TWO NEW DECOMPOSITION ALGORITHMS FOR TRAINING BOUND-CONSTRAINED SUPPORT VECTOR MACHINES*

Lingfeng Niu *, Ruizhi Zhou *, Xi Zhao *, Yong Shi *

Abstract. Bound-constrained Support Vector Machine(SVM) is one of the state-of-art model for binary classification. The decomposition method is currently one of the major methods for training SVMs, especially when the nonlinear kernel is used. In this paper, we proposed two new decomposition algorithms for training bound-constrained SVMs. Projected gradient algorithm and interior point method are combined together to solve the quadratic subproblem efficiently. The main difference between the two algorithms is the way of choosing working set. The first one only uses first order derivative information of the model for simplicity. The second one incorporate part of second order information into the process of working set selection, besides the gradient. Both algorithms are proved to be global convergent in theory. New algorithms is compared with the famous package BSVM. Numerical experiments on several public data sets validate the efficiency of the proposed methods.

Keywords: decomposition algorithm, support vector machine, quadratic programming, global convergence

1 Introduction

Let $S = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_l, y_l)\}$ be a set of training samples belonging to m different classes, where $\mathbf{x}_i \in X \subseteq \mathbb{R}^n$ and $y_i \in Y$ are the input data and corresponding label for sample i , respectively. A classification problem can be interpreted as finding a classifier, which from mathematical point of view is a function that maps an instance

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*Research Center on Fictitious Economy & Data Science, University of Chinese Academy of Sciences, Key Laboratory of Big Data Mining and Knowledge Management, Chinese Academy of Sciences, Beijing, China

$\mathbf{x}$ into an element y of Y . When there are only two labels in the label set Y , the problem is called binary classification. When the possible labels are more than two, the problem is called multi-class classification. We will concentrate on the binary classification problem in this paper.

Support Vector Machine (SVM) [2, 4, 20, 21] is one of the state-of-the-art classifiers nowadays. Subsequently, different improved versions have been proposed [31, 30, 29]. The primal model of the standard SVM is formularized as the following quadratic programming(QP)

$$\min_{\mathbf{w}, b, \xi} \quad \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^l \xi_i \quad (1a)$$

$$s.t. \quad y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + b) \geq 1 - \xi_i, \quad \forall i = 1, \dots, l, \quad (1b)$$

$$\xi_i \geq 0 \quad \forall i = 1, \dots, l, \quad (1c)$$

where, C is a pre-given model parameter, $\phi(\cdot)$ is the mapping from input space to the feature space, which may be infinite dimension. The dual form of (1) can be written as follow, which is always a finite QP:

$$\min_{\alpha} \quad f(\alpha) = \frac{1}{2} \alpha^T \hat{Q} \alpha - \mathbf{1}^T \alpha \quad (2a)$$

$$s.t. \quad \mathbf{y}^T \alpha = 0, \quad (2b)$$

$$\mathbf{0} \leq \alpha \leq \mathbf{C}, \quad (2c)$$

where, $\mathbf{0}$ and $\mathbf{1}$ represent the vector of all zeros and ones, respectively, $\mathbf{C} = C\mathbf{1}$, $\hat{Q}$ is an l by l matrix, $\hat{Q}_{i,j} \equiv y_i y_j K(\mathbf{x}_i, \mathbf{x}_j)$, and $K(\mathbf{x}_i, \mathbf{x}_j) \equiv \phi(\mathbf{x}_i)^T \phi(\mathbf{x}_j)$ is the kernel function. When $K(\cdot, \cdot)$ satisfies Mercer condition [16], matrix $\hat{Q}$ is positive semidefinite.

In this paper, we consider a simple modification model of the standard SVM (1) as follow,

$$\min_{\mathbf{w}, b, \xi} \quad \frac{1}{2} (\|\mathbf{w}\|^2 + b^2) + C \sum_{i=1}^l \xi_i \quad (3a)$$

$$s.t. \quad y_i(\mathbf{w}^T \phi(\mathbf{x}_i) + b) \geq 1 - \xi_i, \quad \forall i = 1, \dots, l, \quad (3b)$$

$$\xi_i \geq 0 \quad \forall i = 1, \dots, l, \quad (3c)$$

The dual form of the above problem is the following QP, which contains bound constraints only:

$$\min_{\alpha} \quad f(\alpha) = \frac{1}{2} \alpha^T Q \alpha - \mathbf{1}^T \alpha \quad (4a)$$

$$s.t. \quad \mathbf{0} \leq \alpha \leq \mathbf{C} \quad (4b)$$

where, $Q_{i,j} \equiv y_i y_j (K(\mathbf{x}_i, \mathbf{x}_j) + 1)$. Suppose the optimal solution of problem (4) is α^* . Then the classifier can be written as

$$H(\mathbf{x}) = \text{sign} \left\{ \sum_{i=1}^l \alpha_i^* y_i (K(\mathbf{x}, \mathbf{x}_i) + 1) \right\}.$$

Similar to the standard SVM, if $\alpha_i^* \neq 0$, the corresponding sample i is called a Support Vector(SV). Furthermore, if $\alpha_i^* = C$, it is called a Bounded Support Vector(BSV). To show the difference with the standard SVM, (3) and (4) are usually called the bound-constrained SVMs.

Bound-constrained SVMs are once proposed independently by Fireß, Cristianini and Campbell in [19], Mangasarian and Musicant in [14] and [15]. Mangasarian and Musicant [14] proves that for the linear kernel case, there exist some kind of equivalence between bound-constrained SVMs and the standard SVMs. The result can be concluded in the following proposition.

Proposition 1. [14] *For the bound-constrained SVMs with linear kernel function $k(\mathbf{x}, \mathbf{y})$, Suppose $(\mathbf{w}, b, \xi)$ is an arbitrary solution of problem (3). If any solution $\hat{\mathbf{v}}$ of the following system:*

$$A^T \mathbf{v} = 0, \mathbf{1}^T \mathbf{v} = b, \mathbf{v} \geq 0,$$

satisfies

$$\mathbf{1}^T \hat{\mathbf{v}} (\mathbf{1}^T \xi - 1) \leq b^2.$$

There must exists a sufficient large C , such that $(\mathbf{w}, b, \xi)$ is also a solution of the standard primal SVM model (1), where $A^T = (x_1, \dots, x_l) \in \mathbb{R}^{n \times l}$, i.e. each row of A is the feature vector of a sample.

Because of the simple formulation and good classification performance of bound-constrained SVM, it draws a lot of attention in the recent years. For the linear kernel case, Mangasarian and Musicant proposed to solve the model (4) by the over-relaxation method in [14]. They proved the global convergence and the linear convergent rate of the algorithm at the same time. Hsieh et al gave a novel dual coordinate descent method and proved the algorithm reaches an ϵ -accurate solution in $O(\log(1/\epsilon))$ iterations in [8]. Several researchers also explore how to train the primal form of (4) and the extended models fast. The existing algorithms can be broadly categorized into two categories: the cutting-plane methods[11, 5, 12, 13, 25], and subgradient methods[3, 17]. For example, in [17], Shalev-Shwartz et al described and analyzed a simple and effective stochastic sub-gradient descent algorithm and prove that the number of iterations required to obtain a solution of accuracy ϵ is $O(1/\epsilon)$. Generally speaking, without counting the loading time, these recent advances on linear classification have shown that training one million instances takes only a few seconds[22].

For the general nonlinear kernel case, the scale of the problems grows quadratically with the number of samples l due to the appearance of kernel matrix. Therefore, for the middle and large scale problem, decomposition techniques are needed to handle the problems arising from the limitation of the memory. Hsu and Lin analyzed in thorough how to design a decomposition algorithm for problem (4) in [9], which mainly discussed how to design an effective working set selection rule based on the existing successful working set selection rule for standard SVMs. They started from applying the working set selection strategy of SVM^{light} [10] to model (4) directly. Step by step, a series of heuristic techniques were added to improve the performance of the selection rule. Different from the designation idea in [9], we will discuss how

to derive effective working set selection rule from the optimization algorithm design point of view. Although the derivation is much more concise than [9], the numerical experiments in section 4 show that the computation efficiency of our new methods is at least as good as Hsu and Lin's method.

The structure of the paper is as follows. In Section 2, after introducing the general decomposition algorithm framework for problem (4), we propose our new working set selection rules. In section 3, we prove that the decomposition algorithms based on our new working set selection strategies are globally convergent. In section 4, we concentrate on the efficient implementation. Numerical experiments on several public data sets are listed. A brief conclusion and some possible future works are given in the last section.

2 The New Decomposition Algorithms

2.1 The decomposition algorithm framework

Because Q_{ij} is in general not zero, Q becomes a fully dense matrix. Due to the density of Q , a prohibitive amount of memory is required to store the matrix. Thus traditional optimization algorithms, which needs the whole Hessian matrix of objective function, cannot be directly used. To conquer this difficulty, several researchers (for example, Osuna et al [24], Platt[26], Joachims[27], Saunders et al[28]) have proposed decomposition methods. The key idea of decomposition is to update a small part of variables at each iteration, and to solve a sequence of constant-size problems. Then, the solution of a large-scale dense problem can be found by solving a number of small problems.

Now we denote the gradient of objective function (4a) as:

$$F(\alpha) = \left(\frac{\partial f(\alpha)}{\partial \alpha_i} \right) = Q\alpha - \mathbf{1}, \quad (5)$$

for any $i \in \{1, \dots, l\}$, the i -th element of $F(\alpha)$ is:

$$F_i(\alpha) = \frac{\partial f(\alpha)}{\partial \alpha_i} = \sum_{j=1}^l Q_{ij}\alpha_j - 1. \quad (6)$$

The index set of variables optimized at a current iteration is called the working set and denoted as $\mathcal{B}$. Let $\mathcal{N} \equiv \{1, \dots, l\}/\mathcal{B}$, and superscript record the iteration number. Using the notation above, the general decomposition method for problem (4) is described in Algorithm 1. Notice that Algorithm 1 is just a framework, to make it be implemented in practice, the specific working set selection rule and QP subproblem solver (8) should be given. In the rest part of this section, we will discuss how to select the working set effectively from the optimization design point of view.

Denote $Qsub(\boldsymbol{\alpha}^{(k)}, \mathcal{B})$ as the optimal objective function value of QP

$$\min_{\mathbf{d}} \quad f(\boldsymbol{\alpha}^{(k)} + \mathbf{d}) - f^{(k)} \quad (7a)$$

$$s.t. \quad -\alpha_i^{(k)} \leq d_i \leq C - \alpha_i^{(k)} \quad \forall i \in \mathcal{B}; \quad (7b)$$

$$d_i = 0, \quad \forall i \notin \mathcal{B}. \quad (7c)$$

If only one iteration is considered, the best working set is $\text{argmin}_{\mathcal{B}: |\mathcal{B}|=n_B} Qsub(\boldsymbol{\alpha}^{(k)}, \mathcal{B})$. However, this choice requires solving $\binom{l}{n_B}$ QPs (7), which is too expensive to manipulate in practice. To decrease the computational cost, we will present two new working set selection methods in the next two subsections.

Algorithm 1 Decomposition Algorithm Framework for Bound-Constrained SVMs

Step 0. Initialization. Given the upper bound for the size of the workings set $n_B \geq 1$, the initial working set $\mathcal{B}^{(0)}$ and the initial point $\boldsymbol{\alpha}^{(0)} = \mathbf{0} \in \mathbb{R}^l$. Set $k = 0$.

Step 1. Subproblem Solution Construct and solve the quadratic subproblem

$$\min_{\{\alpha_i\}_{i \in \mathcal{B}^{(k)}}} \quad \frac{1}{2} \sum_{i,j \in \mathcal{B}^{(k)}} Q_{ij} \alpha_i \alpha_j + \sum_{i \in \mathcal{B}^{(k)}} (F_i^{(k)} - \sum_{j \in \mathcal{B}^{(k)}} Q_{ij} \alpha_j^{(k)}) \alpha_i \quad (8a)$$

$$s.t. \quad 0 \leq \alpha_i \leq C, \quad \forall i \in \mathcal{B}^{(k)}. \quad (8b)$$

Denote the optimal solution as $\{\bar{\alpha}_i\}_{i \in \mathcal{B}^{(k)}}$. Update the iteration point as

$$\alpha_i^{(k+1)} = \begin{cases} \bar{\alpha}_i & \text{if } i \in \mathcal{B}^{(k)}, \\ \alpha_i^{(k)} & \text{if } i \notin \mathcal{B}^{(k)}. \end{cases}$$

Step 2. Gradient Update. Update gradient based on formula (5). Set $k := k + 1$.

Step 3. Working Set Selection. Test whether the iterates should be terminated according to some stopping criteria. If not stop, select at most n_B indices to form a new working set $\mathcal{B}^{(k)}$, and go back to Step 1. Otherwise, output $\alpha^{(k)}$ as the optimal solution and stop.

2.2 Using first order information for working set selection

A straightforward way of simplifying problem (7) is dropping the second order information in the objective function, and confine the incremental variables between 1 and -1, which results in the following Linear Programming(LP):

$$\min_{\{d_i\}_{i \in \mathcal{B}}} \quad \sum_{i \in \mathcal{B}} F_i^{(k)} d_i \quad (9a)$$

$$-\alpha_i^{(k)} \leq d_i \leq C - \alpha_i^{(k)}, \quad \forall i \in \mathcal{B}; \quad (9b)$$

$$-1 \leq d_i \leq 1, \quad \forall i \in \mathcal{B}. \quad (9c)$$

Denote the optimal objective function value of problem (12) as $Lsub(\boldsymbol{\alpha}^{(k)}, \mathcal{B})$. Then, $\operatorname{argmin}_{\mathcal{B}: |\mathcal{B}|=n_B} Lsub(\boldsymbol{\alpha}^{(k)}, \mathcal{B})$ should be a good choice as the working set $\mathcal{B}^{(k)}$. Dividing the constraints (12b) and (12c) by samples, problem (12) can be solved as $|\mathcal{B}|$ independent LPs. In details, for any $i \in \mathcal{B}$,

$$\min_{d_i} d_i F_i^{(k)} \quad (10a)$$

$$s.t. \quad \max\{-1, -\alpha_i^{(k)}\} \leq d_i \leq \min\{1, C - \alpha_i^{(k)}\}. \quad (10b)$$

which solution can be represented as

$$\hat{d}_i = \begin{cases} \max\{-\alpha_i^{(k)}, -1\}, & \text{if } F_i^{(k)} \geq 0, \\ \min\{C - \alpha_i^{(k)}, 1\}, & \text{if } F_i^{(k)} < 0, \end{cases}$$

The above formula can be rewritten as a more compact form:

$$\hat{d}_i = -\operatorname{sign}(F_i^{(k)}) \min\{\frac{1}{2}C(1 - \operatorname{sign}(F_i^{(k)})) - \alpha_i^{(k)}, 1\}.$$

Furthermore, the corresponding optimal objective function value of LP (10) can be represented as

$$-|F_i^{(k)}| \min\{\frac{1}{2}C(1 - \operatorname{sign}(F_i^{(k)})) - \alpha_i^{(k)}, 1\}.$$

Based on these discussion, for $\forall i = 1, \dots, l$, define

$$\ell_i(\boldsymbol{\alpha}) = -|F_i^{(k)}| \min\{\frac{1}{2}C(1 - \operatorname{sign}(F_i^{(k)})) - \alpha_i^{(k)}, 1\}, \quad (11)$$

namely,

$$\ell_i(\boldsymbol{\alpha}) = \begin{cases} \max\{-\alpha_i^{(k)}, -1\}F_i(\boldsymbol{\alpha}), & \text{if } F_i^{(k)} \geq 0, \\ \min\{C - \alpha_i^{(k)}, 1\}F_i(\boldsymbol{\alpha}), & \text{if } F_i^{(k)} < 0, \end{cases}$$

Then,

$$Lsub(\boldsymbol{\alpha}^{(k)}, \mathcal{B}) = \sum_{i \in \mathcal{B}} \ell_i^{(k)}.$$

Therefore, instead of solving $\binom{l}{n_B}$ LPs (10), we can compute $\boldsymbol{\ell}^{(k)}$, and select the indices corresponding to the n_B smallest elements of $\ell_i^{(k)}$ directly to get the value of

$$\min_{\mathcal{B}: |\mathcal{B}|=n_B} Lsub(\boldsymbol{\alpha}^{(k)}, \mathcal{B}).$$

Especially, when $\boldsymbol{\ell}^{(k)} = \mathbf{0}$, no new working set can be selected, and the iterates can be terminated naturally. To summarize, we describe this simple working set selection rule in Algorithm 2.

Algorithm 2 The First-Order Working Set Selection Rule

Compute $\ell^{(k)}$ by formula (11).

if $\ell^{(k)} \neq 0$ **then**

Sort the indices set $\{1, \dots, l\}$ by the values of $\ell^{(k)}$ in increasing order. Choose the first n_B indices to form the working set $\mathcal{B}^{(k)}$.

else

Output $\alpha^{(k)}$ as the optimal solution and terminate the iterates.

end if

In practice, we find that some indices enter and leave the working set lots of times when applying working set selection rule Algorithm 2 directly to Algorithm 1. This causes the decomposition algorithm converging very slow. In order to avoid the zigzagging phenomenon, we keep part of indices from previous working sets at each iteration. To be more precise, at most n_N new indices are allowed to enter into the working set, where $1 \leq n_N \leq n_B$. Other indices are taken from the working set of last iteration. One thing we want to stress is that different techniques of inheriting indices has been used by several state-of-the-art solvers [10, 9, 23]. In this work, we use the inheriting strategy used in [23] to revise our working set selection rules we proposed before. Details are given in Algorithm 3.

Algorithm 3 The Practical First-Order Working Set Selection Rule

Compute $\ell^{(k)}$ by (11).

if $\ell^{(k)} \neq 0$ **then**

i) Set $\hat{\mathcal{B}} = \mathcal{B}^{(k)}$ and $\mathcal{B}^{(k+1)} = \emptyset$.

ii) Sort the indices set $\{1, \dots, l\}$ by the values of $\ell^{(k)}$ in increasing order; Add the first n_N indices to form the working set $\mathcal{B}^{(k+1)}$; Set $\hat{\mathcal{B}} := \hat{\mathcal{B}} \setminus \mathcal{B}^{(k+1)}$.

iii) Fill $j \in \{i | i \in \hat{\mathcal{B}}, 0 < \alpha_i^{(k)} < C\}$, which has the lowest number of consecutive iterations in working set, to $\mathcal{B}^{(k+1)}$ and remove j from $\hat{\mathcal{B}}$ until $|\mathcal{B}^{(k+1)}| = n_B$ or there is no such element.

iv) Fill $j \in \{i | i \in \hat{\mathcal{B}}, \alpha_i^{(k)} = 0\}$, which has the lowest number of consecutive iterations in working set, to $\mathcal{B}^{(k+1)}$ and remove it from $\hat{\mathcal{B}}$ until $|\mathcal{B}^{(k+1)}| = n_B$ or there is no such element.

v) Fill $j \in \{i | i \in \hat{\mathcal{B}}, \alpha_i^{(k)} = C\}$, which has the lowest number of consecutive iterations in working set to $\mathcal{B}^{(k+1)}$ and remove it from $\hat{\mathcal{B}}$ until $|\mathcal{B}^{(k+1)}| = n_B$ or there is no such element.

else

Output $\alpha^{(k)}$ as the optimal solution and terminate the iterations.

end if

2.3 Using second order information for working set selection

In the algorithm described in the last subsection, only first-order information of the objective function (4a) is used for simplicity. However, this does not mean that there is no simple and feasible method which uses second order information for choosing working set. In this subsection, we will consider how to include second order information into the process of working set selection.

Besides keeping both the first order information, we incorporate the diagonal second order information, and get the following revised QP:

$$\min_{\{d_i\}_{i \in \mathcal{B}}} \sum_{i \in \mathcal{B}} (F_i^{(k)} d_i + \frac{1}{2} Q_{ii} d_i^2) \quad (12a)$$

$$-\alpha_i^{(k)} \leq d_i \leq C - \alpha_i^{(k)}, \quad \forall i \in \mathcal{B}; \quad (12b)$$

$$-1 \leq d_i \leq 1, \quad \forall i \in \mathcal{B}. \quad (12c)$$

Denote the optimal objective function value of problem (12) as $DQsub(\alpha^{(k)}, \mathcal{B})$. Since more information of the objective model is used, $\arg\min_{\mathcal{B}: |\mathcal{B}|=n_B} DQsub(\alpha^{(k)}, \mathcal{B})$ should be a better choice of $\mathcal{B}^{(k)}$ than the set selection rule given in the last subsection.

Similar to the discussion of subsection 2.2, dividing the constraints (12b) and (12c) by samples, problem (12) can be solved as $|\mathcal{B}|$ independent QPs. In details, for any $i \in \mathcal{B}$,

$$\min_{d_i} d_i F_i^{(k)} + \frac{1}{2} Q_{ii} d_i^2 \quad (13a)$$

$$s.t. \quad \max\{-1, -\alpha_i^{(k)}\} \leq d_i \leq \min\{1, C - \alpha_i^{(k)}\}. \quad (13b)$$

which solution can be represented as

$$\hat{d}_i = \begin{cases} \max\{-\frac{F_i^{(k)}}{Q_{ii}}, \max\{-\alpha_i^{(k)}, -1\}\}, & \text{if } F_i^{(k)} \geq 0, \\ \min\{-\frac{F_i^{(k)}}{Q_{ii}}, \min\{C - \alpha_i^{(k)}, 1\}\}, & \text{if } F_i^{(k)} < 0, \end{cases}$$

The above formula can be rewritten as a more compact form:

$$\hat{d}_i = \text{mid}\{-\frac{F_i^{(k)}}{Q_{ii}}, \max\{-\alpha_i^{(k)}, -1\}, \min\{C - \alpha_i^{(k)}, 1\}\}.$$

Furthermore, the corresponding optimal objective function value of problem (13) can be represented as $F_i^{(k)} \hat{d}_i + \frac{1}{2} Q_{ii} \hat{d}_i^2$. Based on these discussion, for $\forall i = 1, \dots, l$, define

$$\check{d}_i(\alpha) = \text{mid}\{-\frac{F_i(\alpha)}{Q_{ii}}, \max\{-\alpha_i, -1\}, \min\{C - \alpha_i, 1\}\},$$

and

$$q_i(\alpha) = F(\alpha)_i \check{d}_i(\alpha) + \frac{1}{2} Q_{ii} \check{d}_i(\alpha)^2. \quad (14)$$

Then, $DQsub(\alpha^{(k)}, \mathcal{B}) = \sum_{i \in \mathcal{B}} q_i^{(k)}$. Therefore, instead of solving $\binom{l}{n_B}$ QPs (13), we can compute $\mathbf{q}^{(k)}$, and select the indices corresponding to the n_B smallest elements

of $q_i^{(k)}$ to form $\arg \min_{\mathcal{B}: |\mathcal{B}|=n_B} DQsub(\alpha^{(k)}, \mathcal{B})$ directly. Especially, when $\mathbf{q}^{(k)} = \mathbf{0}$, no new working set can be selected, and the iterates terminate naturally. Similar practical working set selection strategy can also be used here to avoid the zigzagging phenomenon. To summarize, we describe the new working set selection rule in Algorithm 4.

3 Global Convergence Analysis

In this section, we will prove that the decomposition algorithms based on our new working set selection rules (Algorithm 3 or Algorithm 4) are globally convergent.

Algorithm 4 The Practical Second-Order Working Set Selection Rule

Compute $\mathbf{q}^{(k)}$ by (14).

if $\mathbf{q}^{(k)} \neq \mathbf{0}$ **then**

i) Set $\hat{\mathcal{B}} = \mathcal{B}^{(k)}$ and $\mathcal{B}^{(k+1)} = \emptyset$.

ii) Sort the indices set $\{1, \dots, l\}$ by the values of $\mathbf{q}^{(k)}$ in increasing order; Add the first n_N indices to form the working set $\mathcal{B}^{(k+1)}$; Set $\hat{\mathcal{B}} := \hat{\mathcal{B}} \setminus \mathcal{B}^{(k+1)}$.

iii) Fill $j \in \{i | i \in \hat{\mathcal{B}}, 0 < \alpha_i^{(k)} < C\}$, which has the lowest number of consecutive iterations in working set, to $\mathcal{B}^{(k+1)}$ and remove j from $\hat{\mathcal{B}}$ until $|\mathcal{B}^{(k+1)}| = n_B$ or there is no such element.

iv) Fill $j \in \{i | i \in \hat{\mathcal{B}}, \alpha_i^{(k)} = 0\}$, which has the lowest number of consecutive iterations in working set, to $\mathcal{B}^{(k+1)}$ and remove it from $\hat{\mathcal{B}}$ until $|\mathcal{B}^{(k+1)}| = n_B$ or there is no such element.

v) Fill $j \in \{i | i \in \hat{\mathcal{B}}, \alpha_i^{(k)} = C\}$, which has the lowest number of consecutive iterations in working set to $\mathcal{B}^{(k+1)}$ and remove it from $\hat{\mathcal{B}}$ until $|\mathcal{B}^{(k+1)}| = n_B$ or there is no such element.

else

Output $\alpha^{(k)}$ as the optimal solution and terminate the iterations.

end if

Lemma 1. Suppose α is a feasible point of problem (4). α is the KKT point of problem (4) if and only if $\ell(\alpha) = \mathbf{0}$.

Proof. Because α is the KKT point of problem (4), from Kuhn-Tucker theorem [18], we know that there exists Lagrange multipliers $\mathbf{u}$ and $\mathbf{v}$ which satisfies KKT condition:

$$F_i(\boldsymbol{\alpha}) - v_i + u_i = 0, \quad \forall i = 1, \dots, l; \quad (15a)$$

$$v_i \alpha_i = 0, \quad \forall i = 1, \dots, l; \quad (15b)$$

$$u_i(C - \alpha_i) = 0, \quad \forall i = 1, \dots, l; \quad (15c)$$

$$v_i, u_i \geq 0, \quad \forall i = 1, \dots, l. \quad (15d)$$

Because $\boldsymbol{\alpha}$ is a feasible point of problem (4), there are only the following three kinds of possible value for α_i :

(i) If $\alpha_i \in (0, C)$, according to the complementary condition (17b) and (17c), we have $v_i = u_i = 0$. Furthermore, from (17a), we know $F_i(\boldsymbol{\alpha}) = 0$. Therefore, $\ell_i(\boldsymbol{\alpha}) = d_i 0 = 0$.

(ii) If $\alpha_i = 0$, based on the complementary condition (17c), we have $u_i = 0$. Furthermore, according to (17a) and (17d), we have $F_i(\boldsymbol{\alpha}) = v_i \geq 0$. Since the feasible set of problem (10) becomes $\{d_i \mid 0 \leq d_i \leq \min\{C, 1\}\}$ when $\alpha_i = 0$, the corresponding optimal solution is $d_i = 0$. Therefore, $\ell_i(\boldsymbol{\alpha}) = 0F_i(\boldsymbol{\alpha}) = 0$.

(iii) If $\alpha_i = C$, based on the complementary condition (17c), we have $v_i = 0$. Furthermore, according to (17a) and (17d), we have $F_i(\boldsymbol{\alpha}) = -u_i \leq 0$. Since the feasible set of problem (10) becomes $\{d_i \mid -\min\{C, 1\} \leq d_i \leq 0\}$ when $\alpha_i = C$, the corresponding optimal solution is $d_i = 0$. Therefore, $\ell_i(\boldsymbol{\alpha}) = 0F_i(\boldsymbol{\alpha}) = 0$.

Now we prove that if the feasible point of problem (4) $\boldsymbol{\alpha}$ satisfies $\boldsymbol{\ell}(\boldsymbol{\alpha}) = \mathbf{0}$, $\boldsymbol{\alpha}$ must be the KKT point.

For any $i \in \{1, \dots, l\}$, let d_i^* denote the optimal solution of problem (10). Since $\ell_i(\boldsymbol{\alpha}) = F_i(\boldsymbol{\alpha})d_i^* = 0$, we know $F_i(\boldsymbol{\alpha}) = 0$ or $d_i^* = 0$.

(i) If $F_i(\boldsymbol{\alpha}) = 0$, we can choose $u_i = v_i = 0$;

(ii) If $F_i(\boldsymbol{\alpha}) > 0$, we have $d_i^* = -\alpha_i = 0$, namely, $\alpha_i = 0$. Choose $v_i = F_i(\boldsymbol{\alpha})$ and $u_i = 0$;

(iii) If $F_i(\boldsymbol{\alpha}) < 0$, we have $d_i^* = C - \alpha_i = 0$, namely, $\alpha_i = C$. Choose $u_i = -F_i(\boldsymbol{\alpha})$ and $v_i = 0$.

It is not difficult to verify that the above chosen value of $\mathbf{u}$, $\mathbf{v}$ and $\boldsymbol{\alpha}$ satisfy the KKT condition (17). $\square$

Lemma 2. Suppose the kernel function $K(\cdot, \cdot)$ satisfies Mercer condition [16], $\boldsymbol{\alpha}$ is a feasible point of problem (4), then for any $p \in \{1, \dots, l\}$, we have

$$Q_{sub}(\boldsymbol{\alpha}, p) \leq \frac{\ell_p(\boldsymbol{\alpha})}{2} \min\{1, -\frac{\ell_p(\boldsymbol{\alpha})}{2MC^2}\},$$

where, $M = \max_{i=1}^l \{K_{i,i} + 1\}$.

Proof. For any index $p \in \{1, \dots, l\}$, let $\bar{d}_p$ denote the solution of problem (10) when $i = p$. For any $t \in [0, 1]$, $t\bar{d}_p$ stay in the feasible region and $|\bar{d}_p| \leq C$. Furthermore, since kernel function $K(\cdot, \cdot)$ satisfies Mercer condition, we have $K_{p,p} \geq 0$, and $M =$

$\max_{i=1}^l \{K_{i,i} + 1\} \geq 1$. In all,

$$\begin{aligned}
 Qsub(\boldsymbol{\alpha}^{(k)}, p) &\leq \min_{t \in [0,1]} \{ \bar{d}_p F_p(\boldsymbol{\alpha}) t + \frac{1}{2} K_{p,p} \bar{d}_p^2 t^2 \} \\
 &\leq \min_{t \in [0,1]} \{ \ell_p(\boldsymbol{\alpha}) t + \frac{1}{2} K_{p,p} \bar{d}_p^2 t^2 \} \\
 &\leq \min_{t \in [0,1]} \{ \ell_p(\boldsymbol{\alpha}) t + \frac{1}{2} (K_{p,p} + 1) \bar{d}_p^2 t^2 \} \\
 &\leq \min_{t \in [0,1]} \{ \ell_p(\boldsymbol{\alpha}) t + \frac{1}{2} M C^2 t^2 \} \\
 &\leq \frac{\ell_p(\boldsymbol{\alpha})}{2} t^*,
 \end{aligned}$$

where, $t^* = \min\{1, -\frac{\ell_p(\boldsymbol{\alpha})}{2MC^2}\}$. $\square$

Theorem 1. Suppose the kernel function $K(\cdot, \cdot)$ satisfies Mercer condition [16], $n_B \geq n_N \geq 1$. Let $\{\boldsymbol{\alpha}^{(k)}\}$ denote the iterates generated by Algorithm 1 with the working selection rule in Algorithm 3. If $\{\boldsymbol{\alpha}^{(k)}\}$ contains only finite elements, the last iteration point must be the global optima of problem (4). If $\{\boldsymbol{\alpha}^{(k)}\}$ contains infinite elements, any accumulation point is a global optima of problem (4).

Proof. If $\{\boldsymbol{\alpha}^{(k)}\}$ contains finite elements, based on the working selection rule in Algorithm 2, we know the last iteration point must satisfy $\ell(\boldsymbol{\alpha}^{(k)}) = \mathbf{0}$. From lemma 1, it must be a KKT point. Hence, we know that $\boldsymbol{\alpha}$ is a KKT point. Hence, we only discuss the situation of infinite iteration points.

Let $\bar{\boldsymbol{\alpha}}$ be any accumulation of the sequence $\{\boldsymbol{\alpha}^{(k)}\}$. Without loss of generalization, we can assume $\{\boldsymbol{\alpha}^{(k)}\}$ converge to $\bar{\boldsymbol{\alpha}}$ (this requirement always can be obtained by the proper relabeled of the order of iteration points). Because the feasible region of problem (4) is a bounded closed set in $\mathbb{R}^l$ and the iterates generated by the decomposition algorithm are always feasible, we know that $\bar{\boldsymbol{\alpha}}$ is also a feasible point. Furthermore, the value of $f(\bar{\boldsymbol{\alpha}})$ is a finite number.

Let the index p satisfy

$$\ell_p^{(k)} = \min_{1 \leq i \leq l} \ell_i^{(k)} = -\|\ell^{(k)}\|_\infty.$$

Since $n_N \geq 1$, index p must be contained in the selected working set $\mathcal{B}^{(k)}$. Hence,

$$f^{(k+1)} - f^{(k)} \leq Qsub(\boldsymbol{\alpha}^{(k)}, p).$$

From Lemma 2, we have

$$\begin{aligned}
 f^{(k+1)} - f^{(k)} &\leq \frac{1}{2} \ell_p^{(k)} \min\{1, -\frac{\ell_p^{(k)}}{2MC^2}\} \\
 &= -\frac{1}{2} \|\ell^{(k)}\|_\infty \min\{1, \frac{\|\ell^{(k)}\|_\infty}{2MC^2}\}.
 \end{aligned}$$

Sum the above formulae from 0 to s , we get

$$\sum_{k=0}^s \|\ell^{(k)}\|_{\infty} \min\{1, \frac{\|\ell^{(k)}\|_{\infty}}{2MC^2}\} \leq f^{(0)} - f^{(s+1)}.$$

Let $s \rightarrow \infty$, we have

$$\sum_{k=0}^{\infty} \|\ell^{(k)}\|_{\infty} \min\{1, \frac{\|\ell^{(k)}\|_{\infty}}{2MC^2}\} \leq f^{(0)} - f(\bar{\alpha}) < +\infty.$$

Therefore,

$$\|\ell(\bar{\alpha})\|_{\infty} = \lim_{k \rightarrow +\infty} \|\ell(\alpha^{(k)})\|_{\infty} = 0.$$

From lemma 1, we know $\bar{\alpha}$ is the KKT point of problem (4).

Since the kernel function $K(\cdot, \cdot)$ satisfies Mercer condition, problem (4) is a convex problem. Therefore, $\bar{\alpha}$ is a global optima of problem (4). $\square$

Lemma 3. Suppose kernel function $K(\cdot, \cdot)$ satisfies Mercer condition [16], a feasible point of problem (4) α is a KKT point if and only if $\mathbf{q}(\alpha) = \mathbf{0}$.

Proof. Because kernel function $K(\cdot, \cdot)$ satisfies Mercer condition, for all $i \in \{1, \dots, l\}$, $Q_{ii} > 0$. From the definition of $\mathbf{q}(\alpha)$, we know that $\forall i \in \{1, \dots, l\}$, $q_i(\alpha)$ is the optimal objective function value of QP

$$\min_{d_i} F_i(\alpha)d_i + \frac{1}{2}Q_{i,i}d_i^2 \quad (16a)$$

$$s.t. \quad \max\{-1, -\alpha_i\} \leq d_i \leq \min\{1, C - \alpha_i\}. \quad (16b)$$

Let d_i^* denote the optimal solution of (16), we have $q_i(\alpha) = d_i^* F_i(\alpha) + \frac{1}{2}Q_{ii}(d_i^*)^2$.

Firstly, let us prove that any KKT point α of problem (4) satisfies $\mathbf{q}(\alpha) = \mathbf{0}$. Because α is the KKT point of problem (4), from Kuhn-Tucker theorem [18], we know that there exists Lagrange multipliers $\mathbf{u}$ and $\mathbf{v}$ which satisfies KKT condition:

$$F_i(\alpha) - v_i + u_i = 0, \quad \forall i = 1, \dots, l; \quad (17a)$$

$$v_i \alpha_i = 0, \quad \forall i = 1, \dots, l; \quad (17b)$$

$$u_i(C - \alpha_i) = 0, \quad \forall i = 1, \dots, l; \quad (17c)$$

$$v_i, u_i \geq 0, \quad \forall i = 1, \dots, l. \quad (17d)$$

Because α is a feasible point of problem (4), there are only the following three kinds of possible value for α_i :

(i) If $\alpha_i \in (0, C)$, according to the complementary condition (17b) and (17c), we have $v_i = u_i = 0$. Furthermore, from (17a), we know $F_i(\alpha) = 0$. Through simple computation, we get that the optimal objective function value of problem (16) is 0. Therefore, $q_i(\alpha) = 0$.

(ii) If $\alpha_i = 0$, based on the complementary condition (17c), we have $u_i = 0$. Furthermore, according to (17a) and (17d), we have $F_i(\alpha) = v_i \geq 0$. Then, the feasible set of problem (16) becomes $\{d_i \mid 0 \leq d_i \leq \min\{C, 1\}\}$. Since the symmetric axis of quadratic function (16a) is $-\frac{F_i(\alpha)}{2Q_{ii}} \leq 0$, we know the optimal solution is obtained at

zero, namely $q_i(\alpha) = 0$.

(iii) Similar to the discussion in (ii), we can prove that $d_i^* = 0$ and $q_i(\alpha) = 0$.

Now we prove that if a feasible point of problem (4) satisfies $\mathbf{q}(\alpha) = \mathbf{0}$, α is the KKT point of problem (4).

For any $i \in \{1, \dots, l\}$, if $q_i(\alpha) = 0$, we know $d_i^* = 0$ or $d_i^* = -\frac{2F_i(\alpha)}{Q_{ii}}$. In fact, if $d_i^* = -\frac{2F_i(\alpha)}{Q_{ii}} \neq 0$, because of the convexity of the feasible region, it is easy to prove that $q_i(\alpha) < 0$, which is contradict to the known fact of $q_i(\alpha) = 0$. Therefore, $d_i^* = 0$.

(i) If the symmetric axis of (16a) is $-\frac{F_i(\alpha)}{2Q_{ii}} < 0$, because $d_i^* = 0$, there is $\alpha_i = 0$. Set $v_i = F_i(\alpha) > 0$ and $u_i = 0$.

(ii) If the symmetric axis of (16a) is $-\frac{F_i(\alpha)}{2Q_{ii}} > 0$, because $d_i^* = 0$, there is $\alpha_i = C$. Set $u_i = -F_i(\alpha) > 0$ and $v_i = 0$.

(iii) If the symmetric axis of (16a) is $-\frac{F_i(\alpha)}{2Q_{ii}} = 0$. We have $F_i(\alpha) = 0$. Set $u_i = v_i = 0$. It can be checked easily that the above chosen $\mathbf{u}$ and $\mathbf{v}$, together with α satisfy the KKT condition (17). $\square$

Theorem 2. Suppose the kernel function $K(\cdot, \cdot)$ satisfies Mercer condition, $n_B \geq n_N \geq 1$. Let $\{\alpha^{(k)}\}$ denote the iterates generated by Algorithm 1 with working selection rule in Algorithm 4. If $\{\alpha^{(k)}\}$ contains only finite elements, the last iteration point must be the global optima of problem (4). If $\{\alpha^{(k)}\}$ contains infinite elements, any accumulation point is a global optima of problem (4).

Proof. If $\{\alpha^{(k)}\}$ contains finite elements, the last iteration point must satisfy $\mathbf{q}(\alpha^{(k)}) = \mathbf{0}$. From lemma 3, it know that α is a KKT point. Hence, we only discuss the situation of infinite iteration points.

Let $\bar{\alpha}$ be any accumulation point of $\{\alpha^{(k)}\}$. Without loss of generalization, we can assume $\{\alpha^{(k)}\}$ converge to $\bar{\alpha}$ (this requirement always can be obtained by the proper relabeled of the order of iteration points). Because the feasible region of problem (4) is a bounded closed set in $\mathbb{R}^l$ and the iterates generated by the decomposition algorithm are always feasible, we know that $\bar{\alpha}$ is also a feasible point. Furthermore, the value of $f(\bar{\alpha})$ is a finite number.

Let the index p satisfy $q_p^{(k)} = \min_{1 \leq i \leq l} q_i^{(k)} = -\|\mathbf{q}^{(k)}\|_\infty$, where the last equality is based on the definition of ℓ_∞ and the fact of vector $\mathbf{q} \leq \mathbf{0}$. Since $n_N \geq 1$, index p must be contained in the selected working set $\mathcal{B}^{(k)}$. On the other hand, according to the definition of $\mathbf{q}(\alpha)$, we know $Q_{sub}(\alpha, p) = q_p(\alpha)$. Hence,

$$f^{(k+1)} - f^{(k)} \leq Q_{sub}(\alpha^{(k)}, p) = -\|\mathbf{q}^{(k)}\|_\infty.$$

Sum the above formulae from 0 to s ,

$$\sum_{k=0}^s \|\mathbf{q}^{(k)}\|_\infty \leq f^{(0)} - f^{(s+1)}.$$

Let $s \rightarrow \infty$, we have

$$\sum_{k=0}^{\infty} \|\mathbf{q}^{(k)}\|_\infty \leq f^{(0)} - f(\bar{\alpha}) < +\infty.$$

Therefore,

$$\|q(\bar{\alpha})\|_{\infty} = \lim_{k \rightarrow +\infty} \|q(\alpha^{(k)})\|_{\infty} = 0.$$

From lemma 3, we know $\bar{\alpha}$ is the KKT point of problem (4).

Since the kernel function $K(\cdot, \cdot)$ satisfies Mercer condition, problem (4) is a convex problem. Therefore, $\bar{\alpha}$ is a global optima. $\square$

4 Algorithm implementation and numerical experiments

Experiments are carried out in this section to check the efficiency of our new algorithms. We choose the famous package BSVM [9]² as the baseline. Thanks very much to the authors of BSVM for making their source code public available. In order to emphasize the working set selection rules used, we call the implementation of the Algorithm 3 as LBSVM, and call the implementation of the Algorithm 4 as PQB-SVM. Here L means linear and PQ means **p**art of **q**uadratic information is used when selecting the working set.

Besides the working set selection rule, another important component which affects the efficiency of the decomposition algorithm is how to solve the quadratic subproblem (8) at each iteration. According to the structure of the feasible set (8b), we choose projected gradient(PG) algorithm as the default solver for subproblems. However, in some difficult cases, such as when the condition number for the Hessian matrix of (8a) is large, PG method may fail to find the solution satisfied the required accuracy in a given number of iterations. In this difficult case, we call the interior point method(IPM) to solve the QP subproblem (8).

The new algorithms is implemented in C and compile by gcc-4.1.3 with -O3 option. The source code is available upon request from the authors. The underlying PG solver is a modification of PG solver DFGPM in GPDT(Zanni, Serafini and Zanghirati, 7/2/2007 [23]). For the IPM Solver, we use OOQP (Gertz and Wright, 3/9/2008 [6]). A cache is maintained by the Least-Recent-Used principle to store part of the kernel elements which have been calculated.

The cache size for both BSVM, LMSVM and PQBSVM is the default size of BSVM: 100M. The termination accuracy for both algorithms is 10^{-3} . We selected 15 public data sets from text and image classification, hand-written recognition, bioinformation and economics as test problems.³ We summarize the data statistics in Table 1.

Gauss kernel $K(\mathbf{x}, \mathbf{y}) = e^{-\gamma \|\mathbf{x} - \mathbf{y}\|^2}$ are used in the experiment. Model parameter C and kernel parameter γ are selected by the parameter tuning method proposed in [7]. For dataset whose size is larger than 10,000, we random take 5,000 samples as the tuning data set. Otherwise, the whole data set is used. Then the tuning dataset is divided into two parts: 70% for training and 30% for validation. The space we explore

²BSVM can be downloaded from <http://www.csie.ntu.edu.tw/~cjlin/bsvm/>

³“jcn1” is from the International Joint Conference on Neural Networks Competition; “real-sim” is downloaded from <http://www.cs.umass.edu/~mccallum/code-data.html>; “news20”, “rcv1” and “webpage” are down loaded from <http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/>. All the others are from the UCI repository [1].

Table 1: Dataset Statistic and Optimal Parameters

name	#train	#test	#dim.	C	γ
adult	32561	16281	123	2^7	2^{-10}
australian	690	N/A	14	2^{-1}	2^{-2}
breast-cancer	683	N/A	10	2^{-2}	2^{-3}
diabetes	768	N/A	8	2^2	2^{-5}
german	1000	N/A	24	2^7	2^{-10}
heart	270	N/A	13	2^0	2^{-6}
ijcnn1	49990	91701	22	2^5	2^0
ionosphere	351	N/A	34	2^4	2^{-5}
liverdisorder	345	N/A	6	2^6	2^{-2}
mushrooms	8124	N/A	22	2^{-1}	2^{-3}
news20	19996	N/A	1255191	2^5	2^{-5}
rcv1	20242	677399	47236	2^1	2^0
real-sim	72309	N/A	20958	2^5	2^{-1}
sonar	208	N/A	60	2^0	2^{-2}
webpage	49749	14951	300	2^{10}	2^{-10}

for optimal C and γ are in the set $\{2^{-2}, 2^{-1}, \dots, 2^{11}, 2^{12}\}$ and $\{2^{-10}, 2^{-9}, \dots, 2^3, 2^4\}$, respectively. The chosen best kernel and model parameters are given in Table 1.

We increase the size of the working set from 10 to 1280, and we recorded the corresponding results in Table 2. Since LBSVM, PQBSVM and BSVM consider the same model problem (2) and same termination accuracy is used. We do not list prediction accuracy, which is the same for each comparisons presented in Table 2. The column “#WS” represents the size of working set. The column “Time” represents the training time. The shortest training time for each case is marked in bold letter. And in every case, we marked the shortest training time for different working set size in italics. The column “Iter.” is the number of iterations. From Table 2, we can see that, the performance of the new algorithms is at least as good as BSVM. In particular, the PQBSVM algorithm have reached the optimum in most circumstances for all test questions.

Another thing needs to be inspected is the relationship between the training time, the number of iterations and the size of working set. Figure 1 shows the situation for the four test problems: ijcnn1, news20, rcv1 and real-sim, which gives us some intuitive explanation. The number of iterations decreases with the size of the working set. The training time drops with the increasing of working set size at the beginning because of the reduction of iteration numbers. However, when the working set becomes too large, the time for solving the QP subproblem will dominate the training time and the training time starts to increase gradually. This phenomenon meets the general characteristics of decomposition based SVM training algorithm [23]. Perhaps we can exploit and benefit from more larger working set size with more robust inner QP solvers. This topic is one of our future works.

Table 2: Comparisons of BSVM,LBSVM and PQBSVM

Dataset	#WS	Lin's BSVM		LBSVM		PQBSVM	
		time(s)	iter.	time(s)	iter.	time(s)	iter.
adult	10	344.35	14064	279.17	15548	214.35	7163
	20	377.81	6475	247.68	7728	198.48	3985
	40	434.82	2885	186.01	3077	159.49	1627
	80	643.92	1204	161.46	863	155.42	754
	160	1015.46	621	185.08	396	177.66	405
	320	1517.47	387	252.77	261	246.02	202
	640	3831.03	290	749.15	115	753.77	149
	1280	22390.78	197	2781.42	99	1100.64	35
ijcnn1	10	650.7	22794	945.16	58590	356.37	12601
	20	671.26	10009	724.65	14464	301.37	5204
	40	685.16	4049	512.29	4299	269.87	2302
	80	657.55	1600	436.45	1559	239.83	913
	160	574.88	593	363.92	546	196.7	350
	320	528.47	209	284.28	199	163.32	139
	640	596.63	67	372.48	60	256.61	52
	1280	6126.54	32	807.66	27	797.31	24
mushrooms	10	7.31	3160	3.37	902	0.95	145
	20	7.46	1642	3.53	486	1.19	98
	40	8.08	917	3.53	248	1.38	59
	80	8.96	411	4.07	126	1.59	33
	160	10.41	168	4.66	65	2.34	22
	320	14.47	65	7.32	29	3.59	13
	640	33.21	28	20.34	14	8.62	6
	1280	141.85	14	59.8	8	30.36	4
news20	10	2830.06	8070	3246.46	5999	1684.09	2734
	20	2877.79	4088	3234.45	3013	1751.96	1383
	40	2896.77	2054	3296.57	1484	1692	685
	80	2798.33	994	3190.79	742	1736.11	344
	160	2773.31	493	3275.61	372	1691.63	175
	320	3463.19	316	3476.86	192	1831.32	92
	640	3049.63	129	3593.00	96	1973.89	48
	1280	3280.09	60	4412.36	48	2527.88	24
rcv1	10	1380.28	8572	710.2	7681	314.08	3207
	20	1435.63	4456	668.26	3839	321.37	1645
	40	1467.83	2288	676.63	1921	328.41	836
	80	1489.84	1160	661.53	947	324.01	423
	160	1426.52	547	687.94	472	335.43	215
	320	1475.34	285	696.94	231	342.85	107
	640	1817.97	167	742.63	114	367.25	53
	1280	2432.59	89	1061.62	60	484.07	27

Dataset	#WS	Lin's BSVM		LBSVM		PQBSVM	
		time(s)	iter.	time(s)	iter.	time(s)	iter.
real-sim	10	3570.9	14778	2092.14	12331	833.23	4617
	20	3620.44	7475	2018.87	6337	850	2419
	40	3676.19	3794	2056.29	3190	869.82	1249
	80	3603.62	1866	2151.53	1608	884.74	648
	160	3549.23	908	2204.45	817	863.02	316
	320	4263.11	549	2302.09	408	930.37	163
	640	4346.47	261	4611.39	205	1866.31	82
	1280	4482.8	116	5150.1	103	2127.24	42
webpage	10	597.79	64591	2443.66	120330	412.41	12939
	20	728.79	34362	1654.42	40621	434.19	6960
	40	915.8	15449	1422.96	16380	426.42	2668
	80	1096.38	5517	1142.26	6138	310.91	1263
	160	1299.64	1951	924.75	1989	222.45	402
	320	976.65	403	774.38	403	156.46	100
	640	760.68	85	2219.40	245	374.94	45
	1280	2167.48	32	51920.36	1410	1028.9	22

(a) ijcn1

(b) news20

(c) rcv1

(d) real-sim

Figure 1: Working set size to training time and number of iterations.

5 Conclusions and Future Work

We propose two new decomposition algorithms for training bound-constrained SVMs in this paper. These two methods are particularly suitable for training nonlinear SVMs. In the process of selecting working set, when we only use first order information of the objective function, we get a first-order decomposition algorithm. To make the working set selection strategy more effective, we utilize part of second order information and first order information together to choose the working set, which results in Algorithm 4. To the best of our knowledge, this is the first non-SMO decomposition algorithm which can use second order information for selecting working set. Both algorithms are proved to be globally convergent. PG method and IPM are combined together to solve the quadratic subproblems. Experiments on benchmark problems validate the efficiency of the new algorithms.

One of our ongoing work currently is further improving the efficiency of our implementation. We plan to include other types of QP solver for the subproblem solution, which can handle very bad scale problems better. We also consider introducing more careful working set selection strategy to reduce iteration numbers. Last but not least, a careful shrinking strategy should be used to further reduce the number of kernel evaluations.

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