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PRIMAL–DUAL TYPE EVOLUTIONARY MULTIOBJECTIVE OPTIMIZATION

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Abstract. A new, primal-dual type approach for derivation of Pareto front approximations with evolutionary computations is proposed.

At present, evolutionary multiobjective optimization algorithms derive a discrete approximation of the Pareto front (the set of objective maps of efficient solutions) by selecting feasible solutions such that their objective maps are close to the Pareto front. As, except of test problems, Pareto fronts are not known, the accuracy of such approximations is known neither.

Here we propose to exploit also elements outside feasible sets with the aim to derive pairs of Pareto front approximations such that for each approximation pair the corresponding Pareto front lies, in a certain sense, in-between. Accuracies of Pareto front approximations by such pairs can be measured and controlled with respect to distance between such approximations.

A rudimentary algorithm to derive pairs of Pareto front approximations is presented and the viability of the idea is verified on a limited number of test problems.

Keywords: Evolutionary Multiobjective Optimization, lower and upper Pareto front approximation.

1 Introduction

Evolutionary multiobjective optimization (EMO) algorithms (Deb 2001, Coello Coello et al. 2002, Talbi 2009) derive finite approximations of Pareto fronts. Those approximations can be regarded as *lower approximations* (we assume all objectives are maximised), because all their elements are feasible. As, except of test problems, Pareto fronts are in general not known, the exact accuracy of such approximations is known neither and in consequence the accuracy cannot be controlled.

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To heal this, we propose to work with elements outside the feasible solution set (infeasible solutions), with the objective to provide *upper approximations* of Pareto fronts. A pair of a lower and an upper approximation forms an *approximation* of the Pareto front, which accuracy can be controlled by *distance* between the lower and the upper approximation. Thus, the approach proposed realizes the principle of *primal-dual* optimization, an old concept of classical (single objective) optimization used to control accuracy with which the incumbent approximates (in the sense of the objective function value) an optimal solution in case optimization computations are stopped before reaching optimality. No such concept is present as yet in the literature on EMO. Exploiting explicitly infeasible solutions to provide better approximations of Pareto fronts offers a new turn in research in the field.

The outline of the paper is as follows. In Section 2 we provide necessary definitions, in particular we define lower and upper *shells* which yield specific lower and upper approximations of Pareto fronts. Next, in Section 3, we propose an approximation accuracy measure and give a relaxation of the definition of upper shell, which gives rise to a construct more suitable for computations than upper shell itself.

In Section 4 we present a rudimentary evolutionary algorithm for approximating Pareto fronts with given accuracy. An illustrative example is solved in Section 5. Section 6 concludes.

2 Definitions and notation

Mulicriteria Optimization (MO) problem is formulated as:

$$\begin{array}{c} ''max''f(x) \\ x \in X_0 \subseteq \mathcal{R}^n, \end{array} \tag{1}$$

where $f : \mathcal{R}^n \to \mathcal{R}^k; f = (f_1, \ldots, f_k),\ f_i : \mathcal{R}^n \to \mathcal{R},\ i = 1, \ldots, k,\ k \geq 2$, are objective (criteria) functions; $''max''$ denotes the operator of deriving all efficient elements (see the definition below). We assume that X_0 is infinite. We also assume that $X_0 \subset X_{DEC}$, where X_{DEC} (the decision space) is bounded and such that f is meaningful on it.

The *dominance relation* $\prec$ is defined on X_{DEC} as

$$x' \prec x \ \Leftrightarrow \ f(x') \ll f(x),$$

where $\ll$ denotes $f_i(x') \leq f_i(x),\ i = 1, ..., k,$ and $f_i(x') < f_i(x)$ for at least one i.

If $x \prec x'$ then x is *dominated* by x' and x' is *dominating* x.

An element x of X_0 is called *efficient* if

$$\neg\exists_{x' \in X_0} \ x \prec x'.$$

We denote the set of efficient elements by N and set $f(N)$ (the *Pareto front*) by P, $P \subseteq f(X_0)$.

Lower shell is a finite nonempty set $S_L \subseteq X_0$, elements of which satisfy

$$\forall_{x \in S_L} \ \neg\exists_{x' \in S_L} \ x \prec x', \tag{2}$$

(thus no element of S_L is dominated by another element of S_L).

We define *nadir point* y^{nad} as

$$y_i^{nad} = min_{x \in N} f_i(x),\ i = 1, ..., k.$$

Upper shell is a finite nonempty set $S_U \subseteq \mathcal{R}^n \setminus X_0$, elements of which satisfy

$$\forall_{x \in S_U}\ \neg\exists_{x' \in S_U}\ x' \prec x, \tag{3}$$

$$\forall_{x \in S_U}\ \neg\exists_{x' \in N}\ x \prec x', \tag{4}$$

$$\forall_{x \in S_U}\ y^{nad} \ll f(x). \tag{5}$$

3 Approximations of P

Our aim is to approximate P with given accuracy.

To derive S_L which is "close" to N we can use any EMO algorithm (cf. Michalewicz 1996, Deb 2001, Coello Coello et al. 2002, Hanne 2007).

Since the definition of upper shell involves N, this construct is not a suitable approximation of N. A more suitable construct, referring to S_L instead to N, namely *upper approximation* A_U, is obtained by replacing:
condition (3) by

$$\forall_{x \in A_U}\ \neg\exists_{x' \in A_U}\ x' \prec x, \tag{6}$$

condition (4) by

$$\forall_{x \in A_U}\ \neg\exists_{x' \in S_L}\ x \prec x', \tag{7}$$

condition (5) by

$$\forall_{x \in A_U}\ y^{nad}(S_L) \ll f(x), \tag{8}$$

where $y^{nad}(S_L)$ denotes an element of $\mathcal{R}^k$ such that

$$y_i^{nad}(S_L) = min_{x \in S_L} f_i(x),\ i = 1, ..., k,$$

($y^{nad}(S_L)$ varies with S_L).

By definition, upper approximation A_U can contain elements which are dominated by some elements of N, as shown in Figure 1, and certainly such elements are undesirable for the purpose. Condition (8) is meant to limit the domain for such elements. However, as S_L gets "closer" to N and $y^{nad}(S_L)$ gets "closer" to y^{nad}, the chance for such elements being included in A_U gets lower.

With S_L and A_U derived, the accuracy of approximation of P can be measured as

$$acc_P = \max_{x \in S_L} \min_{x' \in A_U} ||f(x) - f(x')||,$$

where $|| \cdot ||$ is a norm. In numerical experiments and applications a form of normalization of acc_P is advisable, cf. Section 5.

In the next section we propose an algorithm for approximating P.

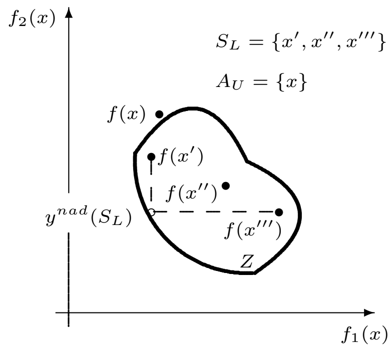

Figure 1: An example where element x dominated by some element of N belongs to A_U.

4 An algorithm for approximating P

The algorithm we propose approximates P within given accuracy.

Let α_P denote the desired value of acc_P.

We limit the domain of searching in $R^n \setminus X_0$ to some set $X_{DEC} = \{x \in \mathcal{R}^n \mid X_i^L \leq x_i \leq X_i^U,\ i = 1, ..., k\}$ such that $X_0 \subseteq int(X_{DEC})$.

To ensure that elements of X_{DEC} generated randomly belong to X_0 with nonzero probability we assume also that set X_0 is k-dimensional.

Algorithm EMO-APPROX

1. $j := 0,\ S_L^j := \emptyset,\ A_U^j := \emptyset$.
2. Select randomly η elements of X_0 and derive S_L^j .
3. Select randomly element x of S_L^j and:
   - 3.1. derive element $x' \in X_{DEC}$ such that $x' \not\prec x$,
   - 3.2. if $x' \in X_0$ then update S_L^j and A_U^j with $S' = S_L^j \cup \{x'\}$, go to 3.4,
   - 3.3. update A_U^j with $A' = A_U^j \cup \{x'\}$,
   - 3.4. if $acc_N \leq \alpha_N$ or $j = j^{max}$ then STOP,
   - 3.5. $j := j + 1$, go to 3.

In Step 2 η is a parameter and derivation of S_L means that selected elements which do not satisfy condition (2) are to be removed.

In substep 3.1 to derive element x' of the required properties, components of x are mutated till $x \in X_{DEC}$ and $x' \not\prec x$ holds. Mutations with probability 0.5 can increase or decrease the value of a randomly selected component. The range of mutations decreases with increasing j. If a mutation increases the i-th component of x then the value of this component after mutation is

$$x_i + (X_i^U - x_i) \times (1 - rnd(0,1)^{2(1-\frac{j}{j^{max}})}),$$

and if this mutation decreases the component then the value of this component after mutation is

$$x_i - (x_i - X_i^L) \times (1 - rnd(0,1)^{2(1-\frac{j}{j^{max}})}).$$

Function $rnd(0,1)$ returns a random number from the range [0,1] with uniform probability. The presented method of mutation and the strategy of decreasing mutation range have been taken from the literature (cf. eg. Michalewicz 1996).

In substep 3.2 the update of S_L^j means that elements of $S' = S_L^j \cup \{x'\}$ which do not satisfy condition (2) are to be removed from S' and only then $S_L^j := S'$. Derivation of A_U^j means that elements of A_U^j which do not satisfy condition (7) with respect to updated S_L^j are to be removed.

In substep 3.3 the update of A_U^j means that elements of $A' = A_U^j \cup \{x'\}$ which do not satisfy conditions (6), (7) and (8) are to be removed from A' and only then $A_U^j := A'$.

There is no guarantee that by each iteration of EMO-APPROX algorithm the approximation accuracy monotonously improves (i.e. on $i+1$-th iteration acc_P takes a smaller value than on iteration i). The phenomenon is illustrated in Figure 2. Indeed, suppose that $S_L = \{a, b\}$, $A_U = \{c, d\}$. Clearly, $acc_P^1 = \max\{||a-c||, ||b-d||\}$ (the superscript indicates the iteration). Including e into S_L causes b to be eliminated from S_L (for e dominates b - condition (2)). Now we have $acc_P^2 = \max\{||a-c||, ||e-d||\}$ and clearly $acc_P^2 \geq acc_P^1$, which means that the approximation accuracy has deteriorated. However, it can be expected that in successive iterations mutations of e or d that local loss of accuracy will be recovered.

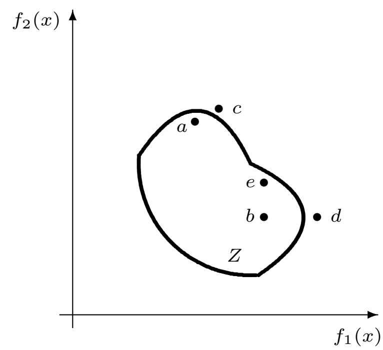

Figure 2: Illustration to possible non-mononotonous behaviour of the algorithm.

5 An illustrative example

We illustrate the behavior of EMO-APPROX algorithm with computations for the test problem taken from Kita et al (1996) (see the references for the link to download)). The problem is as follows

$$''max''(f_1(x), f_2(x)), \text{ where } f_1(x) = -x_1^2 + x_2, \; f_2(x) = \tfrac{1}{2}x_1 + x_2 + 1,$$

subject to

$$\tfrac{1}{6}x_1 + x_2 - \tfrac{13}{2} \leq 0,$$

$$\tfrac{1}{2}x_1 + x_2 - \tfrac{15}{2} \leq 0,$$

$$\tfrac{5}{x_1} + x_2 - 30 \leq 0,$$

$$0 \leq x_i \leq 7 \text{ for } i = 1, 2.$$

We normalized accuracy acc_P as follows

$$acc_P = \max_{x \in S_L} \min_{x' \in A_U} \left(\sum_{i=1}^{k} \left(\frac{f_i(x) - f_i(x')}{s_i^f} \right)^2 \right)^{\frac{1}{2}},$$

where $s_i^f = \max_{x \in S_L} f_i(x) - \min_{x \in S_L} f_i(x), \; i = 1, ..., k$.

We run the algorithm on the test problem with $j^{max} = 9000$, $\eta = 100$, taking three shots of the algorithm behavior and the results it provided at $j = 3000$ and $j = 6000$ and finally at $j = 9000$. As we had no clue what values of parameter α_P to use, we set it to zero and we stopped algorithms after iteration count reached j^{max}. X_{DEC} was assumed to be $[-0.2, 1.2] \times [-0.2, 1.2] \times \ldots \times [-0.2, 1.2]$.

Table 1 shows for each shot values of acc_P, where # f – the number of function f evaluations, $||A_U||$, $||S_L||$ and $||A_U||$ + $||S_L||$ – cardinality of, respectively, A_U, S_L and $A_U \cup S_L$.

In all three shots no element of A_U was dominated by an element of N.

Table 1: Test results.

Shot	j	acc_P	# f	$\|\|S_L\|\|$	$\|\|A_U\|\|$	$\|\|S_L\|\|$ + $\|\|A_U\|\|$
1	3000	0.905	4598	79	63	142
2	6000	0.116	9082	117	86	203
3	9000	0.076	13189	148	91	239

Figure 3 and Figure 4 present, respectively, elements of S_L, A_U and $f(S_L), f(A_U)$ derived for $j = 9000$.

6 Concluding remarks and directions for further research

As said in the Introduction, the approach proposed realizes the principle of *primal-dual* optimization. However, in contrast to the classical (single objective) optimiza-

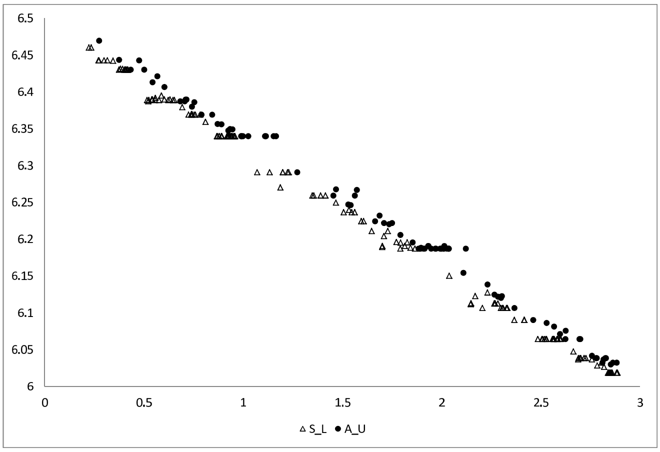

Figure 3: Elements of S_L and A_U.

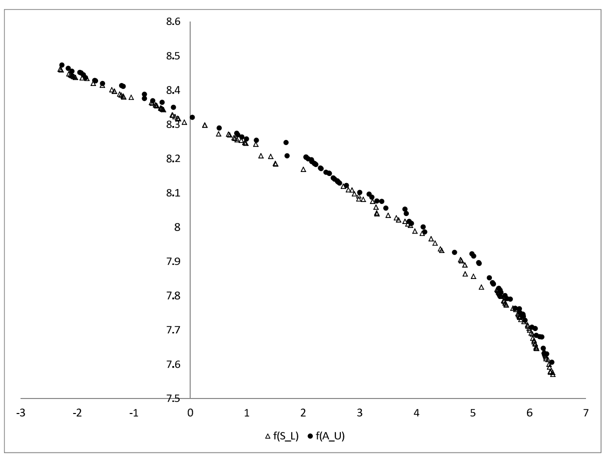

Figure 4: Elements of $f(S_L)$ and $f(A_U)$.

tion, here we do not offer any dual problem. Instead, we offer *constructive duality*, i.e. a construct – upper shell – and its operational counterpart – upper approximation, which can be used, in place of the unknown Pareto front, as a reference to measure the accuracy with which lower shells approximate the Pareto front.

The problem of providing tight approximations of P, being of interest in itself, has an immediate application in Multiple Criteria Decision Making, where accuracy needs to be controlled only locally, as directed by decision maker's preferences (cf. Kaliszewski et al. 2010, Kaliszewski et al. 2012).

Throughout the paper we have assumed that an upper shell exists. It may exists, as in our example, but it may not exist as well (cf. Kaliszewski, Miroforidis 2012). An upper shell does not exist if for no $x' \in X_0$ there exists $x \in \mathcal{R}^n \setminus X_0$ such that $x' \prec x$.

If an upper shell does not exist, it can be replaced by upper shell-like construct built in the objective space R^k, cf. Kaliszewski, Miroforidis 2012.

References

Coello Coello C. A., Van Veldhuizen D.A., and Lamont G.B. (2002), *Evolutionary Algorithms for Solving Multi-Objective Problems*. Kluwer Academic Publishers, New York.

Deb K. (2001), *Multi-objective Optimization Using Evolutionary Algorithms*. John Wiley and Sons, Chichester.

Hanne T. (2007), A multiobjective evolutionary algorithm for approximating the efficient set. European Journal of Operational Research, 176, 1723-1734.

Kaliszewski I., Miroforidis J. (2012), Real and virtual Pareto set upper approximations. In: Multiple Criteria Decision Making '11, eds. T. Trzaskalik, T. Wachowicz, The University of Economics in Katowice, in print.

Kaliszewski I., Miroforidis J., Podkopayev D. (2011), Interactive Multiple Criteria Decision Making based on preference driven Evolutionary Multiobjective Optimization with controllable accuracy – the case of ρ-efficiency. Systems Research Institute Report, RB/1/2011.

Kaliszewski I., Miroforidis J., Podkopayev D. (2012), Interactive Multiple Criteria Decision Making based on preference driven Evolutionary Multiobjective Optimization with controllable accuracy. European Journal of Operational Research, 216, 188-199.

Kita H., Yabumoto Y., Mori N., Nishikawa Y. (1996), Multi-Objective Optimization by Means of the Thermodynamical Genetic Algorithm. In: Parallel Problem Solving from Nature–PPSN IV, Voigt H-M., Ebeling W., Rechenberg I., Schwefel H-P., eds, Lecture Notes in Computer Science, 504-512, Springer-Verlag.

Kita test problem: http://www.lania.mx/ ccoello/EMOO/testfuncs/ , downloaded September 28, 2012.

Michalewicz Z. (1996), *Genetic Algorithms + Data Structures = Evolution Programs*. Springer.

Miroforidis J. (2008), *Decision Making Aid for Operational Management of Department Stores with Multi-objective Optimization and Soft Computing*. PhD Thesis, Systems Research Institute, Warsaw.

Talbi E. (2009), *Metaheuristics: From Design To Implementation*. Wiley.

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