k-1}\right)}{1-\gamma}\right] + \gamma^k \cdot p_t, \\ \sigma^{2\,(M\,\mathrm{Rev})}_k &= VAR_t\left[p_{t+k}\right] = \frac{\sigma^2\left(1-\gamma^{2k}\right)}{1-\gamma^2}. \end{aligned} \tag{7}$$

It results from the fact that ε_{t+m} are independent random variables with the same distribution $\varepsilon_{t+m} : N(0,1)$ for each m=1,2,…,k.

More information on stationary and unstationary econometric time series and on the discussed models might be found in Enders (1995), Metcalf & Hassett (1995) and Weron & Weron (1998).

3. Optimum Mixture of Random Walk and Mean Reversion Models in VaR Long-Term Estimates

For both basic models of time series (described in points 2 and 3) we can determine how fast their variation changes in time depending on the horizon length of the k estimate. For this purpose *Variance Ratio* has been introduced in the methodology by *RiskMetrics™* (*VR*) that is determined[8] as:

$$VR_k = \frac{\sigma^2{}_k}{k \cdot \sigma^2{}_1}, \tag{8}$$

We can look at the presented models of random walk and mean reversion as two extreme poles of the wide spectrum of the possible realization of a financial time series. On the one uttermost pole there is the random walk model whose characteristic feature is its maximum long-term variation ($VR_k = 1$). On the second one there is the mean reversion model which guarantees the minimum long-term variation ($VR_k \to 0$, when $k \to \infty$). The variation of any other time series which determines the prices fluctuation of the researched financial instruments might be described by a hybrid model which is the optimum mixture of the models: *random walk (RW)* and *mean reversion (MRev).* The weights of $\omega \in [0,1]$ and $(1-\omega)$ for the optimum mixture of models are chosen in such a way in order to calibrate the best variation for the hybrid model in relation to the observed historic variance. The historic variance for $k = 1, 2, ..., s$ periods in future is determined with the use of the formula[9]:

$$\sigma_k^2 = \frac{n}{(n-k)(n-k+1)} \sum_{t=k+1}^{n} \left(p_t - p_{t-k} - k \cdot \overline{r_1} \right)^2, \tag{9}$$

where:

n – number of available historic observations,

$\overline{r_1} = \ln(p_t) - \ln(p_{t-1}),\ t = 2, ..., n$ – a mean value for *l*- period logarithmical rates of return for the researched assets.

By using the *Variance Ratio (VR)* the weights for the hybrid model are established by solving the following optimum equation[10]:

$$\sum_{k=1}^{s}\left(VR_k^{(H)}-\omega\cdot VR_k^{(RW)}-(1-\omega)\cdot VR_k^{(M\,\mathrm{Rev})}\right)^2\rightarrow\min, \qquad (10)$$

where:

$VR_k^{(H)}$ – *Variance Ratio* for historic variance calculated on the basis of the formulae (8) and (9),

$VR_k^{(RW)}$, $VR_k^{(M\,\mathrm{Rev})}$ – analogical coefficients for random walk model and mean reversion model (calculated on the basis of the formula (3), (7) and (8)).

In hybrid model it is estimated that natural logarithms of the prices of the researched financial instruments are subjected to normal distribution $p_{t+k}:N\left(\mu_k^{(Mix)},\sigma_k^{(Mix)}\right)$, where the average parameter is $\mu_k^{(Mix)}=\mu_k^{(M\,\mathrm{Rev})}$, which can be determined from the formula (7).
The parameter of the standard deviation can be determined from the following formula:

$$\sigma_k^{(Mix)}=\omega\cdot\sigma_k^{(RW)}+(1-\omega)\cdot\sigma_k^{(M\,\mathrm{Rev})}, \qquad (11)$$

The *Value at Risk* (*VaR*) is sometimes defined in the aspect of the potential losses (or profits) of an investor on the basis of probability of the occurrence in the future of the following scenario: $\mathrm{P}\left(VaR_k^{(d)}<ZS_{t,k}=V_{t+k}-V_t<VaR_k^{(g)}\right)=1-\alpha$[11]. The estimates for k periods in future for lower and upper *VaR* limit can be determined as follows[12]:

estimates for losses (profits):	estimates for prices:	estimates for price logarithms:	
$VaR^{(g)}{}_k=e^{\left(\mu_k+Z_{1-\alpha/2}\cdot\sigma_k\right)}-V_t$,	$VaR^{(g)}{}_k=e^{\left(\mu_k+Z_{1-\alpha/2}\cdot\sigma_k\right)}$,	$VaR^{(g)}{}_k=\mu_k+Z_{1-\alpha/2}\cdot\sigma_k$,	(12)
$VaR^{(d)}{}_k=e^{\left(\mu_k+Z_{\alpha/2}\cdot\sigma_k\right)}-V_t$,	$VaR^{(d)}{}_k=e^{\left(\mu_k+Z_{\alpha/2}\cdot\sigma_k\right)}$,	$VaR^{(d)}{}_k=\mu_k+Z_{\alpha/2}\cdot\sigma_k$.	

where:

$\mu_k=\mu_k^{(Mix)}$, $\sigma_k=\sigma_k^{(Mix)}$ – estimates determined respectively for the mean reversion and standard deviation,

$Z_{\alpha/2}$, $Z_{1-\alpha/2}$ – quantile ranks $(\alpha/2)$ i $(1-\alpha/2)$ for standardized normal distribution.

4. Application of the ClearHorizon™ Methodology Exemplified by the Risk Assessment of Investing in Stocks

The discussed methodology has been worked out for the long-term needs, thus a kind of limit of its application will be a necessity to have the time series with the proper long history. It is required in order to get the stable estimates of parameters for the random walk model and mean reversion model.

The procedure of application of the *ClearHorizon*™ methodology while determining the long-term estimates for the *VaR* will be presented on the example of the estimate for a monthly stock portfolio quotation of the companies on the local floor[13].

On the basis of the available historic data about the prices[14] of stock exchange rates of portfolio components (within the period from May 1995 to August 2007) the parameters of the *random walk* and *mean reversion* models were estimated with the use of the formulae (1) and (5). In order to estimate the model parameters (except for the variation parameter σ) the least- squares method was applied[15]. The variation parameter was estimated as a remainder standard deviation. On the basis of the obtained values of parameters for the monthly fluctuations for logarithms of stock prices from the formulae (3) and (7) the variation σ_k^2 (k=1,...,48) for 4-year estimate horizon was determined. The historic variation of the *k* period was determined with the use of the (9) formula. In order to calibrate the best variation in the hybrid model (compared to the historic variation) two optimizations were solved (10). In order to do so the already calculated values of *Variance Ratio (VR)* from (8) were used. The following weights for the hybrid model were received (which are the optimum mixture of the random walk model and the mean reversion model): ω=0,73, 1–ω=0,27. Figure 1 presents the graphs of *Variance Ratios* for the analyzed portfolio.

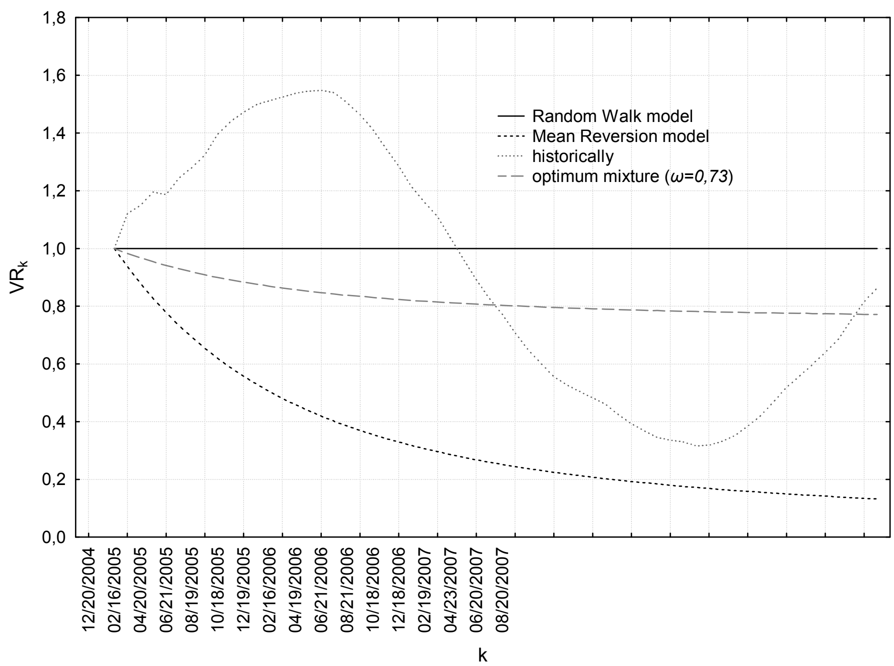

Fig. 1. *K*-values of the monthly *Variance Ratio (VR)* for the analyzed portfolio (on the basis of available monthly portfolio quotations) for the random walk model and the mean reversion model (*t=116*)[16]
Source: own research.

In order to check the quality of the discussed model it is necessary to study the constancy of the parameters of the model in time. In this case the G.C.Chow's model was applied for which, if the hypothesis H_0:$\beta_1=\beta_2=\beta$, the following random variable is true:

$$F = \frac{\dfrac{SS^* - (SS_1 + SS_2)}{k}}{\dfrac{SS_1 + SS_2}{T^* - 2k}}, \tag{13}$$

where:

SS_1 – sum square of rests for the model in the period from l to t;

SS_2 – sum square of rests for the model in the period from t+1 to $T*$;

SS^* – sum square of rests for the model in the period from 1 do $T*$;

k – number of estimated parameters,

has F Fisher-Snedecor distribution with k and T*-2k degrees of freedom.

Below there are presented the values of parameters for the particular classes of models and the values of the *F*-statistics for the Chow's test for the observation *t*=116 for which the example of estimates shown in Figure 2 was determined.

Table 1. Values of parameters for the particular classes of models and the value of *F* statistics for the Chow's test (*t=116*)

Parameters for Random Walk model		Parameters for Mean Reversion model				Parameters for Mix model (hybrid)
μ	σ	α	β	γ	σ	ω
0,00096	0,09925	0,57330	-0,00002	0,93480	0,09766	0,736332

Chow's test for Random Walk model	F critical for α=0,2	F critical for α=0,1	F critical for α=0,05	Chow's test for Mean Reversion model	F critical for α=0,2	F critical for α=0,1	F critical for α=0,05
0,42910	1,65744	2,74045	3,90639	2,25577	1,56734	2,12303	2,66879

Source: own research.

The assessment of stability of the hybrid model was done for the whole analyzed period concluding that the model is stable. The parameters of the component elements of the new model, that is the random walk and the mean reversion were stable as well. It is worth emphasizing that the determined *F* statistics for the discussed period almost always exceeded the critical values for three kinds of importance levels.

Apart from the usual analysis of backtesting, which was excellent, the test of the number of exceedings with the use of LR_{POF} (*Proportion of Failures Test – POF*) was done. LR_{POF} statistics looks as follows:

$$LR_{POF} = -2\ln\left(\frac{(1-q)^{T_0} q^{T_1}}{(1-\hat{q})^{T_0} \hat{q}^{T_1}}\right) \sim \chi_1^2, \tag{14}$$

where:

$\hat{q} = \dfrac{T_1}{T_0 - T_1}$,

$T_1 = \sum_{i=1}^{T} I_t(q)$,

$T_0 = T - T_1$,

T – number of total observations,

T_1 – number of exceedings,

T_0 – number of observations where there was no exceeding.

LR_{POF} has the χ^2 distribution with one degree of freedom.

Table 2. Values of LR_{POF} for the number of exceeding test for α=0,05 (*t=116*)

Random Walk Model		Mean Reversion Model		Mix Model (hybrid)		LR_{POF} statistics for Mix model	
Exceeding over the lower VaR limit	Exceeding over the upper VaR limit	Exceeding over the lower VaR limit	Exceeding over the upper VaR limit	Exceeding over the lower VaR limit	Exceeding over the upper VaR limit	VaR lower limit	VaR upper limit
0	0	0	7	0	2		0,12648

Source: own research.

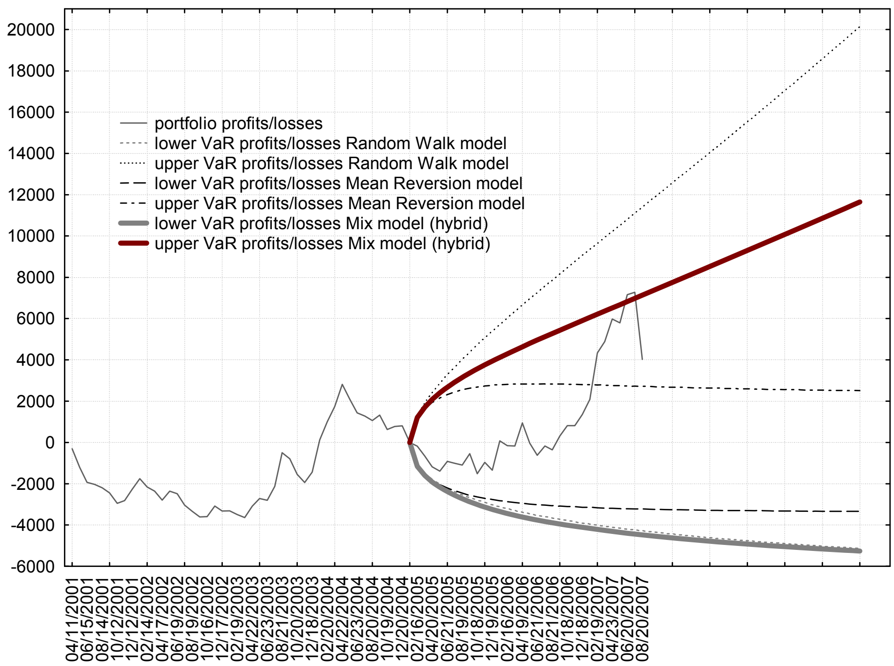

Fig. 2. The *VaR* limits (upper and lower) for the quotation of the discussed basket of stocks estimated on the 90% confidence level and calculated on the following models: the random walk, the mean reversion and the hybrid (*t=116*) with weights ω = 0,73, 1 – ω =0,27

Source: own research.

The long-term estimates for the *VaR* limits for the future quotations of the researched portfolio were determined from the formula (12), taking for the hybrid model the variation determined on the basis of the (11). Figure 2 shows the obtained estimations for the *VaR* limits for the quotation of the considered basket of stocks[17].

Conclusions

From the obtained estimates (Figure 2) we can draw some conclusions. The vast confidence limits for the estimated portfolio values gives the *RW* model. The limits of the estimates determined on the basis of *MRev* model for shorter time series are satisfactory, however, due to the noticeable change of the series structure (July 2007) in longer time horizon are hardly effective. The best calibrated are the estimated *VaR* limits for the hybrid model for which the percentage of exceedings in the given period is small and the cone tails of the estimates show the real change. It seems that it is the hybrid model with the correction (from period to period)[18] of the estimated *VaR* limits that is the most reliable.

Notes

[1] See Best (2000), *RiskMetrics*™ (1996).

[2] See *RiskMetrics*™ (1996), pp.59, 84-87.

[3] See Kim, Mina (2000).

[4] See Kim, Mina (2000), p.3.

[5] See Kim, Mina (2000), p.4.

[6] See Kim, Mina (2000), p.4.

[7] See Kim, Mina (2000), p.5.

[8] See Kim, Mina (2000), p.5.

[9] See Glen (1992), p.147, Kim, Mina (2000), p.9.

[10] See Kim J., Mina J. (2000),, pp. 9]).

[11] Such defined limits for value at risk make up *1*-α percent confidence interval for an investor's future potential losses/profits in time horizon of *k* periods. More information concerning *VaR* can be found e.g. in Best (2000), Pisula & Pisula (2002) and *RiskMetrics*™ (1996).

[12] See Kim, Mina (2000), p.13.

[13] The portfolio block: Bank Millenium, Dębica, Irena, Stalexport and Swarzędz.

[14] In case of prices the closing rates of the considered partnerships creating the portfolio are taken into account.

[15] To determine the model parameters the maximum likelihood method can be applied.

[16] On the graph (for comparison) the estimated values of *VaR* for the historic variation were also presented.

[17] Estimates for *Random Walk model* and *Mean Reversion model* are also determined by the formula (12), taking as parameters μ_k and σ_k the values of parameters determined from the formulas (3) or (7).

[18] Only if the structure of the time series in the consecutive period had changed.

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