DE GRUYTER OPEN
Foundations of Management, Vol. 9 (2017), ISSN 2080-7279
DOI: 10.1515/fman-2017-0020

MODIFICATION OF SHAPLEY VALUE AND ITS IMPLEMENTATION IN DECISION MAKING

Leszek ZAREMBA*, Cezary S. ZAREMBA**, Marek SUCHENEK***

*Academy of Finance and Business Vistula, Institute of Management, Warszawa, Poland
e-mail: l.zaremba@vistula.edu.pl

** Peaceful Games Cezary Zaremba, Warszawa, Poland
e-mail: peacefulgames@wp.pl

*** California State Univ. Dominquez Hills, Computer Science Department, Carson, Calif. USA
e-mail: suchenek@csudh.edu

Abstract: The article presents a solution of a problem that is critical from a practical point of view: how to share a higher than usual discount of $10 million among 5 importers. The discount is a result of forming a coalition by 5 current, formerly competing, importers. The use of Shapley value as a concept for co-operative games yielded a solution that was satisfactory for 4 lesser importers and not satisfactory for the biggest importer. Appropriate modification of Shapley value presented in this article allowed to identify appropriate distribution of the saved purchase amount, which according to each player accurately reflects their actual strength and position on the importer market. A computer program was used in order to make appropriate calculations for 325 permutations of all possible coalitions. In the last chapter of this paper, we recognize the lasting contributions of Lloyd Shapley to the cooperative game theory, commemorating his recent (March 12, 2016) descent from this world.

Keywords: characteristic function, coalition, cooperative game, core of game, imputation, Shapley value.

JEL: C71, F51, F12.

1 Introduction

We consider the decision problem that the importers of raw materials face while minimizing their costs by means of cooperation with their competitors. It has been communicated to the first author by an employee of a Polish consulting company. The solution of the problem that we present here enables the importers to make substantial savings commensurate with their actual contributions to the cooperating group. It turns out that the proposed solution based on the original concept of the Shapley value for co-operative games has one weakness – it discriminates the biggest importer who did not accept that solution. Our approach eliminates this weakness.

The problem we solve in this article has both practical and theoretical ramifications. In particular, it is satisfactory for all the grand coalition members, and not just to some or majority of them. Our improvement is based on appropriate modification of the Shapley value concept for cooperative games (see Comment 1); it was carried out by a computer program used to perform the necessary calculations.

Politics of collective rationality, to which this article is related, has been discussed at several occasions in the economic literature of the subject (cf. Weirich, 2012) where cooperation and rationalism in the language of game theory has been broadly investigated. Some authors pointed out (e.g., Ott, 2016) the problems arising from conflict solving and competitor cooperation by means of strategic alliances and joint decision making.

A popular mathematical model used in problem solving that incorporates cooperative games has been presented in Peleq (2007). Lloyd Shapley (1923 –2016) played a major role in the development of those games. He was awarded a Nobel Prize in 2012 in Economics together with Alvin Roth. A fairly thorough account of Shapley's achievements may be found in Roth (2005), where lasting contribution of Shapley to cooperative game theory was recognized in a form of 20 papers written by 24 authors, who either reviewed or continued the research initiated by Shapley's five remarkable 1953 papers listed in the bibliography. A shorter presentation of Shapley's role in the development of cooperative game theory is given in the last chapter of this article.

Another type of antagonistic games constitutes 2-person, zero-sum games. They may be either static in their nature or dynamic. The last category includes 2-person, zero-sum games where the actions of players are governed by differential equations; see Zaremba (1979,1980, 1982, 1984a, 1984b, 1986, 1989). In such games, there is no place for cooperation, however.

Coming back to our main topic, we assume that the amount of discount offered to each importer by the exporter depends on the total value of the import according to the Table 1.

Table 1. Discount correlation to the import value in millions of PLN

Company size	Bottom limit of the import value	Percentage of discount
very large	700	4
large	300	3
medium	125	2
small	50	1
very small	0	0

The information contained in Table 1 can also be presented graphically, as shown in Fig.1.

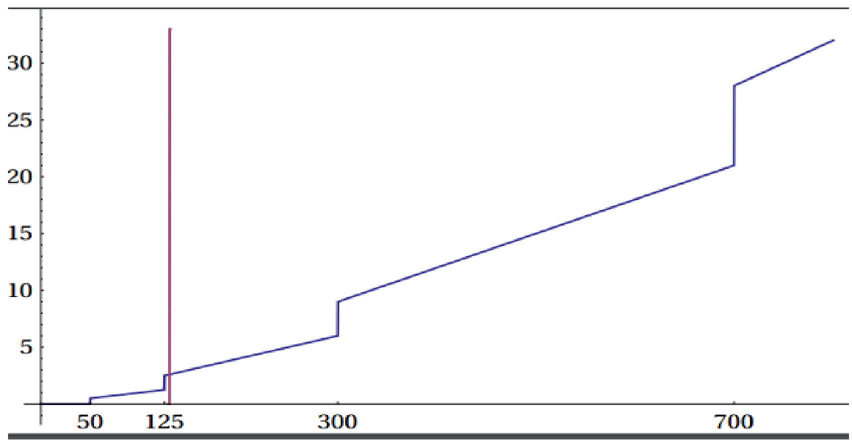

Figure 1. Discount function in millions of PLN

The solution method presented below does not depend on numeric parameters shown in the table above; it may be straightforwardly implemented for any other table presenting the correlation between the discount and the amount of import amount. We assume that there are a number of importers of the raw materials operating on the market, including the existing competitors A, B, C, D and E, whose annual expenditure on purchasing the raw material are shown in the first version in the Table 2a.

Table 2a. The amounts of import for particular companies in millions of PLN

A	B	C	D	E
310	150	130	100	60

We also consider a version Table 2b of the above scenario in order to illustrate the influence of slightly different import quotas on final solution.

Table 2b. The amounts of import for particular companies in millions of PLN

A	B	C	D	E
290	150	130	100	60

A consulting company that provided services to A, B, C, D and E importers recommended that they combine their forces and act as one large importer. This should enable them to obtain the biggest 4% discount as they are going to import raw material totaling 750 million PLN (Table 2a) or 730 million PLN (Table 2b). However, this recommendation gave rise to a complicated problem: how to divide the 4% discount as each of the 5 importers had a different idea of how to share the saved amount of purchase. The said problem may be formulated as follows:

Problem 1: Propose a formula, or a methodology, for the division of saved money by a coalition of 5 importers resulting from obtaining a 4% discount among the coalition members, so that each of them perceives the share he cashes is reflecting adequately their contribution to the coalition and position on the importer market.

Definition 1: The *characteristic function* of an n-person cooperative game with a set C of players (referred to as a grand coalition in this paper) assigns to each subset S of C the maximum value *v*(S) that coalition S can guarantee itself by coordinating the strategies of its members, no matter what the other players do (Thomas 1986, p.86).

Cooperation Games (cf. Malawski, Wieczorek, Sosnowska, 2006, pp.127-150) provide a mathematical model for solving this kind of decision problems. Our approach will benefit from this formalism with discounts playing the role of payouts *v*(S) for all possible coalitions S. Discounts that each coalition is entitled to are presented in Tables 3a and 3b in accordance with the data presented in Table 1 and Tables 2a, 2b, respectively.

In this paper, we are also going to use two classic concepts that are useful while solving the cooperative games, that is, a *core*, first defined explicitly by (Gillies, 1959), and the *Shapley value,* which always exists, is unique and belongs to the core of the game; see Shapley (1953a, 1953d) and for example, Malawski, Wieczorek, Sosnowska (2006, pp.134-136). In the next paragraph, we are going to present a solution to the problem raised by the consulting company by calculating the Shapley value for this cooperative game.

Table 3a. Characteristic function *v*(S) in case of Table 2a in millions of PLN

coalition	discount *v*(S)	coalition S	discount *v*(S)	coalition S	discount *v*(S)
A, B, C, D, E	30	A, B, E	15.6	B, C	5.6
A, B, C, D	20.7	A, C, E	15	B, D	5
A, B, C, E	19.5	B, C, E	10.2	B, E	4.2
A, B, D, E	18.6	A, D, E	14.1	C, D	4.6
A, C, D, E	18	B, D, E	9.3	C, E	3.8
B, C, D, E	13.2	C, D, E	5.8	D, E	3.2
A, B, C	17.7	A, B	13.8	A	9.3
A, B, D	16.8	A, C	13.2	B	3
A, C, D	16.2	A, D	12.3	C	2.6
B, C, D	11.4	A, E	11.1	D	1
				E	0.6

Table 3b. Characteristic function $v(S)$ in case of Table 2b in millions of PLN

coalition	discount $v(S)$	coalition	discount $v(S)$	coalition	discount $v(S)$
A, B, C, D, E	29.2	A, B, E	15	B, C	5.6
A, B, C, D	20.1	A, C, E	14.4	B, D	5
A, B, C, E	18.9	B, C, E	10.2	B, E	4.2
A, B, D, E	18	A, D, E	13.5	C, D	4.6
A, C, D, E	17.4	B, D, E	9.3	C, E	3.8
B, C, D, E	13.2	C, D, E	5.8	D, E	3.2
A, B, C	17.1	A, B	13.2	A	5.8
A, B, D	16.2	A, C	12.6	B	3
A, C, D	15.6	A, D	11.7	C	2.6
B, C, D	11.4	A, E	10.5	D	1
				E	0.6

However, the resulting solution was not accepted by the biggest Polish importer who argued that their biggest input in the 5-person coalition had been underestimated in the Shapley value.

So, we will modify the Shapley value concept for cooperative games in the 3rd chapter (Commentary 1) with the aim to develop a new discount distribution (pay-outs to individual players), which – we claim – will not be questioned by any of the players. The obtained solution is unique and always exists. We will demonstrate that it also belongs to the core (see, Definition 3) of the studied game.

2 Solution of the Problem by means of the Shapley Value

In each n-person game, the following natural question arises: When a coalition is formed, how does it share its payout (reward) between its own individual members?

Definition 2a: (Thomas, 1986, p.90) An *imputation* in an n-person game with characteristic function $v(S)$ defined for all coalitions S is a vector:

$$x = (x_1, x_2, ..., x_n)$$

satisfying:

(i) $x_1 + x_2 + x_3 + x_4 + x_5 = v(N)$, where N is the set of all players;

(ii) $x_i \geq v(i)$ for i = 1, 2,…, n.

In the context studied here, the conditions (i) – (ii) can be formulated as follows:

(a) $x_A + x_B + x_C + x_D + x_E = v\{A, B, C, D, E\}$;

(b) $x_A \geq v(A)$; $x_B \geq v(B)$; $x_C \geq v(C)$; $x_D \geq v(D)$; $x_E \geq v(E)$.

Definition 2b: (Thomas, 1986, p.92) We say that an imputation $x = (x_1, x_2, ..., x_n)$ is rational for coalition S if, and only if the sum of payouts it generates for all members of S is greater or equal to $v(S)$.

Let's note that if a certain imputation:

$$x = (x_1, x_2, ..., x_n)$$

does not satisfy the condition stated in Definition 2b for some coalition S, then S will have no monetary incentive to participate in such share of rewards (discounts in our case). Therefore, we restrict our analysis, without loss of generality, to imputations that are rational for all coalitions S.

Definition 3: (Thomas, 1986, p.92) The set of all imputations which are rational for all coalitions S is called the *core* of game.

The natural candidate for the solution of problem 1 is the Shapley value (see Shapley 1953a and for example Thomas 1986, pp.101-103). It is a concept that in some rational way takes into account what (how much) each of player "contributes" to the largest coalition's reward. In the studied case with 5 players, the Shapley value:

$$x^* = x = (x_A^*, x_B^*, x_C^*, x_D^*, x_E^*)$$

is such an imputation whose coordinates represent an "average" contribution (savings) each of these 5 players brings when it joins the coalition {A, B, C, D, E} at all possible stages of its creation.

Let us explain it. The idea is that the players arrive at game in random order. When any player, for example D, arrives at some already existing coalition S, he brings (contributes) an extra amount to it, namely:

$v(S \cup D) - v(S)$.

Note 1: We should use {D} rather than D, but for simplicity we will not be doing it, hoping it will not cause any misunderstanding.

Suppose that the grand coalition of all 5 importers has been formed in such a way that at first, player D joined player E, next C joined the coalition {E, D}, next B arrived to the coalition {E, D, C}, and finally A joined the coalition {E, D, C, B}. Therefore, when importer E started the creation of coalition {E, D, C, B, A}, his contribution to this largest coalition's reward was just v({E}) = v(E) = 0.6, according to Table 2a.

Note 2: We should remember that coalition {E, D, C, B, A} is the same as coalition {A, B, C, D, E}.

When importer D joined E, he brought an additional award (discount) of *v*{E, D} – *v*({E}) = 3.2 – 0.6 = 2.6 million PLN. Similarly, when C joined the coalition {E, D}, he brought an extra discount of *v(*{E, D, C}) – *v*({E, D}) = 5.8 – 3.2 = 2.6 million according to Table 2a. When B arrived at coalition {E, D, C}, he brought additional discount of:

v({E, D, C, B}) – *v*({E, D, C}) = 13.2 – 5.8 = 7.4 million.

Finally, when A joined coalition {E, D, C, B}, he brought the largest discount of:

v({E, D, C, B, A}) – *v*({E, D, C, B}} = 30 – 13.2 = 16.8 million.

It is easy to notice that the grand coalition can also be formed in 119 = 5! – 1 different ways than the one {E, D, C, B, A}.

Note 3: The foregoing is a sequence and not a set of players that we have examined; for example, {C, B, D, A, E} is another sequence of the same grand coalition.

Performing analogous calculations as above for {C, B, D, A, E}, one can summarize the results obtained above in the Table 4.

Table 4. Contributions of players to greater discount

order of creation of grand coalition	contribution of player A	contribution of player B	contribution of player C	contribution of player D	contribution of player E	total
{E, D, C, B, A}	16.8	7.4	2.6	2.6	0.6	30
{C, B, D, A, E}	9.3	3	2.6	5.8	9.3	30

Following this procedure of Shapley for all 118 = 5! – 2 remaining permutations, and next averaging the results obtained in each column (for each importer), we arrive at the Shapley imputation:

$x = (x_A*, x_B*, x_C*, x_D*, x_E*)$.

Proposition 1: The Shapley value for the cooperative game with characteristic function presented in Table 3a is the imputation $x = (x_A*, x_B*, x_C*, x_D*, x_E*)$ = (11,68; 5,89; 5,14; 4,11; 3,18). Dividing the coordinates of this vector by the amounts of import for the companies A, B, C, D, E, one obtains the Shapley value in percentage terms (**3.77**%; 3.93%; 3.95%; 4.11%; **5.29**%). One could understand why the biggest importer A felt disappointed and discriminated when he was told that the Shapley imputation for him was considerably less than $310 \cdot 4\% = 12.4$ million he was almost certain to receive, hoping for even more because he had the largest contribution to the import value of 750 million. It is known (see, e.g., Malawski, Wieczorek, Sosnowska, 2006, pp.134-136) that the proposition below is true. In order to gain a greater insight, we will however demonstrate a proof in this particular case.

Proposition 2: The Shapley value for the cooperative game with characteristic function presented in Table 3a belongs to the core of this game.

Proof. It is enough to verify (see Table 5a below) that the discounts (11.68; 5.89; 5.14; 4.11; 3.18) offered by Shapley solution:

$x = (x_A*, x_B*, x_C*, x_D*, x_E*)$

to each importer sum up for all 31 = $2^5 - 1$ coalitions to greater amounts than those specified by the characteristic function (Table 3a). We leave to the reader a verification of this claim just for one coalition {A, C, E}. In fact, Shapley's solution offers for {A, C, E} the total discount of:

20 million = 11.68 + 5.14 + 3.18

while the characteristic function is not so much generous. Indeed, according to assumptions made in Table 1 and Table 2a, we have v {A, C, E} = 15 million = $(310 + 130 + 60) \cdot 3\%$. Let us recall that v(S) is the maximum value that coalition S can guarantee itself by coordinating the strategies of its members, no matter what the other players will do.

Table 5a. Discounts attributable to each coalition according to Shapley value and Table 2a

coalition	discount v(S)	coalition	discount v(S)	coalition	discount v(S)
A, B, C, D, E	30	A, B, E	20.75	B, C	11.03
A, B, C, D	26.82	A, C, E	20	B, D	10
A, B, C, E	25.89	B, C, E	14.21	B, E	9.07
A, B, D, E	24.86	A, D, E	18.97	C, D	9.25
A, C, D, E	24.11	B, D, E	13.18	C, E	8.32
B, C, D, E	18.32	C, D, E	12.43	D, E	7.29
A, B, C	22.71	A, B	17.57	A	11.68
A, B, D	21.68	A, C	16.82	B	5.89
A, C, D	20.93	A, D	15.79	C	5.14
B, C, D	15.14	A, E	14.86	D	4.11
				E	3.18

This way we proved the proposition with the help of a computer program which performed all the calculations. One can carry out a similar reasoning for the data presented in Table 2b, thus arriving at the following proposition.

Proposition 3: The Shapley value for the cooperative game with characteristic function presented in Table 3b is the imputation = (10.47; 6; 5.24; 4.22; 3.27). Dividing the coordinates of this vector by the amounts of import for particular companies A, B, C, D and E, one obtains the Shapley value in percentage terms (**3.61**%; 4%; 4.03%; 4.22%; **5.47**%).

In this case, the disappointment of importer A was smaller than previously because according to Table 2b, he himself was not entitled to receive a 3% discount. However, he still questioned why the smallest importers got the highest discounts. Let's note here that the 5 payoffs (discounts) to the players A, B, C, D and E do not sum up this time to 30 million PLN, but to 29.2 million PLN because $730 \cdot 4\% = 29.2$.

Proposition 4: The Shapley value for the cooperative game with characteristic function presented in Table 3b belongs to the core of this game.

Once again, knowing that the fact above is true, we will demonstrate it in the particular example studied. Clearly, the proof will again follow the same lines as in the case of Proposition 2 with Table 3a and Table 5a replaced by Table 3b and Table 5b (below), respectively.

Commentary 1: In Chapter 3, we modify the Shapley idea, by taking into account not only all possible ways (permutations) by which the full coalition can be created (as Shapley did), but also we take into account all possible ways by which the remaining (smaller) coalitions could have been created.

Table 5b. Discounts attributable to each coalition according to Shapley value and Table 2b

coalition	discount v(S)	coalition	discount v(S)	coalition	discount v(S)
A, B, C, D, E	29.20	A, B, E	19.75	B, C	11.24
A, B, C, D	25.93	A, C, E	18.99	B, D	10.22
A, B, C, E	24.99	B, C, E	14.52	B, E	9.28
A, B, D, E	23.97	A, D, E	17.97	C, D	9.46
A, C, D, E	23.21	B, D, E	13.5	C, E	8.52
B, C, D, E	18.74	C, D, E	12.74	D, E	7.5
A, B, C	21.71	A, B	16.47	A	10.47
A, B, D	20.69	A, C	15.71	B	6.00
A, C, D	19.93	A, D	14.69	C	5.24
B, C, D	15.46	A, E	13.75	D	4.22
				E	3.28

3 Solution of Problem 1 by means of market strength (MS) method

Now we will propose another solution concept to Problem 1 by means of the so-called market strength method. This method will produce such imputation $(\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D, \hat{x}_E)$ of discounts which fully takes into account the real strength of each of importers A, B, C, D and E on the raw material market. It turns out that the solution $(\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D, \hat{x}_E)$ assigns higher discount to the largest importer than the Shapley solution.

Similarly, as in the Shapley's approach, the MS method requires the computation of average marginal discount (contribution) that each importer brings to the existing coalition when he joins it. But unlike as in Chapter 2, this time we take into account all 325 coalitions (of all possible sizes), not just the biggest one, as was the case with the methods based on the Shapley value. The information gained this way therefore yields more information about the strength of each importer on the raw material market than in the Shapley's approach.

That information (knowledge) is next utilized in the construction of a new imputation:

$$(\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D, \hat{x}_E)$$

we propose as a solution. Consider thus:

$1 = \binom{5}{0}$ 5-member coalition,

$5 = \binom{5}{1}$ 4 - member coalitions,

$10 = \binom{5}{2}$ 3-member coalitions,

$10 = \binom{5}{3}$ 2-member coalitions and finally

$5 = \binom{5}{4}$ importers A, B, C, D and E, which formally may be viewed as 1-member coalitions.

The number of all sizes' coalitions is:

$$\binom{5}{0}5! + \binom{5}{1}4! + \binom{5}{2}3! + \binom{5}{3}2! + \binom{5}{4}1 = 1 \cdot 120 + 5 \cdot 24 + 10 \cdot 6 + 10 \cdot 2 + 5 \cdot 1 = 325$$

Our computer program thus analyzed 325 permutations, averaging all marginal contributions the players bring when they join an already existing coalition. We demonstrate this by examining just 10 listed below permutations representing 10 ways to create a coalition:

ACEBD (9.3; 4.5; 3.9; 10.5; 1.8);
ACEDB (9.3; 12; 3.9; 3; 1.8);
EBCA (9.3; 3.6; 6; -; 0.6); EBCD (-; 3.6; 6; 3; 0.6);
EBD (-; 3.6; -; 5.1; 0.6); ECA (11.2; -; 3.2; -; 0.6);
CA (10.6; -; 2.6; -; -); DB (-; 4; -; 1; -);
B (-; 3; -; -; -); D (-; -; -; 1; -).

Let us now explain what do we mean by writing EBD (-; 3.6; -; 5.1; 0.6). Dash on the place of the first coordinate means that the first importer (A) does not participate in this coalition. Similarly, dash on the place of third coordinate means that the third importer (C) does not participate in coalition EBD either, which is obviously true. The second coordinate (equal to 3.6) indicates that the second importer (B) contributes (brings marginal discount) at average of 3.6 million when joining an existing coalition of any size (4 or 3 or 2 or 1 or 0).

Computing the average strength of player A (the same remark applies also to any other importer) on the raw materials market, the computer program adds up the marginal contributions (extra discounts) that importer A brought to the already existing coalitions when he joined them. The sum of all the marginal contributions divided by the number of such instances yields the average strength of importer A on the market.

To clarify this point even further, let us assume that there are only 10 (listed above) ways the coalitions can be formed; in reality, there are 325 of them. Then the average contribution of player A would be equal to (9.3+9.3+9.3+11.2+10.6)/5 = (49.7)/5 = 9.94. Indeed, we divide by 5 (not by 10) the sum of all contributions of player A because A joined existing coalitions only 5 times.

We are now about to define this new solution determined by the MS method. In case of data displayed in Table 2a, our computer program calculated the average strengths for importers A, B, C, D and E on the raw material market obtaining the following imputation (10.99; 4.99; 4.29; 3.18; 2.32) of discounts. Since the sum of such calculated contributions (extra discounts) equals 25.77, which is less than 30 = $4\% \cdot (310+150+130+60)$, we should proportionally increase all coordinates of the vector (10.99; 4.99; 4.29; 3.18; 2.32) to receive another imputation for which the sum of its coordinates will be equal to 30. As a result, we will obtain the imputation (**12.795**; 5.81; 4.995; 3.70; **2.70**).

Proposition 5: The solution $(\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D, \hat{x}_E)$ of the cooperative game problem 1 with characteristic function given in Table 3a, determined according to the MS method proposed in this article, is the following imputation of discounts (**12.795**; 5.81; 4.995; 3.70; **2.70**). Dividing the coordinates of this vector by the amounts of import for companies A, B, C, D and E, one arrives at the solution (**4.13**%; 3.87%; 3.84%; 3.70%; **4.50**%).

Comparing this new imputation (**12.795**; 5.81; 4.995; 3.70; **2.70**) with the Shapley value (11.68; 5.89; 5.14; 4.11; 3.18), one can see that importer A became the key beneficiary of MS method. Clearly, the same conclusion holds when we compare these two solutions in percentage terms, that is, (**4.13**%; 3.87%; 3.84%; 3.70%; **4.50**%) with the Shapley solution (**3.77**%; 3.93%; 3.95%; 4.11%; **5.29**%).

Note 4: One might argue that it is not altogether clear that A should be the main beneficiary. Assuming so seems similar to the Marxian economy that distributes rewards according to the investment effort and resources. However, if E with its "small" contribution increased everybody else's reward by a large margin, he would be in a good negotiating position to command a lion's share of these rewards. If, however, he has to compete with other "small" importers, then his rewards should be equal to the minimum acceptable by any of these "small" importers. Perhaps, the assumption that coalitions split their rewards regardless of what other actors on the market are doing is a root cause of this flaw.

In this way, we have arrived at a new result.

Proposition 6: The solution $(\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D, \hat{x}_E)$ of the cooperative game problem 1 with characteristic function given in Table 2a, determined according to the **MS** method, belongs to the core of the game.

Proof. It is enough to make sure that the discounts $(\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D, \hat{x}_E) =$ (**12.795**; 5.81; 4.995; 3.70; **2.70**) sum up for all $31 = 2^5 - 1$ coalitions to greater amounts (Table 6a) than those featured in Table 3a.

We leave to the reader justification of this claim for coalition {A, C, E} only. In fact, by summing up the coordinates $\hat{x}_A$, $\hat{x}_C$, $\hat{x}_E$, we obtain 12.79 + 4.99 + 2.70 = 20.48, which is more than the discount v{A, C, E} = 15 million = $(310+130+60) \cdot 3\%$ listed in Table 3a.

Table 6a. Discounts attributable to each coalition according to Shapley value and Table 2a

coalition	discount	coalition	discount	coalition	discount
A, B, C, D, E	30	A, B, E	21.3	B, C	10.8
A, B, C, D	27.29	A, C, E	20.48	B, D	9.51
A, B, C, E	26.29	B, C, E	13.5	B, E	8.51
A, B, D, E	25	A, D, E	19.19	C, D	8.69
A, C, D, E	24.18	B, D, E	12.21	C, E	7.69
B, C, D, E	17.2	C, D, E	11.9	D, E	6.40
A, B, C	23.59	A, B	18.6	A	12.79
A, B, D	22.3	A, C	17.78	B	5.81
A, C, D	21.48	A, D	16.49	C	4.99
B, C, D	14.5	A, E	15.49	D	3.70
				E	2.70

One can carry out a similar reasoning for data presented in Table 2b.

In this case, the computer program determined the following average marginal contributions of 5 importers: **(9.65; 5.15; 4.44; 3.34; 2.48)**. Since the sum of these is equal to 25.06 < 29.2 = $4\% \cdot (290+150+130+100+60)$, we increase proportionally all coordinates of the vector (9.65; 5.15; 4.44; 3.34; 2.48) in order to receive another imputation for which the sum of its coordinates will be equal to 29.2. As a result, we obtain the imputation (**11.24**; 6; 5.17; 3.89; **2.89**). This way, with the help of our computer program, we have arrived at the following result.

Proposition 7: The solution $(\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D, \hat{x}_E)$ of the cooperative game problem 1 with characteristic function given in Table 3b, determined according to the MS method, is the following imputation of discounts (**11.24**; 6; 5.17; 3.89; **2.89**). Dividing these discounts by the amounts of import for companies A, B, C, D and E, one gets a solution (**3.88**%; 4%; 3.98%; 3.89%; **4.82**%).

Comparison of the imputation (**11.24**; 6; 5.17; 3.89; **2.89**) with the Shapley imputation (10.47; 6; 5.24; 4.22; 3.28) shows that importer A became the key beneficiary of the MS method. The same conclusion holds if we compare these two solutions in percentage terms, that is, (**3.88**%; 4%; 3.98%; 3.89%; **4.82**%) with Shapley solution (**3.61**%; 4%; 4.03%; 4.22%; **5.47**%).

It is clear that in case of data shown in Table 2b, importer A did not expect discount as high as in the case of Table 2a because this time he was not eligible to get a 3% discount by itself (290 < 300); see Table 1.

Proposition 8: The solution $(\hat{x}_A, \hat{x}_B, \hat{x}_C, \hat{x}_D, \hat{x}_E) =$ (**11.24**; 6.00; 5.17; 3.89; **2.89**) to the cooperative game with characteristic function given in Table 2b, determined according to MS method, belongs to the core of the game.

The proof follows the same lines as the proof of Proposition 2 with Table 3a and Table 5a replaced by Table 3b and Table 5b, respectively.

Corollary 1: Propositions 1 and 3 show that the Shapley value concept, although successful in many applications, was not acceptable by importer A in the problem studied. However, a natural modification of this notion has led to a solution that – as we claim – is now acceptable to all players (importers) since it was derived based on the full information concerning the strengths of all players on the raw material market.

4 Contribution of Lloyd Shapley to game theory

In Roth's opinion (see Roth, 2005), who is a Nobel prize winner in Economics in 2012, in the years following the work of von Newmann and Morgenstern, Lloyd Shapley became the very personification of game theory due to the role he played in shaping the agenda for game-theoretic research.

The authors of this paper feel an obligation to honor L. Shapley, who left this world in March 2016, for his lasting contributions to cooperative game theory. The below description of Shapley's achievements is partially drawn on a comprehensive survey given in Roth (2005), supplemented by publications which appeared after 2005.

In 1953, it was Shapley who for the first time proposed to assign in a numerical way to each player in an n-person game the "value" of playing that game. The function he derived for this purpose became known as the Shapley value. In consecutive years, the same value function was rederived from apparently different assumptions by other people.

An interesting example of an n-person game where the Shapley value can be easily implemented is the United Nations Security Council. It consists of 15 members among whom five of them are permanent members having a veto, while 10 are rotating members. The voting rule means that a motion is passed if it receives 9 votes and 0 vetoes. This game can be modeled by assigning $v(S) = 1$, if S contains all 5 permanent members plus $\geq$ 4 rotating members, and $v(S) = 0$ otherwise. It can be shown (Roth, 2005, p.7) that the Shapley value of a rotating member is 0.00186, while the Shapley value of a permanent member is 0.196; clearly, those 15 numbers sum up to 1.

Analogous calculations, based on the existing in 1954 rules of the Security Council, are presented in Shapley and Shubik (1954). That article was the first in which the Shapley value was applied to simple games that are natural models for voting rules. By definition, a *simple game* is one represented by a characteristic function $v(\cdot)$ which takes only the values 1 and 0. In such games, a coalition S is said to be *winning* if $v(S) = 1$, and loosing if $v(S) = 0$. When the Shapley value is employed to simple games, it is called the Shapley-Shubik index.

In an article (Shapley 1953e), he began to study games where the actions of the players determine not only their payoffs but also the transition probabilities to a subsequent stage of the game. It was in fact an extension of the Markovian decision processes known from the operations research literature. He proved the existence of a value for such games as well as stationary optimal strategies. A quite large amount of publications followed this paper; see Gillette (1957) who extended Shapley's results to undiscounted games under ergodicity assumptions, and many other articles, for example, by Blackwell and Ferguson (1968), who solved the problem stated by Gillette. Almost 30 years later, Mertens (1981) and Newman proved the existence of a value for infinitely repeated stochastic games continuing the research initiated by L. Shapley.

A similar problem concerning the voting rules in European Union was tackled by Felsenthal and Machover (1977); see also their book published in 1998. Interesting results related to voting power and power index were obtained by Holler (1982, 1984, 1995, 1997). However, in some cooperative games, the players' choices are influenced or restricted by political institutions. For such games Steunenberg, Schmidtchen and Kobolt (1999) studied the concept of *strategic power index*.

Somewhat similar approaches to simple games were also offered by other authors; for example, Brams and Lake (1977) and Owen (1977) who modified the Shapley value in order to take into account the possibility that some players (due to personal or political affinities) are more likely to act together than others. See also the contributions of Johnston (1978), Deegal and Packel (1979), Holler and Packel (1983).

Polish authors have some contributions in this exciting area too; for example, Mercik (1997, 1999), Mercik and Kolodziejczyk (1986) and Sosnowska (1997). Her latest article published in 2016 was devoted to parliamentary elections in Poland in which she employed the Shapely value concept. Homenda (2009) discusses among others the difference between voting weight and voting power, as well as the meaning of winning and blocking coalitions, initiative and preventive power of a player. An interesting modification of the Shapley value was offered by Nowak and Radzik (1994) who called their solution "solidarity" value. Their notion was introduced axiomatically satisfying the efficiency, additivity and symmetry axioms, and additionally, an extra postulate saying that the players who contribute more than an average to a given coalition S may support the "weaker" players from S.

Some authors rejected the requirement assumed so far that the probability of forming a coalition is the same for all coalitions S. Their goal was to define a power index for each player and a solution to such games. See Aumann and Dreze (1975), Owen (1977), and Carreras and Owen (1988).

Assessing how much influence a voting system can give to voters became an important issue in evaluating the legislative reapportionment schemes when a USA court ruled that valid schemes must guarantee voters equal representation. In court decisions related to these issues, Banzhaf (1965, 1968) introduced a measure of voter influence called Banzhaf index which gained a measure of legal authority, particularly in New York State; see also Colleman (1971), Lucas 1983 and Shapley (1977).

Instead of looking at random orders of players joining already existing coalitions, as Shapley did, the Banzhaf index takes into account only such coalitions S that S – {i} is losing, while S is winning; the full mathematical treatment of the Banzhaf index can be found in (Dubey and Shapley, 1979). Despite the fact that the Shapley-Shubik and Banzhaf index are very similar, it may happen in particular instances (games) that they the rank voters differently; see also the book by Shubik (1982). Applications of Banzhaf index and Shapley value in the analysis of European Central Bank are presented in Sosnowska (2014).

In 1994 Aumann and Hart edited the 2nd issue of "Handbook of Game Theory …", which contains several articles on diversified areas of research where game theory is applicable. For example, Geanakoplos (1994, pp.1438-1495) devoted his article to common knowledge; Friedman, a well-known expert in differential games, published his article on (1994, pp.782-798); Dutta and Radner showed applicability of game theory to moral hazard problems (1994, pp.870-900). In the 3rd volume of the same handbook by Aumann and Hart (2002), the reader will find articles on many other topics. They include results obtained by Aumann and Heifetz (2002, pp.1665-1685) on incomplete information, and an article of Raghavan (2002, pp. 1687-1718) on non-zero-sum two-person games, where the concept of Nash equilibrium is employed.

The reader will find there an article of Merttens (2002, pp.1809-1827) on stochastic games, as well as results of Avenhaus, von Stengel and Zamir on inspection games (2002, pp.1947-1984), which – as they demonstrate – not only concern arms control and disarmament issues but also theoretical approaches to auditing/accounting in insurance, as well as environmental surveillance topics. Let us add now two more articles from Vol. 3 of this handbook, namely a paper on variations of Shapley's value by Monderer and Samet (2002, pp.2055-2075), and an article on analysis of legal rules by means of game-theoretic approaches (2002, pp.2229-2269).

In the article "Weighted Shapley values" (Roth, 2005, Chapter 6), the authors Ehud Kalai and Dov Samet report on such generalizations of Shapley's approach that include non-anonymous players. As a matter of fact, it was Shapley who in his Ph.D. dissertation introduced for the first time the non-anonymity by assigning different weights to the players. However, Kalai and Samet investigate the more general lexicographic weight systems.

Their "partnership consistency" axiom refers to the players who are only valuable to a coalition S if and only if they all are in S together. As a result of their studies, different players may be viewed as "blocks" of various sizes. A result of Monderer, Samet, and Shapely (1992) demonstrate that the set of weighted Shapley values always contains the core of the game.

A decision problem arising in the field of organization and evolution of labor market for medical interns and residents is successfully tacked by Roth (1984) in the framework of game theory.

Interesting results can also be found in Roth (2005, Chapter 8), where Uriel Rothblum studies 3 formulas for the Shapley value. It is worth to add at this place that the random-order representation employed by Shapley has no special (important) status, although it proved to be useful in facilitating computations in certain cases, such as for example the Security Council. Each of the 3 proposed by Rothblum formulas is calculated as an average taken over coalitions of the same size, and can be equally useful in applications for other cases of cooperative games.

In an article of Aumann and Myerson (Roth, 2005, Chapter 12), the authors propose another (novel) approach to the question of how cooperation between players may be organized. They no longer look solely at disjoint coalitions but investigate the bilateral links between players. In their approach, the players may choose whether and with whom to establish links. They extend the concept of the Shapely value to evaluate payoffs to the players from each set of links.

There is also a substantial amount of publications related to the Shapley value concerning the so-called *large games*. They arise in economic models of perfect information in which agents (players) are negligibly small compared to the size of the market. It is known that as the number of players increases, the core shrinks in the limit to contain just the competitive allocations. The literature on the Shapley value for large games includes, among others, the following papers by Shapley and his collaborators: Mann and Shapley (1960, 1962), Shapley (1961a, 1961b), Shapley and Shapiro (1978), and Milnor and Shapley (1978).

One of the first papers to apply the Shapely value to cost allocation decision problems belongs to Shubik (1962**)**. His goal was to design a reward system for managers. The risks they undertake must be consistent with the senior management attitude to the expected return and its variance of the studied company. In this paper, Shubik examines the control problems which arise if joint costs are assigned by various internal pricing conventions.

An article by Young in Aumann and Hart (1994, pp.1194-1230) concerns cost allocation problem which is presented as a kind of game in which costs (as well as benefits) are shared among different parts of an organization. The organization would like to have an efficient and equitable allocation mechanism that provides appropriate incentives to its all parts. Young demonstrates in what way the cooperative game theory provides tools for analysis of these issues. He is convincing the reader that cooperative game theory and cost allocation are closely intertwined in practical applications, and even that the Shapley value have long been used implicitly by some companies.

This subject became interesting to game theorists as well as to accountants; see Moriarity (1983), Zaremba L. and C. Zaremba (2010). Allocation costs for telephone calls were studied by Billera, Heath and Raanan (1978) as an application of Aumann and Shapley's research on nonatomic games in which the role of players is occupied by instants of telephone time.

In Roth's opinion, Shapley's contributions to game theory would have been significant even if he would only introduce the value of a game with some basic applications of it. But he also had other research achievements, which we will shortly list below. For example, the first definition of the core of a game belongs to both Shapley (1953c) and Gillies (1953a, b). Later research of Shapley concerning this subject has led to the notion of *balancedness*, which is related to the theory of linear programming; see Bondareva (1963) and Shapley (1967a).

In 1962, Gale and Shapley introduced a new type of games concerning the stability of marriage, called "marriage" games. A few years later, Shapley and Shubik (1969a) demonstrated that the so-called *transferable utility* games are "totally balanced", which means that the subgame for any subset of players is balanced, too. These games can be formulated as exchange economies with continuous and concave utility functions.

In 1971, Shapley introduced a new class of games called *convex games* and proved that such games have non-empty core. One year later, Shapely and Shubik introduced and studied so-called "assignment" games that appeared to be a subject of study for economists interested in labor markets; see, Roth and Sotomayor (1990).

Shapley explored the stable sets for several classes of games and demonstrated how to better understand the strategic possibilities facing coalitions of players; see, for example, a paper of Shapley (1953d) on "quota" games, and another paper by Shapley (1959) concerning the symmetric market games. In his papers on simple games with empty cores, which refer to modeling of political and committee decision making, Shapley (1962a, 1963, 1964a, 1967b) demonstrated how successfully he applies his earlier results on the stable sets.

It is worth to mention a recent extension of the Shapley value to the so-called procedural value for transferable utility games proposed by Malawski (2013), where the players can share their marginal contributions with their predecessors. A somewhat similar solutions concept, called "egalitarian" value was recently investigated by Casajus and Huettner (2013).

Finally, as we read from the introduction to a very recent book by Hooler and Owen (2013), the publications on power indices and coalition forming have multiplied. Especially popular are the applications of these notions to European Union. In the closing remarks, they express an opinion that there is an edge of "aesthetics" in the study of power indices which attracts even pure mathematicians.

One of the articles in this book (Mercik, 2013) proposes a voting model for a legislature composed of several disjoint and cohesive subgroups to introduce a new concept of power index. The cohesiveness of each subgroup is measured by a probability that its members vote the same way as their leader. The relative power of the cabinet is a function of the sizes and cohesiveness of the groups supporting that cabinet. Mercik's theory is illustrated by an analysis of the Polish government.

5 Bibliography

[1] Aumann, R., Dreze, J., 1975. Solutions of cooperative games with coalition structures. *International Journal of Game Theory*, (4), pp.180-192.

[2] Aumann, R., Hart, S., 1994. *Handbook of Game Theory with Economic Applications,* Vol. 2. North-Holland: Elsevier.

[3] Aumann, R., Hart, S., 2002. *Handbook of Game Theory with Economic Applications,* Vol. 3. North-Holland: Elsevier.

[4] Aumann, R., Heifetz, A., *Incomplete Information*. In: R. Aumann and S. Hart, eds., 2002. *Handbook of Game Theory with Economic Applications,* Vol. 3. North-Holland: Elsevier, pp.1665-1685.

[5] Aumann, R., Myerson, R., 2005. *Endogenous Formation of Links Between Players and of Coalitions: An Application of the Shapley Value.* In: Roth A. *Shapley Value – Essays in Honor of L. Shapley.* Cambridge: Cambridge Univ. Press, pp.175-194.

[6] Avenhaus, R., von Stengel, B., and Zamir, S., *Inspection Games*. In: R., Aumann and S., Hart, eds., 2002. *Handbook of Game Theory with Economic Applications,* Vol. 3. North-Holland: Elsevier, pp.1947-1984.

[7] Banzhaf, J.F., III, 1965. Weighted Voting Doesn't Work: A Mathematical Analysis. *Rutgers Law Review*, 19, pp.317-343.

[8] Banzhaf, J.F., III, 1968. One Man, 3312 Votes: A Mathematical Analysis of the Electoral College. *Villanova Law Review*, 13, pp.303-346.

[9] Billera, L., Heath, D., and Raanan, J., 1978. Internal Telephone Billing, Rates: A Novel Application of Non-Atomic Games Theory. *Operations Research*, 26, pp.956-965.

[10] Blackwell, D., and Ferguson, T., 1968. The Big Match. *Annals of Mathematical Statistics,* 39, pp.159-163.

[11] Bondareva, O., 1963. Some Applications of Linear Programming Methods to the Theory of Cooperative Games. *Problemy Kibernetiki*, 10, pp.119-139 (in Russian).

[12] Brams, S., Lake, M., 1977. Power and Satisfaction in a Representative Democracy. In: *Game Theory and Political Science Conference*. Hyannis MA., USA.

[13] Carreras F. Owen G. 1988. Evaluation in Catalonian Parliament 1980-1984. *Mathematical Social Science*, 15, pp.87-92.

[14] Casajus, A., Huettner, F., 2013. Null Players, Solidarity and the Egalitarian Shapely Values. *Journal of Mathematical Economics*, 49(1), pp.58-61.

[15] Colleman, J., 1971. Control of Collectivities and Power of a Collectivity to Act. In: *Social Choice*, B. Lieberman, ed., Gordon and Breach, pp.269-300.

[16] Deegal, J., Packel, E., 1979. A New Index of Power for Simple n-Person Games. *International Journal of Game Theory*, 7, pp.113-123.

[17] Dubey, P., Shapley, L., 1979. Mathematical Properties of the Banzhaf Power Index. *Mathematics of Operations Research*, 4, pp.99-131.

[18] Dutta, P., Radner R., *Moral Hazard*. In: R. Aumann and S. Hart, eds., 1994. *Handbook of Game Theory with Economic Applications,* Vol.2. North-Holland: Elsevier, pp.870-900.

[19] Felsenthal, D., Machover, M., 1977. The Weighted Rule in the EU's Council of Ministers, 1958-1995; Intentions and Outcomes. *Electoral Studies*, (16), pp.33-47.

[20] Friedman, A., 1994. *Differential Games*. In: R., Aumann and S., Hart, eds. *Handbook of Game Theory with Economic Applications,* Vol. 2. North-Holland: Elsevier, pp.782-798.

[21] Gale, D., Shapley, L., 1962. College Admissions and Stability of Marriage. *American Mathematical Monthly*, 69, pp.9-15.

[22] Geanakoplos, J., 1994. *Common Knowledge*. In: R., Aumann and S., Hart, eds. *Handbook of Game Theory with Economic Applications,* Vol. 2. North-Holland: Elsevier, pp.1438-1495.

[23] Gillette, D., 1957. Stochastic Games with Zero Stop Probabilities. In: *Contribution to the Theory of Games.,* M., Dresher, A., Tucker and P., Wolfe, eds., vol. III *Ann. Math. Studies*, 39. Princeton: Princeton University Press, pp.179-187.

[24] Gillies, D., 1959. Solutions to General non-zero-sum Games. In: *Contributions to the Theory of Games,* Vol. 4, *Ann. Math. Studies*, (40), eds. A. Tucker and D. Luce. Princeton: Princeton University Press, pp.47-85.

[25] Gillies, D., 1953a. Locations of Solutions. In: *Report of an Informal Conference on the theory of N-person games*, ed. H.W. Kuhn, Princeton University.

[26] Gillies, D., 1953b. Some Theorems on N-person Games, Ph. D. dissertation. Department of Mathematics, Princeton University.

[27] Holler, M., 1982. Forming Coalitions and Measuring Voting Power. *Political Studies,* 30(2), pp.266-231.

[28] Holler, M., Packel, E., 1983. Power, Luck and the Right Index. *Journal of Economics (Zeitschrift fur Nationalekonomie)*, 43(1), pp.21-29.

[29] Holler, M., 1984. A Public Good Power Index. In: *Coalitions and Collective Actions*, ed., M. Holler. Wurzburg and Wein: Physica-Verlag, pp.51-59.

[30] Holler, M., Owen, G., 2013. Power Indices and Coalition Formation. *Springer Science & Business Media* (386 pp.).

[31] Holler, M., Li, X., 1995. From Public Good Index to Public Value: An Axiomatic Approach and Generalization. *Control & Cybernetics*, 24(2), pp.257-270.

[32] Holler, M., 1997. Power, Monotocity and Expectations. *Control & Cybernetics*, 26(4), pp.605-608.

[33] Homenda, W., 2009. Decision Making in Voting Games: An Insight into Theory and Practice. In: *MDAI'09 Proceedings of the 6th International Conference on Modelling Decisions for Artificial Intelligence*. Awaji Island, Japan, Nov. 30 – Dec. 02, pp.60-71.

[34] Johnston, R., 1978. On the Measurement of Power: Some Reactions to Laver. *Environment and Planning A*, 10, pp.907-914.

[35] Lucas, W., 1983. Measuring Power in Weighted Voting Systems. In: *Political and Related Models*, S. Brams, W. Lucas, P. Straffin, eds., Berlin: Springer, pp.183-238.

[36] Malawski, M., Wieczorek, A., and Sosnowska, H., 2006. *Konkurencja i kooperacja - teoria gier w ekonomii i naukach społecznych* (*Competition and Cooperation – Theory of Games in Economics and Social Science*). Warszawa: Wydawnictwo Naukowe PWN.

[37] Malawski, M., 2013. Procedural Values for Cooperative Games. *International Journal of Game Theory,* 42(1), pp.305-324.

[38] Mann, I., Shapley, L., 1960. Values of Large Games IV: Evaluating the Electoral College by Monte Carlo Techniques. RM-2651, *Rand Corporation*, Santa Monica.

[39] Mann, I., Shapley, L., 1962. Values of Large Games VI: Evaluating the Electoral College Exactly. RM-3158, *Rand Corporation*, Santa Monica.

[40] Mercik, J., Kolodziejczyk, W., 1986. Taxonomy Approach to a Cabinet Formation. *Mathematical Social Science,* 12, pp.159-167.

[41] Mercik, J., 1997. Power and Expectations. *Control & Cybernetics*, 26(4), pp.617-622.

[42] Mercik, J., 1999. *Siła i oczekiwania; Decyzje grupowe* (*Power and Expectations; Group Deci-

sions). Warszawa-Wrocław: Wydawnictwo Naukowe PWN.

[43] Mercik, J., 2013. Index of Power for Cabinet. In: Holler and Owen, eds., *Power Indices and Coalition Formation*, pp.371-380. New York: Springer Science+ Business Media.

[44] Mertens, J-F., Neyman, A., 1981. Stochastic Games. *International Journal of game Theory*, 10, pp.53-66.

[45] Mertens, J-F., 2002. Stochastic Games. In: R. Aumann and S. Hart, eds., 2002. *Handbook of Game Theory with Economic Applications,* Vol.3. North-Holland: Elsevier, pp.1809-1827.

[46] Milnor, J., Shapley, L., 1978. Values of Large Games II: Oceanic Games. *Mathematics of Operations Research*, 3, pp. 290-307; initially in: *Rand Corporation, RM-2649*, Santa Monica,1961.

[47] Monderer, D., Samet, D., and Shapely, L., 1992. Weighted Values and the Core. *Journal of Political Economy*, 21, pp.27-39.

[48] Monderer, D., Samet, D., 2002. *Variations on the Shapley Value*. In: R. Aumann and S. Hart, eds., 2002. *Handbook of Game Theory with Economic Applications,* Vol. 3. North-Holland: Elsevier, pp.2055-2075.

[49] Moriariti, S. ed., 1983. *Joint Cost Allocation*. Tulsa: Univ. of Oklahoma Press.

[50] Nowak, A., Radzik, T., 1994. A Solidarity Value for n-Person Transferable Utility Games. *International Journal of Game Theory*, 23(1), pp.43-48.

[51] Ott, U., 2006. *International Joint Ventures: An Interplay of Cooperative and Noncooperative Games under Incomplete Information*. Houndmills Basingstoke: Palgrave Macmillan.

[52] Owen, G., 1977. *Values of Games with a Priori Unions*. In: R. Henn and O. Moeschlin. *Mathematical Economics and Game Theory*. Berlin: Springer Verlag, pp.76-88.

[53] Peleq, B., 2005. *Introduction to the Theory of Cooperative Games*, 2 ed. Berlin: Springer.

[54] Raghavan, T., *Non-sero-sum Two-Person Games*. In: R. Aumann and S. Hart, eds., 2002. *Handbook of Game Theory with Economic Applications,* Vol. 3. North-Holland: Elsevier, pp.1687-1718.

[55] Roth, A., 1984. The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory, *Journal of Political Economy*, 92, pp.991-1016.

[56] Roth, A., Sotomayor, M., 1990. Two-Sided Matching: A Study in Game-Theoretic Modelling and Analysis. *Econometric Society Monograph*. Cambridge: Cambridge University Press.

[57] Roth, A., 2005. *The Shapley Value: Essays in Honor of Lloyd S. Shapley*. Cambridge: Cambridge Univ. Press.

[58] Shapley, L., 1953a. *A Value for n-Person Games*. In: eds., H.W. Kuhn and A.W. Tucker. *Contributions to the Theory of Games*, Vol. 2. Ann. Math. Studies 28. Princeton: Princeton Univ. Press, pp.307-317.

[59] Shapley, L., 1953b. Adaptive and Non-Adaptive Set Functions. Ph.D. thesis. Department of Mathematics, Princeton Univ.

[60] Shapley, L., 1953c. *Open Questions*. In: ed., H.W. Kuhn. *Report of an Informal Conference on the Theory of n-Person Games.* Princeton University.

[61] Shapley, L., 1953d. *Quota Solutions of n-Person Games*. In: eds., H.W. Kuhn and A.W. Tucker. Contributions to the theory of Games, Vol. 2, *Ann. Math. Studies* 28, Princeton Univ. Press, pp.343-359.

[62] Shapley, L., 1953. Stochastic Games. *Proceedings of the National Academy of Sciences*, 39, pp.1095-1100.

[63] Shapley, L., Shubik M., 1954. A Method for Evaluating a Distribution of Power in a Committee System. *American Political Science Review*, 48, pp.787-792.

[64] Shapley, L., 1959. The Solution of a Symmetric Market Game. *Annals of Mathematics Studies*, 40, pp.145-162.

[65] Shapley, L., 1961a. Values of Large Games III: A Cooperation with 2 Large Stockholders. RM-26-50 *Rand Corporation*, Santa Monica.

[66] Shapley, L., 1961b. Values of Large Games IV: An 18-Person Market Game. RM-2860 *Rand Corporation*, Santa Monica.

[67] Shapley, L., 1962a. Compound Simple Games, I: Solutions of Sums and Products. RM -3192-PR, *The Rand Corporation*, Santa Monica.

[68] Shapley, L., 1963). Compound Simple Games, II: Some General Composition Theorems. RM -3643-PR, *The Rand Corporation*, Santa Monica.

[69] Shapley, L., 1964a. Solutions of Compound Simple Games, I: Solutions of Sums and Products. *Annals of Mathematics Studies*, 52, pp.267-305.

[70] Shapley, L. 1964b. Values of Large Games VII: A General Exchange Economy with Money. RM-4248, *Rand Corporation*, Santa Monica.

[71] Shapley, L., 1967a. On Balanced Sets and Cores. *Naval Research Logistics Quarterly*, 14, pp.453-460.

[72] Shapley, L., 1967b. On Committees. In: eds., F. Zwicky and A. Wilson. *New Methods of Thought and Procedure*. New York: Springer, pp.246-270.

[73] Shapley, L., Shubik, M., 1969. On Market Games. *Journal of Economic Theory*, 1, pp.9-25.

[74] Shapley, L., 1971. Cores of Convex Games. *International Journal of Game Theory*, 1, pp.11-26.

[75] Shapley, L., Shubik, M., 1972. The assignment Game I: The Core. *International Journal of Game Theory*, 1, pp.111-130.

[76] Shapley, L., 1977. A Comparison of Power Indices and a Nonsymmetric Generalization. Paper P-5872. The Rand Corporation, Santa Monica.

[77] Shapley, L., Shapiro, N., 1978. Values of Large Games I: A Limit Theorem. *Mathematics of Operations Research*, 3, pp.1-9; initially in *Rand Corporation*, *RM-2648*, Santa Monica (1960).

[78] Shubik, M., 1962. Incentives, Decentralized Control, the Assignment of Joint Costs and Internal Pricing. *Management Science*, 8(3), pp.325-343.

[79] Shubik, M., 1982. *Games Theory in the Social Science*. Cambridge: MIT Press.

[80] Sosnowska, H., 1997. Coalitions and Rationality, *Control & Cybernetics* 26(4), pp.623-624.

[81] Sosnowska, H., 2014. Banzhaf Value for Games Analyzing Voting with Rotation. *Operations Research & Decisions*, 24(4), pp.75-88.

[82] Sosnowska, H., 2016. Approval Voting as a Method of Prediction in Political Votings. Case of Polish Elections. *LNCS Transactions on Computational Collective Intelligence*, 23, pp.17-28.

[83] Steunenberg, B., Schmidtchen, D., and Kobolt, C., 1999. Strategic Power in the European Union: Evaluating the Distribution of Power in Policy Games. *Journal of Theoretical Politics*, 11(3), pp.339-366.

[84] Thomas L., 1986. *Games, Theory & Applications*. Chichester: Ellis Horwood.

[85] Weirich, P., 2012. *Collective Rationality: Equilibrium in Cooperative Games*. Oxford: Oxford Univ. Press.

[86] Young, H., 1994. *Cost Allocation*. In: Aumann and Hart, eds., *Handbook of Game Theory with Economic Applications,* Vol.2. North-Holland: Elsevier, pp.1194-1230.

[87] Zaremba, L., 1979. On the Existence of Value in the Varaiya-Lin Sense in Differential Games of Pursuit. *Journal of Optimization Theory & Applications*, 29, pp.135-145.

[88] Zaremba, L., 1980. On the Existence of Value in Pursuit-Evasion Games with Restricted Coordinates, *Journal of Optimization Theory & Applications*, Vol. 30, pp.451-470.

[89] Zaremba, L., 1982. Existence of Value in Differential Games with Fixed Time Duration. *Journal of Optimization Theory & Applications*, Vol. 38, pp.581-598.

[90] Zaremba, L., 1984a. Existence of Value in Games Governed by Generalized Differential Equations. *Systems & Control Letters*, Vol. 4, pp.237-341

[91] Zaremba, L., 1984b. Existence of Value in Generalized Pursuit-Evasion Games. *SIAM Journal on Control & Optimization* Vol. 22, pp.894-901.

[92] Zaremba, L., 1986. Existence Theorems for Games of Survival. *Journal of Optimization Theory & Applications*, Vol. 48, pp.431-446.

[93] Zaremba, L., 1989. Optimality Principles of Dynamic Programming in Differential Games. *Journal of Mathematical Analysis and Applications*, Vol. 138, pp.43-51.

[94] Zaremba, L., Zaremba, C., 2010. Rozwiązanie uogólnionego problemu optymalnej alokacji zasobów (Solution to Generalized Optimal Resource Allocation Problem). *Zarządzanie Zmianami – Biuletyn Naukowy i Informacyjny Wyższej Szkoły Zarządzanie – the Polish Open University*, 6, pp.1-19.